3-Zebra Trees

Oluwatobi Aderotoye, Dennis Davenport, Shakuan Frankson,
and Kanasia McTyeire
Mathematics Department
Howard University, Washington, DC
oluwatobi.aderotoye@bison.howard.edu
dennis.davenport@howard.edu
shakuan.frankson@bison.howard.edu
kanasia.mctyeire@bison.howard.edu

Abstract

An ordered tree, also known as a plane tree or a planar tree, is defined recursively as having a root and an ordered set of subtrees. A *3-zebra tree* is an ordered tree where all edges connected to the root (call this height 1) are tricolored as are all edges at odd height. The edges at even height are all black as usual. In this paper we show that the number of *3-zebra trees* with n edges is the number of Schröder paths with bicolored level steps.

1 Introduction

A *3-zebra tree* is an ordered tree where every edge at the root (height 1) is either red, black or white, as are all edges at odd heights.

In our research several sequences were found which were already in the Online Encyclopedia of Integer Sequences (OEIS) [8]; the A-numbers in this paper are from this source.

The companion concept is that of *little 3-zebra trees* where the tricolored edges are at even heights.

This paper generalizes an article by L. Shapiro, D. Davenport, and L. Woodson titled "Zebra Trees", see [3], where a *zebra tree* is an ordered tree where edges at the root are black or white, as are those at odd heights. An obvious question is what happens when odd edges are $k-colored$? This question is investigated in this article. In Section 2, we consider the case when the odd heights are $3-colored$ (called *3-zebra trees*). We prove that the generating function counting such trees also counts Schröder paths with bicolored level steps. Using $RiordanArrays$ we show that the average root degree of 3-*zebra trees* is $2\sqrt{3}+1$. In Section 3 we examine the boundary of 3-*zebra trees* and find the proportion of such trees whose left boundary edge is length 1. In the Section 4, we investigate general k-*zebra trees.*

2 3-*Zebra Trees*

Theorem 1. *Let $\mathcal{A}$ be the generating function counting 3-zebra trees and $\mathcal{B}$ be the generating function counting little 3-zebra trees. Then,*

a. $\mathcal{A}(x) = \mathcal{A} = \frac{1-2x-\sqrt{1-8x+4x^2}}{2x}$ *and*

b. $\mathcal{B}(x) = \mathcal{B} = \frac{1+2x-\sqrt{1-8x+4x^2}}{6x}$

Proof. Consider the following figure. Figure 1 is used to count *3-zebra trees* and figure 2 is used to count *little 3-zebra trees.*

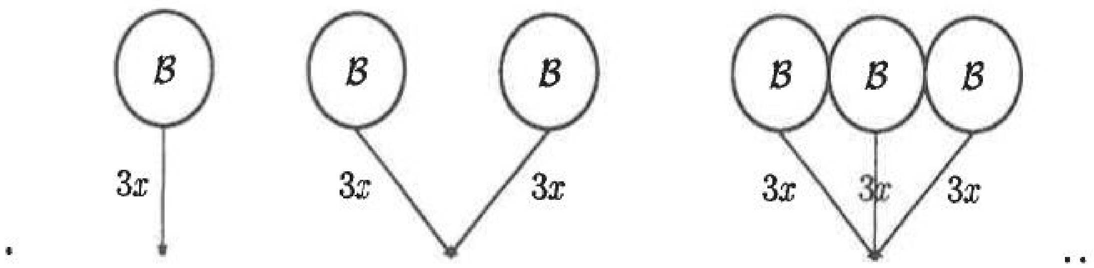


Figure 1: Counting *3-Zebra* Trees

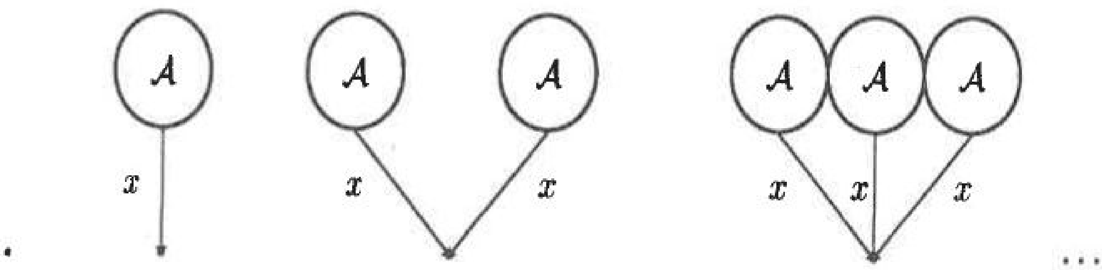


Figure 2: Counting *Little 3-Zebra* Trees

From Figure 1, we get

$$\mathcal{A} = 1 + 3x\mathcal{B} + (3x)^2\mathcal{B}^2 + (3x)^3\mathcal{B}^3 + ... = \frac{1}{1-3x\mathcal{B}} \tag{1}$$

From Figure 2, we get

$$\mathcal{B} = 1 + x\mathcal{A} + x^2\mathcal{A}^2 + x^3\mathcal{A}^3 + ... = \frac{1}{1-x\mathcal{A}} \tag{2}$$

Using equations 1 and 2, we get

$$\mathcal{A} = \frac{1 - x\mathcal{A}}{1 - x\mathcal{A} - 3x}.$$

Hence,

$$x\mathcal{A}^2 + (2x - 1)\mathcal{A} + 1 = 0. \tag{3}$$

The result for $\mathcal{A}$ follows from equation 3 and the quadratic formula. A similar argument gives $\mathcal{B}$. □

Expanding the generating functions gives the following:

$$\mathcal{A} = 1 + 3x + 12x^2 + 57x^3 + 300x^4 + 1686x^5 + \ldots [A047891] \tag{4}$$
$$\mathcal{B} = 1 + x + 4x^2 + 19x^3 + 100x^4 + 562x^5 + \ldots [A007564] \tag{5}$$

The following can easily be proved using *Theorem 1.* The formulas will be useful in studying *3-zebra trees.*

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be as defined in* **Theorem 1.** *Then,*

1. $\mathcal{B} = \frac{\mathcal{A}+2}{3}$,
2. $\mathcal{A}^2 = \frac{\mathcal{A}-2x\mathcal{A}-1}{x}$.

The next theorem is known, but the proof is possibly new.

Theorem 3. *$\mathcal{A}$ counts ordered trees with tricolored leaves.*

Proof. Let T be the generating function counting ordered trees with tricolored leaves. Consider the following figure.

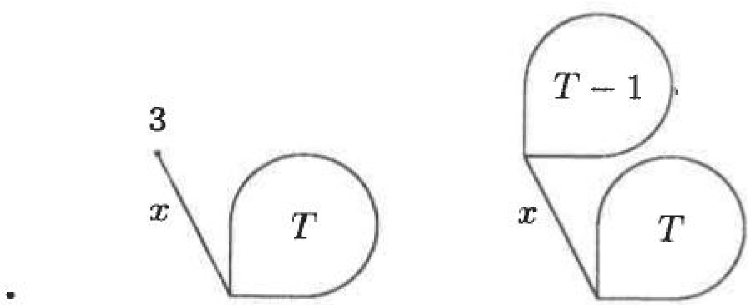


Figure 3: Ordered Trees With Tricolored Leaves

The figure gives all possible ordered trees with tricolored leaves. From this we get

$$\begin{aligned} T(x) &= 1 + 3xT + xT(T-1) \\ &= 1 + 2xT + xT^2 \end{aligned}$$

The results follows by solving for T using the quadratic formula. □

The Schröder numbers S_n count many objects, such the number of lattice paths in the Cartesian plane that start at $(0,0)$, end at $(2n,0)$, do not go below the x−axis, and are composed only of steps $(1,1)$ (up), $(1,-1)$ (down), and $(2,0)$ (horizontal). Such paths are called *Schröder paths*. The generating function for the *Schröder Numbers* is

$$\begin{aligned} S(x) &= \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x} \\ &= 1 + 2x + 6x^2 + 22x^3 + 90x^4 + 394x^5 + \cdots [A006318] \end{aligned}$$

Theorem 4. *The generating function counting Schröder paths with bicolored level steps is $\mathcal{A}$. And the generating function counting such paths with no level steps on the $x-axis$ is $\mathcal{B}$.*

Proof. The proof is similar to the classical proof of finding the generating function counting Schröder paths. Let $G(x)$ be the generating function counting Schröder paths with bicolored level steps. And let $F(x)$ be the generating function counting such paths with no level steps on the $x-axis$. Recall that a prime Schröder path is either simply a horizontal step of length 2 or an up step followed by a Schröder path above the $x-axis$, followed by a down step. In our case the horizontal step has weight $2x$. So, the generating function for prime paths is $2x + xG(x)$.

Thus

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} \Big(2x + xG(x)\Big)^n \\ &= \frac{1}{1 - 2x - xG(x)} \end{aligned}$$

Hence we get

$$G(x) = \frac{1 - 2x - \sqrt{1 - 8x + 4x^2}}{2x} = \mathcal{A}$$

By a similar argument we get

$$F(x) = \frac{1 + 2x - \sqrt{1 - 8x + 4x^2}}{6x} = \mathcal{B}$$

□

In our calculations for **Theorem 6** we will use Riordan Arrays. In 1991, Shapiro, Getu, Woan, and Woodson introduced a group of infinite lower triangular matrices called the Riordan group, see [7]. The elements of the group are defined by two power series $g = 1 + g_1x + g_2x^2 + ...$ and $f = f_1x + f_2x^2 + f_3x^3 + ...$ with $f_1 \neq 0$, where the coefficients of g gives the left most column and the k^{th} column is given by the coefficients of $g \cdot f^k$, for $k = 0, 1, 2, 3, ...$ The power series f is called the multiplier function. Let $d_{n,k}$ be the coefficient of x^n in $g(x)(f(x))^k$. Then $D = (d_{n,k})_{n,k\geq 0}$ is a Riordan array and an element of the Riordan group. We write $D = (g(x), f(x))$.

Theorem 5. *(The Fundamental Theorem of Riordan Arrays): Let $A(x) = \sum_{k=0}^{\infty} a_k x^k$ and $B(x) = \sum_{k=0}^{\infty} b_k x^k$ and let A and B be the column vectors $A = (a_0, a_1, a_2, \cdots)^T$ and $B = (b_0, b_1, b_2, \cdots)^T$. Then $(g, f)A = B$, if and only if $B(x) = g(x)A(f(x))$.*

The proof of the following is similar to that of **Theorem 3** from [3].

Theorem 6. *The average degree of the root for 3-zebra trees is $2\sqrt{3} + 1 \approx 4.464$.*

Proof. Our proof is motivated by Figure 1, where the first diagram gives all 3-zebra trees with root degree one, the second those with root degree 2, the third those with root degree 3, and so on. This gives us the following Riordan array where the multiplier function is $3x\mathcal{B}$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 3 & 0 & 0 & 0 & 0 & \\ 0 & 3 & 9 & 0 & 0 & 0 & \\ 0 & 12 & 18 & 27 & 0 & 0 & \cdots \\ 0 & 57 & 81 & 81 & 81 & 0 & \\ 0 & 300 & 414 & 405 & 324 & 243 & \\ & & & \cdots & & & \end{bmatrix} = (1, 3x\mathcal{B})$$

Recall that $\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots$.

Thus using **The Fundamental Theorem of Riordan Arrays**, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 3 & 0 & 0 & 0 & 0 & \\ 0 & 3 & 9 & 0 & 0 & 0 & \\ 0 & 12 & 18 & 27 & 0 & 0 & \cdots \\ 0 & 57 & 81 & 81 & 81 & 0 & \\ 0 & 300 & 414 & 405 & 324 & 243 & \\ & & & \cdots & & & \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 21 \\ 129 \\ 786 \\ 4854 \\ \cdots \end{bmatrix}$$

$$\begin{aligned} &= (1, 3x\mathcal{B})\,\frac{x}{(1-x)^2} = \frac{3x\mathcal{B}}{(1-3x\mathcal{B})^2} \\ &= 3x\mathcal{B}\mathcal{A}^2 = 3x\left(\frac{\mathcal{A}+2}{3}\right)\mathcal{A}^2 \\ &= x(2\mathcal{A}+\mathcal{A}^2)\mathcal{A} = (\mathcal{A}-1)\mathcal{A} \\ &= \mathcal{A}^2 - \mathcal{A} \end{aligned}$$

Thus the average root degree is

$$\frac{[x^n](\mathcal{A}^2 - \mathcal{A})}{[x^n]\mathcal{A}}.$$

Note that for $n \geq 1$,

$$\begin{aligned} [x^n]\mathcal{A}^2 &= [x^n]\frac{\mathcal{A} - 1 - 2x\mathcal{A}}{x} \\ &= [x^n]\left[\frac{\mathcal{A}}{x} - 2\mathcal{A}\right] \\ &= a_{n+1} - 2a_n \end{aligned}$$

Thus

$$\begin{aligned} \frac{[x^n](\mathcal{A}^2 - \mathcal{A})}{[x^n]\mathcal{A}} &= \frac{a_{n+1} - 2a_n - a_n}{a_n} \\ &= \frac{a_{n+1}}{a_n} - 3 \to 4 + 2\sqrt{3} - 3 = 1 + 2\sqrt{3} \approx 4.464 \end{aligned}$$

□

The asymptotic result comes from $2x\mathcal{A} = 1 - 2x - \sqrt{1 - 8x + 4x^2}$ having its smallest singularity at $1 - \frac{1}{2}\sqrt{3}$, together with the ratio test.

Using similar calculations and Figure 2, we get the following theorem on the average root degree for the little 3-zebra trees.

Theorem 7. *The average degree of the root for little 3-zebra trees is* $\frac{2}{3}\sqrt{3} + 1 \approx 2.155$.

By way of comparison, the average degree at the root for ordered trees approaches 3, see [5]; and the average degree at the root for zebra trees approaches 3.828, see [3].

3 The Boundary of 3-Zebra Trees

In [4] the notion of the boundary of ordered trees was introduced. The definition is given below.

Definition 1. The *right boundary* of an ordered tree is the rightmost path that emanates from the root and terminates at the right most leaf. The *left boundary* is defined similarly. The *boundary* of an ordered tree is the union of the left and right boundaries.

As in [4], we start by examining the left boundary edges. We consider two cases, when the left boundary edge terminates at an even height and when it terminates at an odd height. Finding our generating functions is motivated by the following figure.

Using Figure 4, the generating function counting trees with odd left boundary height is

$$\begin{aligned} O &:= 3xA + 3^2x^3BA^2 + 3^4x^5A^3B^2 + \ldots \\ &= \frac{3xA}{1-3x^2AB} \\ &= 3x + 9x^2 + 45x^3 + 234x^4 + 1314x^5 + 7713x^6 + \ldots \end{aligned}$$

and the generating function counting trees with even left boundary height is

$$\begin{aligned} E &:= 1 + 3x^2AB + 3^2z^4A^2B^2 + 3^3z^6A^3B^3 + \ldots \\ &= \frac{1}{1-3x^2AB} \\ &= 1 + 3x^2 + 12x^3 + 66x^4 + 372x^5 + 2199x^6 + \ldots \end{aligned}$$

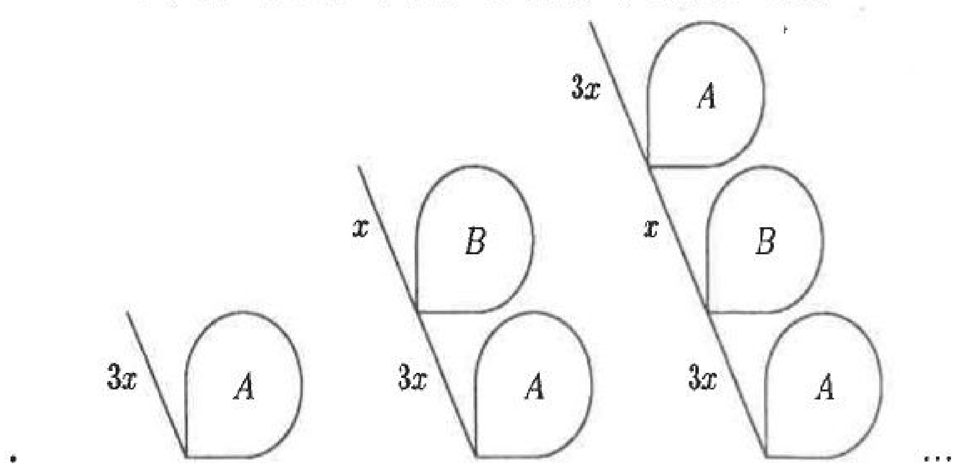


Figure 4: Left Boundary Edges

Combining the two cases gives

$$\frac{3xA}{1-3x^2AB} + \frac{1}{1-3x^2AB} = \frac{1+3xA}{1-3x^2AB} = A, \text{ as expected.} \tag{6}$$

Theorem 8. *The proportion of 3-zebra trees whose left boundary edge is of length* 1 *approaches* $2(3-2\sqrt{2}) \approx 0.879$.

Proof.

$$\frac{[x^n]3xA}{[x^n]A} = \frac{3A_{n-1}}{A_n}$$
$$\to \frac{3}{4+2\sqrt{3}} = 3(1-\frac{1}{2}\sqrt{3}) \approx 0.879.$$

□

4 General k-Zebra Trees

A *k-zebra tree* is an ordered tree where every edge at the root (height 1) is one of k colors, as are all edges at odd heights. The proof of the following is similar to that of *Theorem*1.

Theorem 9. *Let* $\mathcal{A}_\mathcal{K}$ *be the generating function counting k-zebra trees and* $\mathcal{B}_\mathcal{K}$ *be the generating function counting little k-zebra trees. Then,*

a. $\mathcal{A}_\mathcal{K}(x) = \mathcal{A}_\mathcal{K} = \dfrac{1-x(k-1)-\sqrt{1-2x(k+1)+x^2(k-1)^2}}{2x}$ *and*

b. $\mathcal{B}_\mathcal{K}(x) = \mathcal{B}_\mathcal{K} = \dfrac{1+x(k-1)-\sqrt{1-2x(k+1)+x^2(k-1)^2}}{2kx}$

Proof. Consider the following figure that is similar to Figure 1 and 2 in section 1. Figure 5 is used to count *k-zebra trees* and figure 6 is used to count *little k-zebra trees.*

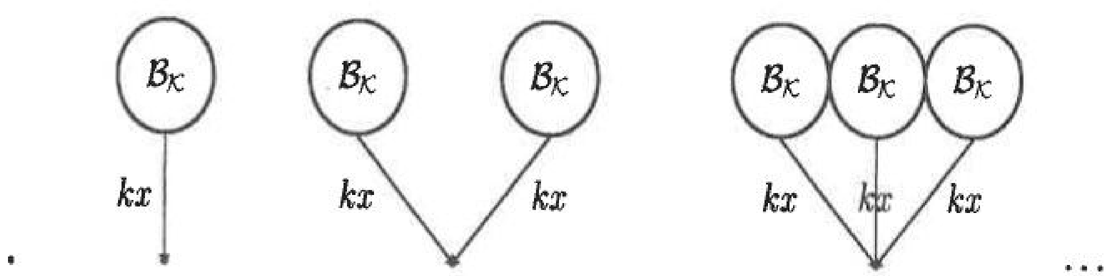


Figure 5: Counting *K-Zebra* Trees

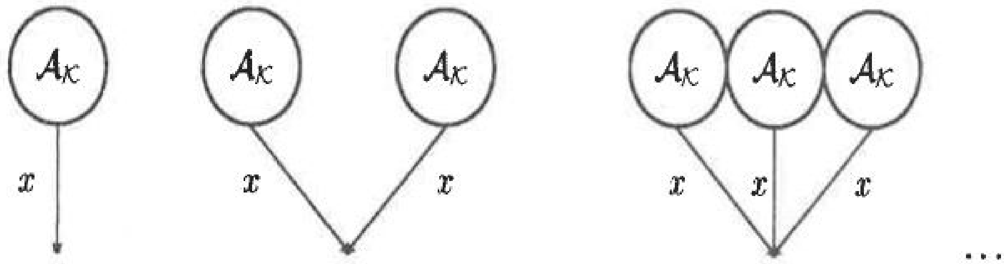


Figure 6: Counting *Little K-Zebra* Trees

From Figure 5, we get

$$\mathcal{A}_\mathcal{K} = 1 + kx\mathcal{B}_\mathcal{K} + (kx)^2{\mathcal{B}_\mathcal{K}}^2 + (kx)^3{\mathcal{B}_\mathcal{K}}^3 + ... = \frac{1}{1 - kx\mathcal{B}_\mathcal{K}} \tag{7}$$

From Figure 6, we get

$$\mathcal{B}_\mathcal{K} = 1 + x\mathcal{A}_\mathcal{K} + x^2{\mathcal{A}_\mathcal{K}}^2 + x^3{\mathcal{A}_\mathcal{K}}^3 + ... = \frac{1}{1 - x\mathcal{A}_\mathcal{K}}. \tag{8}$$

Using equations 7 and 8, we get

$$\mathcal{A}_\mathcal{K} = \frac{1 - x\mathcal{A}_\mathcal{K}}{1 - x\mathcal{A}_\mathcal{K} - kx}.$$

Hence,

$$x{\mathcal{A}_\mathcal{K}}^2 + (kx - x - 1)\mathcal{A}_\mathcal{K} + 1 = 0. \tag{9}$$

The result for $\mathcal{A}_\mathcal{K}$ follows from equation 9 and the quadratic formula. A similar argument gives $\mathcal{B}_\mathcal{K}$. □

Note that when $k = 1$ we get $\mathcal{A}_1 = \mathcal{B}_1 = \frac{1-\sqrt{1-4x}}{2x}$, which is the generating function counting Dyck paths. This makes sense, since Dyck paths have no level steps and the 1−zebra tree is simply an ordered rooted tree.

Expanding the generating functions gives the following:

$$\begin{aligned} \mathcal{A}_\mathcal{K} &= 1 + kx + (k^2 + k)x^2 + (k^3 + 3k^2 + k)x^3 + (k^4 + 6k^3 + 6k^2 + k)x^4 + \quad (10) \\ \mathcal{B}_\mathcal{K} &= 1 + x + (k + 1)x^2 + (k^2 + 3k + 1)x^3 + (k^3 + 6k^2 + 6k + 1)x^4 + \ldots \quad (11) \end{aligned}$$

The following, which is similar to *Theorem 2* in section 1, can easily be proved using *Theorem 9*. Hence, we get similar relationships between the generating function counting little k-zebra trees and big k-zebra trees as is seen with the generating functions for little Schröder numbers and big Schröder numbers.

Theorem 10. *Let $\mathcal{A}_{\mathcal{K}}$ and $\mathcal{B}_{\mathcal{K}}$ be as defined in* **Theorem 9**. *Then,*

1. $\mathcal{B}_{\mathcal{K}} = \frac{\mathcal{A}_{\mathcal{K}}+(k-1)}{k}$

2. ${\mathcal{A}_{\mathcal{K}}}^2 = \frac{\mathcal{A}_{\mathcal{K}}-(kx-x)\mathcal{A}_{\mathcal{K}}-1}{x}$

The proofs of the following theorems are similar to those of *Theorem 3* and *Theorem 4*; simply change the 3 in the *Figure 3* to k for *Theorem 3* and the weight of the level steps to kz for *Theorem 4*.

Theorem 11. *$\mathcal{A}_k$ counts ordered trees with $k-$colored leaves.*

Theorem 12. *The generating function counting Schröder paths with $k-1$ colored level steps is $\mathcal{A}_{\mathcal{K}}$. And the generating function counting such paths with no level steps on the $x-axis$ is $\mathcal{B}_{\mathcal{K}}$.*

5 Bijections

We can easily find a bijection between Schröder paths with $k-1$ colored level steps and ordered trees with k colored leaves. We will illustrate the bijection using the following tree with $k-$colored leaves.

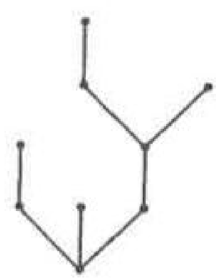
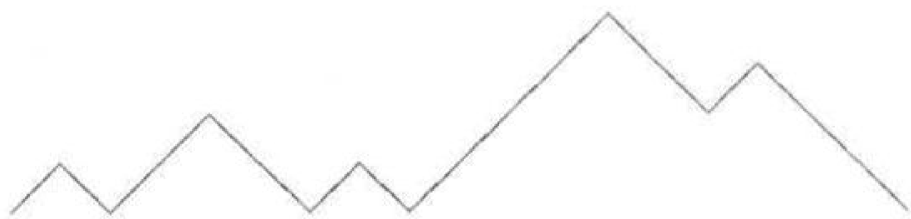


Figure 7: Ordered Tree With $k-$colored Leaves

Using the preorder traversal algorithm, with U denoting up and D down, we get the following $UUDDUDUUUUDDUDDD$. This corresponds to the Schröder path given below.



Each peak corresponds to a leaf in the ordered tree. Leaves that are colored k are mapped to the path with a peak, while those that are colored i for $1 \leq i \leq k-1$ are mapped to the path with level step of the same color.

We could not find a bijection involving the k-zebra trees. That is an ongoing project.

6 Acknowledgments

The research was partially supported by NSF grant DUE-1356481. During the summer of 2017 the three HU students did an REU funded by the grant. We would also like to thank Professor Lou Shapiro of Howard University and Professor Leon Woodson of Morgan State University for their helpful comments. It was Professor Woodson who suggested the proof of *Theorem 3*.

References

[1] *P. Barry*, On the Inverses of a Family of Pascal-Like Matrices Defined by Riordan Arrays, Journal of Integer Sequences, Vol. 16 (2013), 13.5.6.

[2] *G. S. Cheon and L. Shapiro*, The Uplift Principle for Ordered Trees, Applied Mathematics Letters 25, (2012), 1010-1015

[3] *D. Davenport, L. Shapiro, and L. Woodson*, Zebra Trees, Congressus Numerantium 228 (2017), 75-84

[4] *D. Davenport, L. Pudwell, L. Shapiro, and L. Woodson*, The Boundary of Ordered Trees, Journal of Integer Sequences, Vol. 18 (2015) Article 15.5.8

[5] *N. Dershowitz and S. Zaks*, Enumeration of Ordered Trees, Discrete Mathematics, Vol. 31 (1980) 9-28

[6] *E. Schröder*, Vier Combinatorische Probleme, Z. fur Math. Phys. 15 (1870), 361-376

[7] *L. W. Shapiro, S. Getu, W. Woan and, L. C. Woodson*, The Riordan Group, Discrete Applied Mathematics 34 (1991), 229-239.

[8] *Sloane's Online Encyclopedia of Integer Sequences*, www.research.att.com/njas/sequences/

[9] *R. Stanley*, Hipparchus, Plutarch, Schröder, and Hough, American Mathematical Monthly 104 (1997), 344-350.