# Maximum Rectilinear Crossing Numbers of Polyhex Graphs

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**Abstract**

A polyhex is a set of hexagons of the euclidean tessellation of the plane by congruent regular hexagons. Then a polyhex graph has the vertexpoints of the hexagons as its vertices and the sides of the hexagons as is edges. A rectilinear drawing of a graph in the plane uses straight line segments for the edges. Partial results are given for the maximum number of crossings over all rectilinear drawings of a polyhex graph.

## 1 Introduction

A rectilinear drawing $D(G)$ of a graph $G$ is a realization of $G$ in the plane where the vertices of $G$ are mapped into distinct points, also called vertices of $D(G)$, and where the edges are mapped into straight line segments, also called edges of $D(G)$, connecting the corresponding vertices in such a way that two edges have at most one point in common, either at a vertex or a crossing (point of intersection). The maximum number of crossings over all rectilinear drawings $D(G)$ is denoted by $\overline{CR}(G)$.

A polyhex (hexagon polyomino, hexagon animal) is a set of hexagons of the euclidean tessellations of the plane by congruent regular hexagons such that the set of hexagons and its complement are edge-connected. Then a polyhex graph $P$ has the vertexpoints of the hexagons as its vertices and the sides of the hexagons as its edges. In honor of the 50th anniversary of the Southeastern Conference on Combinatorics, Graph Theory, and Computing

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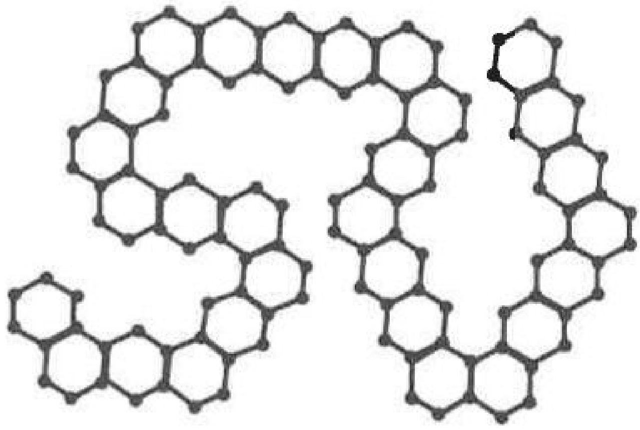

Figure 1: Polyhex graph $S(29, 13)$ with $\overline{CR}(S(29, 13)) = 10, 284$.

Figure 1 presents an example of a polyhex graph with 29 hexagons. This graph can have 10, 284 crossings in maximum.

In this paper we will consider $\overline{CR}(P)$ for polyhex graphs $P$ continuing earlier work in [1] for polyominoes (squares), in [2] for polyiamonds (triangles), and in [3] for game boards as chessboard like parts of all three euclidean tessellations of the plane.

## 2 Two gluing theorems

For certain polyhex graphs it is possible to glue together their extremal drawings such that the resulting drawing has the maximum rectilinear crossing number of the corresponding polyhex graph too.

**Theorem 1.** *Let $G_1$ and $G_2$ be polyhex graphs with $e_1$ and $e_2$ edges and with extremal drawings as in Figure 2 having $\overline{CR}(G_1)$ and $\overline{CR}(G_2)$ crossings. Then for $G_1 + G_2$, i.e. the graph that results when $G_1$ and $G_2$ are glued along the gluing edge $AB$, it holds*

$$\overline{CR}(G_1 + G_2) = \overline{CR}(G_1) + \overline{CR}(G_2) + (e_1 - 1)(e_2 - 1) - 2.$$

*Proof.* If the two extremal drawings $D(G_1)$ and $D(G_2)$ from Figure 2 are combined along the edge $AB$ as depicted in Figure 3 then all $e_1 - 1$ edges of $G_1$ (excluding $AB$) and all $e_2 - 1$ edges of $G_2$ (excluding $AB$) determine pairwise crossings besides the two pairs of adjacent edges incident to $A$ and to $B$. Thus $(e_1 - 1)(e_2 - 1) - 2$ crossings are added to $\overline{CR}(G_1) + \overline{CR}(G_2)$. Since these occur, there are no misses other than those within $D(G_1)$ and $D(G_2)$. Therefore, this gives the asserted maximum.

□

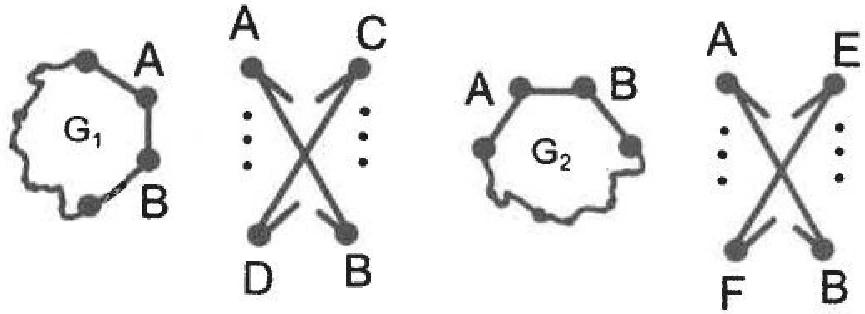


Figure 2: $G_1$ and $G_2$ with extremal $D(G_1)$ and $D(G_2)$.

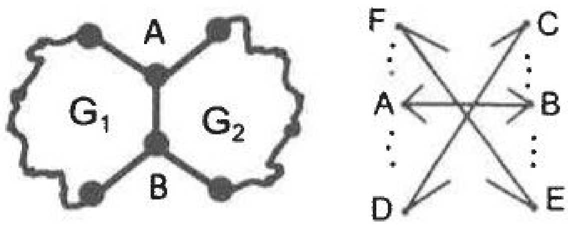


Figure 3: $G_1 + G_2$ and its extremal drawing.

In the following a miss is a nonadjacent edge pair which does not determine a crossing.

**Remark 1.** *We may ask whether we can prove that any extremal drawing of a polyhex graph P (or even of a bipartite graph) has a bipartite extremal drawing as in Figure 2.*
*In any case, if there exists such an extremal drawing then there are exactly two gluing edges. By symmetry of P there may be other pairs too, but not at the same time.*

**Theorem 2.** *If a hexagon H is glued along AB to a polyhex graph P where the second gluing edge of H is adjacent to the opposite edge of AB, i.e. CD or EF in Figure 4, then the subdrawing D(H) of D(P+H) with* $\overline{CR}(P+H)$ *crossings has only six crossings and* $\overline{CR}(P+H) = \overline{CR}(P) + 5e - 1$*, where e is the number of edges of P.*

*Proof.* Consider $P$ and $H$ as in Figure 4. If $D(H)$ has $CD$ as a gluing edge and a maximum of 7 crossings then $D(H)$ is unique as in the middle of Figure 4 and $AB$ cannot be the second gluing edge of Theorem 1. If a gluing edge $AB$ would be done corresponding to Theorem 1, then that edge of $P$ being incident to $B$ would induce at least two misses ($AE, EF$, or $CD, ED$). However, that $D(H)$ on the right of Figure 4 has six crossings,

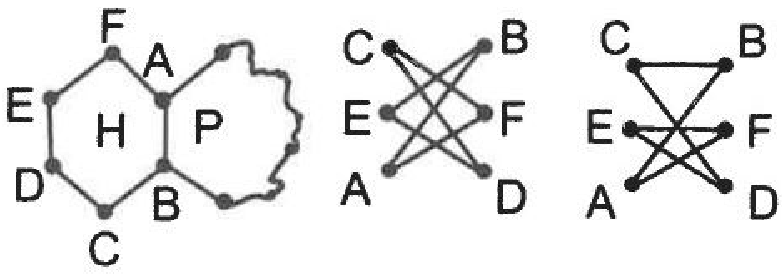


Figure 4: Gluing of $H$ by nonopposite edges.

i.e. one miss only in addition, and both $AB$ and $CD$ are possible as gluing edges corresponding to Theorem 1. Thus

$$\overline{CR}(P+H) = \overline{CR}(P) + 6 + 5(e-1) - 2 = \overline{CR}(P) + 5e - 1.$$

□

A snake is a polyhex with a path as its skeleton where each hexagon determines a vertex and edges are between hexagons having an edge in common. In other words, a snake has two hexagons at its two ends and each further hexagon has exactly two neighboring hexagons at one edge and either at its opposite edge or at one of the two edges which is adjacent to the opposite one.
A general upper bound for $\overline{CR}(G)$ for a graph $G$ with vertex set $V$, edge set $E$, and vertex degrees $deg(v)$ is the so-called thrackle number $t(G)$ which counts the number of nonadjacent edge pairs of $G$, that is,

$$\overline{CR}(G) \leq t(G) = \binom{|E|}{2} - \sum_{v \in V} \binom{deg(v)}{2}.$$

For a polyhex graph $P_n$ with $n$ hexagons the subthrackle number $s(P_n)$ improves the upper bound as follows,

$$\overline{CR}(P_n) \leq s(P_n) = t(P_n) - 2n,$$

since for each of the $n$ hexagons there are two misses in its unique extremal rectilinear drawing (see Lemma 1 in [1], [4], and [6]).

**Corollary 1.** *For a snake polyhex graph $S(n,h)$ with $n$ hexagons where $h \leq n-2$ of them have nonopposite neighbors it holds*

$$\overline{CR}(S(n,h)) = \frac{1}{2}(25n^2 - 15n + 4) - h.$$

*Proof.* The snake $S(n, h)$ has $5n+1$ edges, $4n+2$ vertices, $2n+2$ of them of degree 3 and $2n+4$ of degree 2. Then the subthrackle number $s$ is

$$s(S(n,h)) = \binom{5n+1}{2} - \binom{3}{2}(2n-2) - (2n-4) =$$

$$= \frac{1}{2}(25n^2 - 15n + 4).$$

Furthermore, by Theorem 2 there is exactly one further miss for each of the $h$ hexagons and thus $\overline{CR}(S(n,h)) = s(S(n,h)) - h$. □

The example in Figure 1 shows $S(29,13)$ implying $\overline{CR}(S(29,13)) = 10,284$. For $h = 0$ the $n$ hexagons are positioned in a straight line and the subthrackle bound is attained,

$$\overline{CR}(S(n,0)) = s(S(n,0)) = \frac{1}{2}(25n^2 - 15n + 4).$$

It may be remarked that using the gluing theorems we can get $\overline{CR}$ for a chain of polyhexes glued along the two gluing edges and connected to arbitrarily long snakes. However, it is an open problem to get an octopus, i.e. to have three or more snakes growing in more than two directions.

# 3 Small polyhex graphs

All polyhexes with up to 4 hexagons are depicted in Figure 5 where the numbers following the names are the corresponding values of $\overline{CR}$.

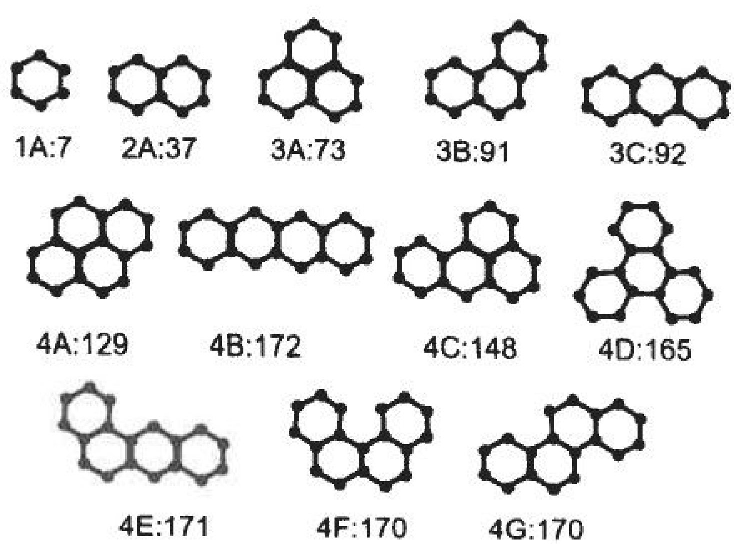


Figure 5: All polyhex graphs with $n \leq 4$ hexagons and their values of $\overline{CR}$.

For all polyhex graphs which are snakes Corollary 1 can be applied to determine the values of $\overline{CR}$ as given in Figure 5. Graphs $3A$, $4A$, $4C$, and $4D$ remain.

3A: This is a game board graph and it was proved in [3] that $\overline{CR}(3A) = 73$.

4C: Since there exists an extremal drawing of $3A$ having the appropriate gluing edges another hexagon can be glued to $3A$ to obtain by Theorem 1 that $\overline{CR}(4C) = 73 + 7 + (15-1)(6-1) - 2 = 148$.

4A: The lower bound follows from Figure 6.

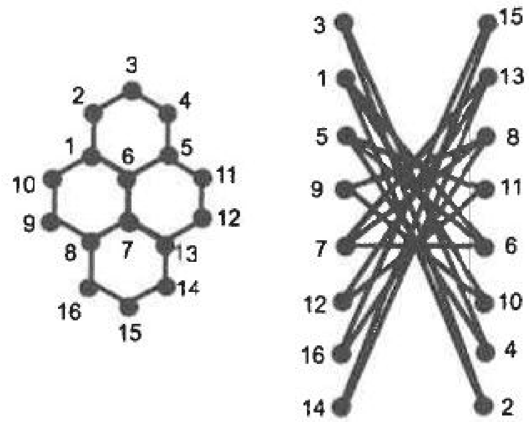


Figure 6: $\overline{CR}(4A) \geq 129$, gluing edges 2, 3 and 14, 15.

For the upper bound we use the thrackle numbers $t(4A) = 143$ and $t(3A) = 84$. Since $\overline{CR}(3A) = 73$ there are at least 11 misses in a subdrawing $3A$. An additional hexagon completing $3A$ to $4A$ has either at least 3 misses or 2 misses. At least 3 misses yield $\overline{CR}(4A) \leq 143 - 11 - 3 = 129$. If the hexagon has 2 misses then it has a unique subdrawing $D(C_6)$ of $D(4A)$ and there are 3 edges of $4A$ connected to 3 of its vertices as in Figure 7 with all three possible labelings of the $D(C_6)$.

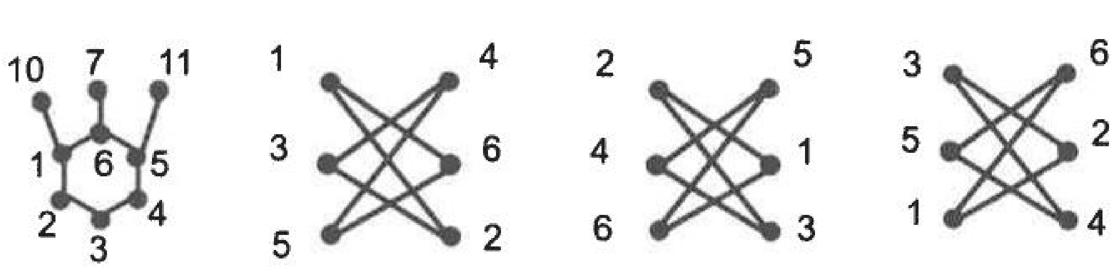


Figure 7: Additional hexagon with 2 misses.

Since $6,7\|2,3 \vee 3,4$, $1,10\|3,4 \vee 4,5$, and $5,11\|1,2 \vee 2,3$ we have at least one further miss so that $\overline{CR}(4A) \leq 143 - 11 - 2 - 1 = 129$.

<u>4D</u>: The lower bound follows from Figure 8.

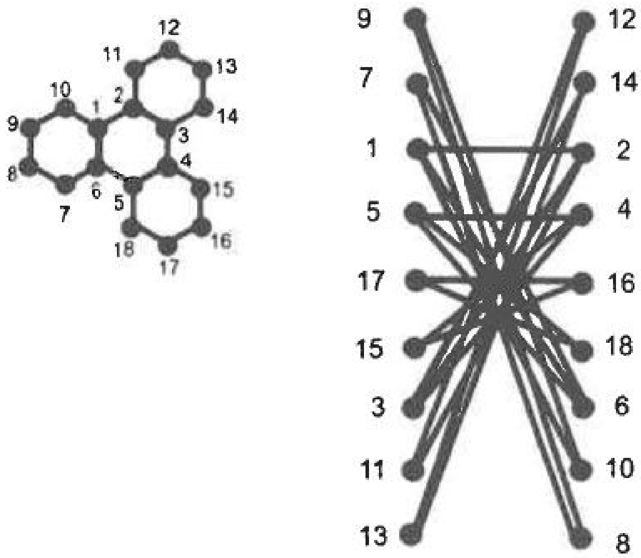


Figure 8: $\overline{CR}(4D) \geq 165$, gluing edges 8, 9 and 12, 13.

The subthrackle number is $s(4D) = 180 - 8 = 172$ so that for the upper bound 7 further misses are needed. Note, that misses within one of the four hexagons may have been counted already by $s(4D)$.
(I) The center hexagon has 7 crossings. Then there is a unique drawing of this $C_6$. Use the labeling as in Figure 9.

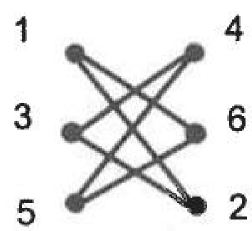


Figure 9: Labeling used for the unique $D(C_6)$ of the center hexagon.

Consider the paths 6, 7, 8 and 3, 14, 13. If both determine at least 4 misses then $\overline{CR}(4D) \leq 172 - 8 = 164$. For 6, 7, 8 there are only 7 cases with less than 4 misses (see Figure 10).

If in addition the path 1, 10, 9 has at least 2 misses then together with 6, 7, 8 we have at least 4 misses. There are only two cases of 1, 10, 9 with at most one crossing (see Figure 11).

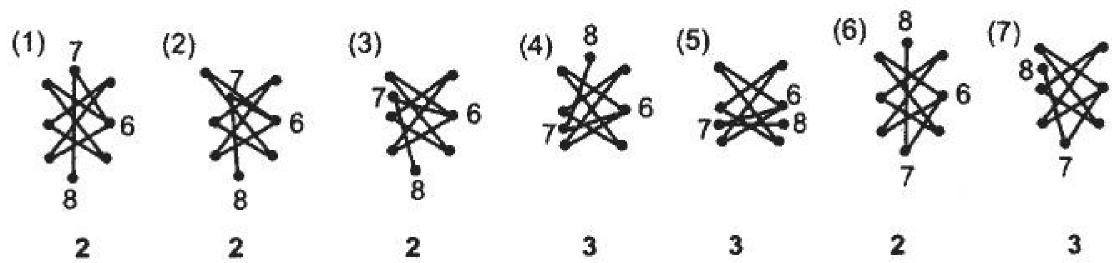


Figure 10: Possible paths 6, 7, 8 with the numbers of misses in bold.

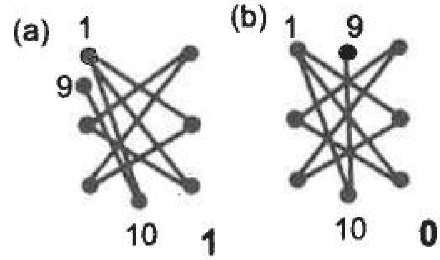


Figure 11: Paths 1, 10, 9 with at most 1 miss in bold.

For $(a)$ with (1) to (7) there are always at least 4 misses since for each of (1) to (3) we have $1,2\|8,9$ and for (6) we have $3,4\|8,9$ as additional misses so that 4 misses occur. For $(b)$ with (4) to (7) additional misses are $1,2\|8,9$, $2,3\|8,9$, $2,3\|8,9$, and $5,6\|8,9$, respectively. Together there remain the three cases of Figure 12 for paths $6,7,8,9,10,1$ with 2 misses each. The corresponding three cases with 2 misses each for the path $3,14,13,12,11,2$ are depicted in Figure 13. For all six pairs of $(b,i)$ and $(d,j)$ we have $2+2+2=6$ misses since always $3,4\|6,7$ and $1,10\|2,11$ as well. Then the seventh miss for $\overline{CR}(4D) \leq 172-7=165$ is guaranteed since either $9,10\|11,12$, or 9 is to the left of 11 so that $2,11\|8,9$, or 12 is to the right of 10 so that $1,10\|12,13$ (see Figures 12 and 13).

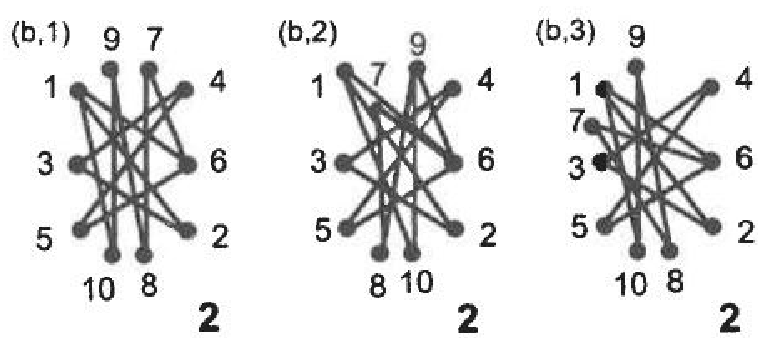


Figure 12: Paths 6, 7, 8, 9, 10, 1 with two misses.

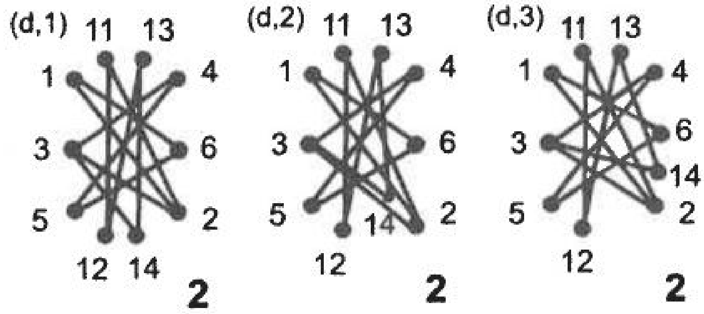


Figure 13: Paths 3, 14, 13, 12, 11, 2 with two misses.

(II) The center hexagon has 6 crossings so that $\overline{\overline{CR}}(4D) \leq 172 - 1 = 171$. There is a unique drawing of this $C_6$ depicted in Figure 14. This can be checked in [7] where only one of the 29 nonisomorphic drawings can be drawn rectilinear or it can be directly proved that the 3 possibilities of an edge with 2 crossings cannot be completed to a drawing where each edge has exactly 2 crossings, and that all completions of the 3 possibilities of an edge with 3 crossings only once result in a drawing with 6 crossings. With the labeling of Figure 14 we have 4 possible directions to insert 5, 18 as in Figure 15.

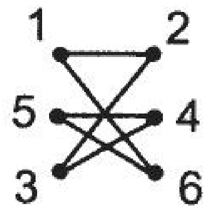


Figure 14: The unique $D(C_6)$ with 6 crossings.

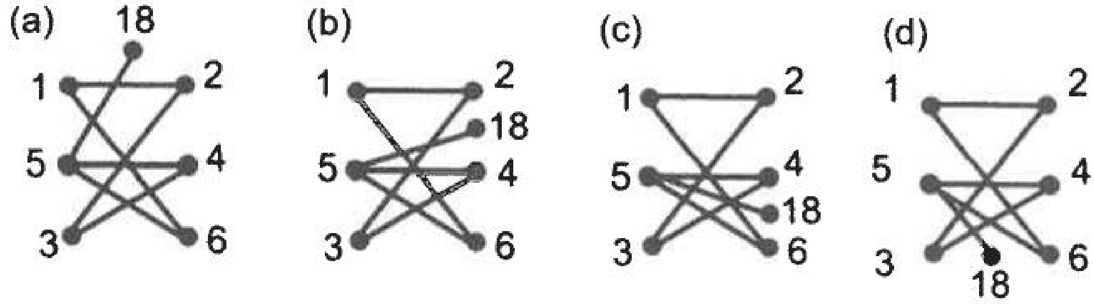


Figure 15: Edge 5, 18 in the center hexagon.

For each case there are at least the following 3 misses: $(a)$ $5, 18 \| 2, 3$;

$5, 18 \| 3, 4$; $2, 11 \| 4, 5 \vee 5, 18$. (*b*) $5, 18 \| 3, 4$; $5, 18 \| 1, 2$; $17, 18 \| 1, 2 \vee 3, 4$. (*c*) $5, 18 \| 1, 2$; $17, 18 \| 1, 2 \vee 5, 6$; $6, 7 \| 1, 2 \vee 1, 10 \| 5, 6 \vee 6, 7 \| 1, 10$. (*d*) $5, 18 \| 1, 6$; $5, 18 \| 1, 2$; $6, 7 \| 4, 5 \vee 5, 18$. In all cases the path 5, 18, 17 and the edges 2, 4 and 6, 7 have been used. If then the path 4, 15, 16 and the edges 1, 10 and 3, 14 are used then due to symmetry we have again 3 misses in all 4 cases. Thus together we have $3 + 3 = 6$ misses yielding $\overline{CR}(4D) \leq 171 - 6 = 165$. It may be remarked that there may be another labeling in Figures 14 and 15 where 1, 6 is the labeling of the edge without a crossing. However, then the path 3, 14, 13 with edges 6, 7 and 2, 11 and the path 4, 15, 16 with edges 1, 10 and 5, 18 determine the corresponding 6 misses in each case.

(III) The center hexagon has $i$ crossings, $i \leq 5$. Then $\overline{CR}(4D) \leq 172 - (7 - i) = 165 + i$ so that $i$ misses are needed. From [7] or directly proved all rectilinear $D(C_6)$s with $i \leq 5$ crossings are depicted in Figure 16.

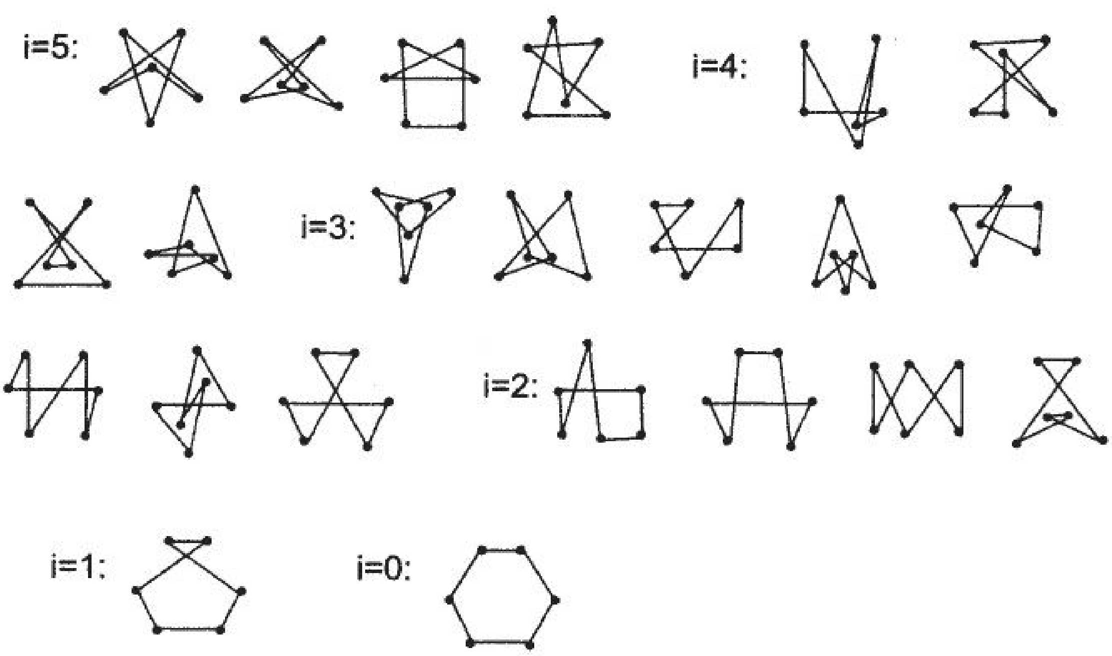


Figure 16: Rectilinear $D(C_6)$s with $i$ crossings, $i \leq 5$.

For each of these drawings it can easily be checked that at least i misses are determined between the edges of the center $C_6$ with vertices 1, 2, 3, 4, 5, 6 and the 6 edges incident to the vertices of the center $C_6$ (see Figure 8).

Now all values $\overline{CR}$ asserted in Figure 5 are determined. There are 22 polyhexes with five hexagons. For 19 of them, $\overline{CR}$ can be settled by gluing and with the results of Figure 5. There remain open the three polyhexes of Figure 17.

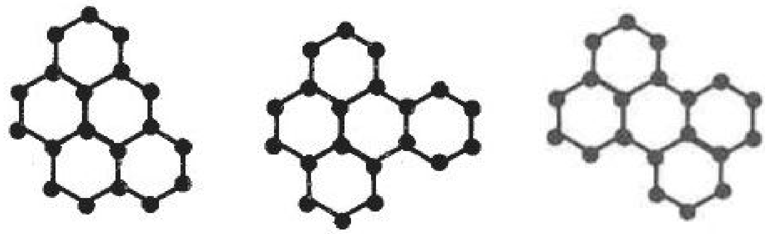

Figure 17: Polyhexes with 5 hexagons and $\overline{CR} = ?$

# 4 Order of magnitude

What about maximum and minimum of $\overline{CR}(P)$?

**Theorem 3.** *For a polyhex graph $P$ it holds max $\overline{CR}(P) = \mathcal{O}(\frac{25}{2}n^2)$ and min $\overline{CR}(P) = \mathcal{O}(\frac{9}{2}n^2)$.*

*Proof.* From [4] it follows that the maximum and minimum number of the edges of a polyhex with $n$ hexagons is of order $5n$ and $3n$, respectively. Thus the thrackle numbers give the order of the corresponding upper bounds of $\overline{CR}(P)$ to be $\frac{25}{2}n^2$ and $\frac{9}{2}n^2$. By the linear snake $\frac{25}{2}n^2$ is attained (see Corollary 1). In [3] it was proved that $\overline{CR}(B_k)$ is at least of order $\frac{243}{96}k^4$ where $B_k$ is a game board of order $k$ with $n = \frac{3}{4}k^2$ hexagons and having a minimum number of edges. Then $k^2 = \frac{4}{3}n$ implies $\frac{243}{96}k^4 = \frac{9}{2}n^2$. □

In future work $\overline{CR}$ for polyomino like parts of platonic solids and of noneuclidean tessellations of the plane can be discussed.

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