The pseudograph threshold number $\pi(r,s,a,t)$

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1 Introduction

A pseudograph is a graph in which multiple edges and multiple loops may be allowed. The degree $d(v)$ of a vertex v in a pseudograph is the number of edges incident with v, where a loop counts as two edges. A $(d,d+s)$-graph is a graph in which the degree of each vertex v satisfies $d \le d(v) \le d+s$. An $(r,r+a)$-factor is a spanning $(r,r+a)$ subgraph. An $(r,r+a)$-factorization of a graph G is the expression of G as the union of edge disjoint $(r,r+a)$-factors.

Earlier papers on factorizing pseudographs include [13] and [14] by Hilton and [15] by Hilton and Rajkumar. There was a mistake in [14] by Hilton and this caused an error in [15] by Hilton and Rajkumar. Unfortunately, to rectify these errors is a non-trivial exercise, and this was the motivation for the present rather long paper. Where results were correct in the earlier papers and in a suitable form to be quoted without causing the reader any difficulty we have quoted them. But to preserve coherence we have included the main development in this paper.

1.1 The threshold numbers

We let $\pi(r,s,a,t)$, the **pseudograph threshold number**, to be the least value of d, say $d=d_0$, such that every $(d,d+s)$-simple graph with $d \ge d_0$ has an $(r,r+a)$-factorization with x factors for at least t values of x. In the case when there is no such d_0, we put $\pi(r,s,a,t)=\infty$.

The **multigraph threshold number**, $\mu(r,s,a,t)$ is defined similarly except that we consider $(d,d+s)$-multigraphs (without loops), instead of pseudographs.

Similarly we have the **simple graph threshold number** $\sigma(r,s,a,t)$ where we consider $(d,d+s)$-simple graphs.

We also have the **bipartite graph threshold number** $\beta(r,s,a,t)$, where we consider $(d,d+s)$-bipartite multigraphs, and **the simple bipartite graph threshold number** $\beta_s(r,s,a,t)$, where we consider $(d,d+s)$-simple bipartite graphs.

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There are some relationships between these threshold numbers which can be deduced immediately.

Lemma 1. *For $r \geq 0, s \geq 0, a \geq 0, t \geq 1$, we have*

$$\beta_s(r,s,a,t) \leq \sigma(r,s,a,t) \leq \mu(r,s,a,t) \leq \pi(r,s,a,t).$$

Lemma 1 is almost obvious. It is also almost obvious that if $\rho \leq r$ and $r+a \leq \rho+\alpha$ then any $(r, r+a)$-factor of a pseudograph is a $(\rho, \rho+\alpha)$-factor. From this simple fact we deduce:

Lemma 2. *Let ρ, r, s, a, α, t be integers with ρ, r, a, α, t positive and s non-negative, Let $\rho \leq r \leq r+a \leq \rho+\alpha$. Then*

$$\pi(r,s,a,t) \geq \pi(\rho,s,\alpha,t)$$

Proof. We note that a graph could have a $(\rho, \rho+\alpha)$-factorization, and yet not have an $(r, r+a)$-factorization. But if a graph has an $(r, r+a)$-factorization then it does have a $(\rho, \rho+\alpha)$-factorization. □

Lemma 2 has two useful consequences.

Corollary 3. *Let r, s, a, t be integers with r, a, t positive and s non-negative. Then*

(i) $\pi(r,s,a,t) \geq \pi(r,s,a+1,t)$

(ii) $\pi(r,s,a,t) \leq \pi(r+1,s,a-1,t)$.

The same result holds for the same reason in the case of all the other threshold functions: $\beta_s(r,s,a,t), \beta(r,s,a,t), \sigma(r,s,a,t), \mu(r,s,a,t)$.

Another inequality which is almost obvious is:

Lemma 4. *Let r, s, a, t be integers with r, a, t all positive and s non-negative. Then*

$$\pi(r,s,a,t) \leq \pi(r,s+1,a,t).$$

The first result on $(r, r+a)$-factorizations seems to be the following one due to Akiyama, Avis and Era [1] in 1980.

Theorem 5. *Every regular pseudograph is $(1,2)$-factorable. In particular, if r is an odd integer then every r-regular pseudograph can be decomposed into $\frac{r+1}{2}$ $(1,2)$-factors.*

In the case when r and a are both even we have the following striking result, due to Hilton.

Theorem 6. *Let integers r, a, t be positive and s be non-negative and r and a both even and positive. Then*

$$\begin{aligned}\beta(r,s,a,t) &= \beta_s(r,s,a,t) = \sigma(r,s,a,t) &= \mu(r,s,a,t) = \pi(r,s,a,t) \\ & &= r\left\lceil\frac{tr+s-1}{a}\right\rceil + (t-1)r.\end{aligned}$$

It is convenient to define

$$N(r,s,a,t) = \left\lceil\frac{tr+s-1}{a}\right\rceil + (t-1)r$$

when r, s, a, t are integers with r, a, t positive and s non-negative.

The pseudograph threshold numbers are completely determined in the following two theorems.

Theorem 7. *Let r, s, a and t be integers with r and t positive, $a \geq 2$ and s non-negative.*

(1) If r and a are both even, then

$$\pi(r,s,a,t) = N(r,s,a,t).$$

(2) If r and a are both odd then

$$\pi(r,s,a,t) = \begin{cases} N(r+1,s,a-1,t)-1 & if\ (r+1)t+s \not\equiv 2 \\ & (mod\, a-1), \\ N(r+1,s,a-1,t)-(r+1)-1 & if\ (r+1)t+s \equiv 2 \\ & (mod\, a-1). \end{cases}$$

(3) If r is odd and a is even, $a \geq 4$, then

$$\pi(r,s,a,t) = \begin{cases} N(r+1,s,a-2,t)-1 & if\ (r+1)t+s \not\equiv 2,3 \\ & (mod\, a-2), \\ N(r+1,s,a-2,t)-(r+1)-1 & if\ (r+1)t+s \equiv 2,3 \\ & (mod\, a-2). \end{cases}$$

(4) If r is even and a is odd, then

$$\pi(r,s,a,t) = \begin{cases} N(r,s,a-1,t) & if\ rt+s \not\equiv 2\ (mod\, a-1), \\ N(r,s,a-1,t)-r & if\ rt+s \equiv 2\ (mod\, a-1). \end{cases}$$

For $a = 0$ or 1, or $a = 2$, r odd we give the results in Theorem 8. Note that we use the notation $\pi(r, s, a, t) = \infty$ when there is no finite threshold number for the given value of r, s, a and t.

If $a = 2$ then $\pi(r, s, a, t)$ is given in Theorem 7 if r is even, but if r is odd it is given below in Theorem 8.

Theorem 8. *Let r, s and t be integers with r and t positive and s non-negative. Then*

$$\pi(r, s, 0, t) = \infty$$

and

$$\pi(r, s, 1, t) = \begin{cases} 2 & \text{if } r = 2, s = 0 \text{ and } t = 1, \\ 1 & \text{if } r = 1, s = 0 \text{ and } t = 1, \\ \infty & \text{otherwise,} \end{cases}$$

and if r is odd, then

$$\pi(r, s, 2, t) = \begin{cases} \infty & \text{if } r \geq 1, \text{ and } s > 1 \text{ or } t > 1, \\ 1 & \text{if } r = 1,\ s \in \{0, 1\} \text{ and } t = 1. \end{cases}$$

As already remarked, the earlier published versions of these theorems were wrong. The original error was in Theorem 24 of the paper by Hilton [14], with similar errors in Theorems 27 and 30 of that paper. Corrected versions of these theorems are given in this paper as Theorems 13, 16 and 19 respectively. The error in the paper by Hilton [14] affected the determination of $\pi(r, s, a, t)$ in the earlier paper by Hilton and Rajkumar [15]. In particular, part (3) of Theorem 7 is a corrected result of the similar theorem in [15].

We remark that the topic of $(r, r + a)$-factorization of graphs has been considered by various Japanese Mathematicians. Akiyama and Kano [4] have recently published a book "Factors and factorization". The main topic is just factors, and they do not specifically aim to evaluate threshold numbers. They do not draw much distinction between simple graphs, pseudographs and multigraphs, and the parameter t is not mentioned.

One result of Kano [16] in 1985 is of particular interest:

Theorem 9. *Let a and b be even integers such that $0 \leq a \leq b$, and let $n \geq 1$ be an integer. Then the pseudograph G can be decomposed into n (a, b)-functions if and only if G is an (an, bn)-graph.*

Theorem 9 is a corollary of Theorem 10.

Theorem 10. *Let d, r, x be positive integers and let a and s be non-negative integers. Let r and a be even. Then every $(d, d+s)$-pseudograph is $(r, r+a)$-factorizable with x factors if and only if*

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}.$$

Proof. (1). If d and s are even this is proved in [15].
(2). If d and s are both odd, if

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$$

then

$$\frac{(d-1)+(s+1)}{r+a} \leq x \leq \frac{d-1}{r}$$

so, by part (1) above, every $(d-1, (d-1)+(s+1))$-pseudograph is $(r, r+a)$-factorizable with x factors. But a $(d, d+s)$-pseudograph is a $(d-1, (d-1)+(s+1))$-pseudograph. Therefore every $(d, d+s)$-pseudograph is $(r, r+a)$-factorizable with x factors.

Conversely, if every $(d, d+s)$-pseudograph is $(r, r+a)$-factorizable with x factors, then it is proved in [15] that

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}.$$

(3). If d is odd and s is even and $\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$, then

$$\frac{(d-1)+(s+2)}{r+a} \leq x \leq \frac{d-1}{r},$$

so every $(d-1, (d-1)+(s+2))$-pseudograph is $(r, r+a)$-factorable with x factors. But every $(d, d+s)$-pseudograph is a $(d-1, (d-1)+(s+2))$-pseudograph, so every $(d, d+s)$-pseudograph is $(r, r+a)$-factorable with x factors.

The converse is true as in part (2) above.

(4). If d is even and s is odd and $\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$, then

$$\frac{d+(s+1)}{r+a} \leq x \leq \frac{d}{r},$$

so every $(d, d+(s+1))$-pseudograph is $(r, r+a)$-factorable with x factors. But a $(d, d+s)$-pseudograph is a $(d, d+(s+1))$-pseudograph, so every $(d, d+s)$-pseudograph is $(r, r+a)$-factorable with x factors.

The converse is true as in part (2) above again. □

2 Bounds for $\pi(r, s, a, t)$ when r, a are not both even

We first find reasonable bounds when r and a are not both even. Later, in the next section, we refine those bounds to obtain actual evaluations.

The earlier errors in the paper [14] by Hilton are manifested here in Theorem 13 (Case 3(iii)) where originally it was not noticed that the assumption $(r+1)t+s-1 \not\equiv 2 (\text{mod } a-1)$ (or $(r+1)t+s \not\equiv 3(\text{mod } a-1)$) was needed. It is manifested similarly in Theorem 16 (Case 3 (iii)) and Theorem 19 (Case 3 (iii)). This oversight affects the contents of Section 3, making it much longer, and it also affects the final result obtained there, in particular the evaluation of $\pi(r,s,a,t)$ when r is odd and a is even.

Let us bound $\pi(r,s,a,t)$ when r and a are both odd.

Lemma 11. *Let r,s,a,t be integers with r,t positive, $a \geq 3$ and s non-negative. Let r,a be odd and s be even, let $(r+1)t+s \not\equiv 2 \ (\text{mod } a-1)$. Then*

$$\pi(r+1,s,a-1,t)-1 \leq \pi(r,s,a,t) \leq \pi(r+1,s,a-1,t).$$

Note that, as $r+1$ and $a-1$ are both even, $\pi(r+1,s,a-1,t)$ is evaluated in Theorem 6.

Proof. By Lemma 2, $\pi(r,s,a,t) \leq \pi(r+1,s,a-1,t)$.

To prove the other inequality, let $d = \pi(r+1,s,a-1,t)-2$, so that, by the formula in Theorem 6, d is even. Let G be the $(d,d+s)$-pseudograph with two components, G_1 and G_2, where G_1 has one vertex on which are placed $\frac{d}{2}$ loops, and G_2 has one vertex on which are placed $\frac{d+s}{2}$ loops. Since r and a are both odd and all the edges of G are in fact loops, any $(r,r+a)$-factor of G is actually an $(r+1,r+a)$-factor, i.e. an $((r+1),(r+1)+(a-1))$-factor.

By Theorem 10, it follows that for any $((r+1),(r+1)+(a-1))$-factorization of G into x $((r+1),(r+1)+(a-1))$-factors,

$$\frac{d+s}{(r+1)+(a-1)} \leq x \leq \frac{d}{r+1}.$$

Since $d = \pi(r+1,s,a-1,t)-2$, it follows from Theorem 6 (since s, $r+1$ and $a-1$ are even and positive) that

$$d = (r+1)\left\lceil \frac{t(r+1)+s}{a-1} \right\rceil + (t-1)(r+1) - 2,$$

so

$$\frac{d}{r+1} = \left\lceil \frac{t(r+1)+s}{a-1} \right\rceil + (t-1) - \frac{2}{r+1}.$$

Therefore

$$x \leq \left\lceil \frac{t(r+1)+s}{a-1} \right\rceil + (t-2).$$

We also have that

$$d+s = (r+1)\left\lceil \frac{t(r+1)+s}{a-1} \right\rceil + (t-1)(r+1)+s-2,$$

so that

$$d+s = \frac{(r+1)}{(a-1)}(t(r+1)+s+c) + (t-1)(r+1)+s-2,$$

where $0 \leq \frac{c}{2} \leq \frac{a-1}{2} - 1$ and $a-1 \mid (r+1)t+s+c$. Therefore

$$\begin{aligned} d+s &= \{(r+1)+(a-1)\}\frac{t(r+1)+s+c}{a-1} - (t(r+1)+s+c) \\ +(t-1)(r+1)+s-2 &= (r+a)\frac{(t(r+1)+s+c)}{a-1} - (r+1)-c-2 \end{aligned}$$

so that

$$\begin{aligned} \frac{d+s}{(r+1)+(a-1)} &= \frac{t(r+1)+s+c}{a-1} - \frac{r+c+3}{r+a} \\ &= \left\lceil \frac{t(r+1)+s}{a-1} \right\rceil - \frac{r+c+3}{r+a}. \end{aligned}$$

Since $0 \leq \frac{c}{2} \leq \frac{a-1}{2} - 1$ and $(r+1)t+s \not\equiv 2 \ (\text{mod } a-1)$, it follows that $r+c+3 < r+a$, and so

$$x \geq \left\lceil \frac{t(r+1)+s}{a-1} \right\rceil.$$

There are therefore only $t-1$ values that x can take, so there do not exist t values of x for which G has an $((r+1),(r+1)+(a-1))$-factorization into x $((r+1),(r+1)+(a-1))$-factors. Therefore there do not exist t values of x for which G has an $(r,r+a)$-factorization into x $(r,r+a)$-factors. It follows that $d < \pi(r,s,a,t)$.

We now deduce that $\pi(r+1,s,a-1,t)-1 \leq \pi(r,s,a,t)$, so that

$$\pi(r+1,s,a-1,t)-1 \leq \pi(r,s,a,t) \leq \pi(r+1,s,a-1,t).$$

□

The missing case of Lemma 11, when $(r+1)t+s \equiv 2 \ (\text{mod } a-1)$, is covered less well by Lemma 12:

Lemma 12. *Let r,s,a,t be integers with r,t positive, $a \geq 3$ and $s \geq 2$. Let r,a be odd and $(r+1)t+s \equiv 2 \ (\text{mod } a-1)$ (so that s is even). Then*

$$\pi(r+1,s-2,a-1,t)-1 \leq \pi(r,s,a,t) \leq \pi(r+1,s-2,a-1,t)+(r+1).$$

Note that $\pi(r+1, s-2, a-1, t)$ can be written down explicitly using Theorem 6.

Proof.

$$
\begin{aligned}
& \pi(r+1, s-2, a-1, t) - 1 & \\
\leq\ & \pi(r, s-2, a, t) & \text{by Lemma 11,} \\
\leq\ & \pi(r, s, a, t) & \text{by Lemma 4,} \\
\leq\ & \pi(r, s+2, a, t) & \text{by Lemma 4 again,} \\
\leq\ & \pi(r+1, s+2, a-1, t) & \text{by Lemma 11,} \\
=\ & (r+1)\left\lceil \tfrac{t(r+1)+(s+2)-1}{a-1} \right\rceil + (t-1)(r+1) & \text{by Theorem 6,} \\
=\ & (r+1)\left\lceil \tfrac{t(r+1)+(s-2)-1}{a-1} \right\rceil + (t-1)(r+1) + (r+1) & \text{since } (r+1)t+s \\
& & \equiv 2 \pmod{a-1}, \\
=\ & \pi(r+1, s-2, a-1, t) + (r+1) & \text{by Theorem 10.}
\end{aligned}
$$

□

Theorem 13. *Suppose $r \geq 1$, $a \geq 3$ are odd, and $s \geq 0$, $t \geq 1$. If $(r+1)t + s \not\equiv 1, 2, 3 \pmod{a-1}$ then*

$$\pi(r+1, s, a-1, t) - 1 \leq \pi(r, s, a, t) \leq \pi(r+1, s, a-1, t).$$

If $(r+1)t + s \equiv i \in \{1, 2\} \pmod{a-1}$ and $s \geq i$, then

$$\pi(r+1, s-i, a-1, t) - 1 \leq \pi(r, s, a, t) \leq \pi(r+1, s-i, a-1, t) + r + 1.$$

If $(r+1)t + s \equiv 3 \pmod{a-1}$ then

$$\pi(r+1, s, a-1, t) - 1 \leq \pi(r, s-3, a, t) + (r+1) \leq \pi(r+1, s, a-1, t)$$

and

$$\pi(r+1, s, a-1, t) - 1 \leq \pi(r, s+1, a, t) \leq \pi(r+1, s, a-1, t).$$

Note that the outer bounding terms are given explicitly in each case in Theorem 6.

Proof. We consider various cases.

Case 1: $(r+1)t + s \equiv 2 \pmod{a-1}$.
In this case s is even and the theorem follows from Lemma 12.

Case 2: $(r+1)t + s \not\equiv 2 \pmod{a-1}$ *and s is even.*
In this case we also have that $(r+1)t + s \not\equiv 1 \pmod{a-1}$ and so the theorem follows from Lemma 11.

Case 3: $(r+1)t + s \not\equiv 2 \pmod{a-1}$ *and s is odd.*

Case 3(i) $(r+1)t + s \equiv 1 \pmod{a-1}$.

Then

$$
\begin{aligned}
& \pi(r+1, s-1, a-1, t) - 1 & \\
\leq\ & \pi(r, s-1, a, t) & \text{by Lemma 11,} \\
\leq\ & \pi(r, s, a, t) & \text{by Lemma 4,} \\
\leq\ & \pi(r, s+1, a, t) & \text{by Lemma 4 again,} \\
\leq\ & \pi(r+1, s-1, a-1, t) + r + 1 & \text{by Lemma 12 since } (r+1)t + (s+1) \\
& & \equiv 2 \pmod{a-1}.
\end{aligned}
$$

Case 3(ii)
$(r+1)t + s \equiv 3 \pmod{a-1}$. Then

$$
\begin{aligned}
& \pi(r+1, s, a-1, t) - 1 & \\
=\ & (r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1) - 1 & \text{by Theorem 6,} \\
=\ & (r+1)\left\lceil \tfrac{t(r+1)+(s-3)-1}{a-1} \right\rceil + (t-1)(r+1) - 1 + (r+1) & \\
=\ & \pi(r+1, s-3, a-1, t) - 1 + (r+1) & \text{by Theorem 6,} \\
\leq\ & \pi(r, s-3, a, t) + (r+1) & \text{by Lemma 11 since} \\
& & (r+1)t + s - 3 \not\equiv 2 \pmod{a-1}, \\
\leq\ & \pi(r+1, s-3, a-1, t) + (r+1) & \text{by 11 again,} \\
=\ & (r+1)\left\lceil \tfrac{t(r+1)+(s-3)-1}{a-1} \right\rceil + (t-1)(r+1) + r + 1 & \text{by Theorem 6,} \\
=\ & (r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1) & \\
=\ & \pi(r+1, s, a-1, t). &
\end{aligned}
$$

Also

$$
\begin{aligned}
& \pi(r+1, s, a-1, t) - 1 & \\
=\ & (r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1) - 1 & \text{by Theorem 6,} \\
=\ & (r+1)\left\lceil \tfrac{t(r+1)+(s+1)-1}{a-1} \right\rceil + (t-1)(r+1) - 1 & \text{since } (r+1)t+s \\
& & \equiv 3 \pmod{a-1}, \\
=\ & \pi(r+1, s+1, a-1, t) - 1 & \text{by Theorem 6,}
\end{aligned}
$$

$$\begin{aligned}
&\le \pi(r,s+1,a,t) && \text{by Lemma 11,}\\
&\le \pi(r+1,s+1,a-1,t) && \text{by Lemma 11 again,}\\
&= (r+1)\left\lceil\tfrac{t(r+1)+(s+1)-1}{a-1}\right\rceil+(t-1)(r+1) && \text{by Theorem 6,}\\
&= (r+1)\left\lceil\tfrac{t(r+1)+s-1}{a-1}\right\rceil+(t-1)(r+1) && \text{since } (r+1)t+s \equiv 3(\text{mod } a-1),\\
&= \pi(r+1,s,a-1,t).
\end{aligned}$$

Case 3(iii)

$(r+1)t+s\not\equiv 1$ (mod $a-1$) *and* $(r+1)t+s\not\equiv 3$ (mod $a-1$).

Then

$$\begin{aligned}
&\pi(r+1,s,a-1,t)-1\\
&= (r+1)\left\lceil\tfrac{t(r+1)+s-1}{a-1}\right\rceil+(t-1)(r+1)-1 && \text{by Theorem 6,}\\
&= (r+1)\left\lceil\tfrac{t(r+1)+(s-1)-1}{a-1}\right\rceil+(t-1)(r+1)-1 && \text{since } (r+1)t+s-1 \not\equiv 1(\text{mod } a-1),\\
&= \pi(r+1,s-1,a-1,t)-1 && \text{by Theorem 6,}\\
&\le \pi(r,s-1,a,t) && \text{by Lemma 11 since } (r+1)t+(s-1)\not\equiv 2(\text{mod } a-1),\\
&\le \pi(r,s,a,t) && \text{by Lemma 4,}\\
&\le \pi(r,s+1,a,t) && \text{by Lemma 4 again,}\\
&\le \pi(r+1,s+1,a-1,t) && \text{by Lemma 11, since } (r+1)t+(s+1)\not\equiv 2(\text{mod } a-1),\\
&= (r+1)\left\lceil\tfrac{t(r+1)+(s+1)-1}{a-1}\right\rceil+(t-1)(r+1) && \text{by Theorem 6,}\\
&= (r+1)\left\lceil\tfrac{t(r+1)+s-1}{a-1}\right\rceil+(t-1)(r+1) && \text{since } (r+1)t+s \not\equiv 1(\text{mod } a-1),\\
&= \pi(r+1,s,a-1,t) && \text{by Theorem 6.}
\end{aligned}$$

□

Our results and proofs in the remaining cases, when one of r and a is even and the other is odd are very similar to the case when both r and a are odd. The reader might feel like breezing through our accounts of these cases, but we include all the details so that proofs can be checked easily.

We look next at the case when r is even and a is odd.

Lemma 14. *Let r, s, a, t be integers with r, t positive, $a \ge 3$, and s non-negative. Let r and s be even and a be odd. Let $rt + s \not\equiv 2$ (mod $a-1$). Then*

$$\pi(r,s,a-1,t)-1 \le \pi(r,s,a,t) \le \pi(r,s,a-1,t).$$

Proof. By Lemma 2, $\pi(r,s,a,t) \le \pi(r,s,a-1,t)$.

To prove the other inequality, let $d = \pi(r,s,a-1,t)-2$, so that d is even. Let G be the $(d, d+s)$-pseudograph with two components, G_1 and G_2, where G_1 has one vertex on which are placed $\frac{d}{2}$ loops, and G_2 has one vertex on which are placed $\frac{d+s}{2}$ loops. Since r is even and a is odd, any $(r, r+a)$-factor of G is actually an $(r, r+(a-1))$-factor.

By Theorem 10, it follows that, for any $(r, r+(a-1))$-factorization of G into x $(r, r+(a-1))$-factors,

$$\frac{d+s}{r+(a-1)} \le x \le \frac{d}{r}.$$

Using Theorem 6 and using the facts that a is odd and r and s are even,

$$\frac{d}{r} = \left\lceil\frac{tr+s}{a-1}\right\rceil + t - 1 - \frac{2}{r},$$

so that

$$x \le \left\lceil\frac{tr+s}{a-1}\right\rceil + t - 2.$$

We also have that

$$\begin{aligned}
d+s &= r\left\lceil\frac{tr+s}{a-1}\right\rceil + (t-1)r - 2 + s\\
&= \frac{r}{a-1}(tr+s+c) + (t-1)r + s - 2.
\end{aligned}$$

where $0 \le \frac{c}{2} \le \frac{a-1}{2} - 1$, c is even and $a-1 \mid rt+s+c$. Therefore

$$d+s = \frac{r+(a-1)}{a-1}(tr+s+c) - r - c - 2$$

so that

$$\begin{aligned}
\frac{d+s}{r+(a-1)} &= \frac{tr+s+c}{a-1} - \frac{r+c+2}{r+(a-1)}\\
&= \left\lceil\frac{tr+s}{a-1}\right\rceil - \frac{c+r+2}{r+(a-1)}.
\end{aligned}$$

But $c = a-3$ if and only if $rt+s \equiv 2 \pmod{a-1}$ so that, since $rt+s \not\equiv 2 \pmod{a-1}$,

$$x \geq \left\lceil \frac{tr+s}{a-1} \right\rceil,$$

and so there are at most $t-1$ possible values of x.

Therefore there do not exist t values of x for which G has an $(r, r+a)$-factorization into x $(r, r+a)$-factors. Therefore

$$d < \pi(r, s, a, t)$$

and so

$$\pi(r, s, a-1, t) - 1 \leq \pi(r, s, a, t).$$

□

The missing case of Lemma 14, when $rt+s \equiv 2 \pmod{a-1}$ is covered in Lemma 15.

Lemma 15. *Let r, s, a, t be integers with r, t positive, $a \geq 3$, a odd and $s \geq 2$. Let r be even and $rt+s \equiv 2 \pmod{a-1}$ (so that s is even). Then*

$$\pi(r, s-2, a-1, t) - 1 \leq \pi(r, s, a, t) \leq \pi(r, s-2, a-1, t) + r.$$

Proof.

$$\begin{aligned}
& \pi(r, s-2, a-1, t) - 1 & \\
\leq\ & \pi(r, s-2, a, t) & \text{by Lemma 14,} \\
\leq\ & \pi(r, s, a, t) & \text{by Lemma 4,} \\
\leq\ & \pi(r, s+2, a, t) & \text{by Lemma 4 again,} \\
\leq\ & \pi(r, s+2, a-1, t) & \text{by Lemma 14,} \\
=\ & r\left\lceil \tfrac{tr+(s+2)-1}{a-1} \right\rceil + (t-1)r & \text{by Theorem 6,} \\
=\ & r\left\lceil \tfrac{tr+(s-2)-1}{a-1} \right\rceil + (t-1)r + r & \text{since } rt+s \equiv 2 (\text{mod } a-1). \\
=\ & \pi(r, s-2, a-1, t) + r. &
\end{aligned}$$

□

Theorem 16. *Suppose $r \geq 1$ is even, $a \geq 3$ is odd and $s \geq 0$, $t \geq 1$. If $rt+s \not\equiv 1, 2, 3 \pmod{a-1}$ then*

$$\pi(r, s, a-1, t) - 1 \leq \pi(r, s, a, t) \leq \pi(r, s, a-1, t).$$

If $rt+s \equiv i \in \{1, 2\} \pmod{a-1}$ and $s \geq i$, then

$$\pi(r, s-i, a-1, t) - 1 \leq \pi(r, s, a, t) \leq \pi(r, s-i, a-1, t) + r.$$

If $rt+s \equiv 3 \pmod{a-1}$ then

$$\pi(r, s, a-1, t) - 1 \leq \pi(r, s-3, a, t) + r \leq \pi(r, s, a-1, t)$$

and

$$\pi(r, s, a-1, t) - 1 \leq \pi(r, s+1, a, t) \leq \pi(r, s, a-1, t).$$

The bounding terms in each case are given explicitly by Theorem 6.

The proof of Theorem 16 follows that of Theorem 13, but uses Lemmas 14 and 15 instead of 11 and 12.

Proof. We consider various cases.

Case 1:
$rt+s \equiv 2 \pmod{a-1}$. In this case s is even and the theorem follows from Lemma 15.

Case 2:
$rt+s \not\equiv 2 \pmod{a-1}$ *and s is even.* In this case we also have that $rt+s \not\equiv 1 \pmod{a-1}$ and so the theorem follows from Lemma 14.

Case 3:
$rt+s \not\equiv 2 \pmod{a-1}$ *and s is odd.*

Case 3(i)
$rt+s \equiv 1 \pmod{a-1}$. Then

$$\begin{aligned}
& \pi(r, s-1, a-1, t) - 1 & \\
\leq\ & \pi(r, s-1, a, t) & \text{by Lemma 14,} \\
\leq\ & \pi(r, s, a, t) & \text{by Lemma 4,} \\
\leq\ & \pi(r, s+1, a, t) & \text{by Lemma 4 again,} \\
\leq\ & \pi(r, s-1, a-1, t) + r & \text{by Lemma 15 since} \\
& & rt + (s+1) \equiv 2 (\text{mod } a-1).
\end{aligned}$$

Case 3(ii)
$rt+s \equiv 3 \pmod{a-1}$. Then

$$
\begin{aligned}
&\pi(r,s,a-1,t)-1 && \\
=\;& r\left\lceil \tfrac{tr+s-1}{a-1}\right\rceil+(t-1)r-1 && \text{by Theorem 6,}\\
=\;& r\left\lceil \tfrac{tr+(s-3)-1}{a-1}\right\rceil+(t-1)r-1+r && \\
=\;& \pi(r,s-3,a-1,t)-1+r && \text{by Theorem 6,}\\
\le\;& \pi(r,s-3,a,t)+r && \text{by Lemma 14 since}\\
& && rt+s-3\not\equiv 2(mod\ a-1),\\
\le\;& \pi(r,s-3,a-1,t)+r && \text{by Lemma 14 again,}\\
=\;& r\left\lceil \tfrac{tr+(s-3)-1}{a-1}\right\rceil+(t-1)r+r && \text{by Theorem 6,}\\
=\;& r\left\lceil \tfrac{tr+s-1}{a-1}\right\rceil+(t-1)r && \\
=\;& \pi(r,s,a-1,t) && \text{by Theorem 6.}
\end{aligned}
$$

Also

$$
\begin{aligned}
&\pi(r,s,a-1,t)-1 && \\
=\;& r\left\lceil \tfrac{tr+s-1}{a-1}\right\rceil+(t-1)r-1 && \text{by Theorem 6,}\\
=\;& r\left\lceil \tfrac{tr+(s+1)-1}{a-1}\right\rceil+(t-1)r-1 && \text{by Lemma 14 since}\\
& && rt+s\equiv 3(\text{mod } a-1),\\
=\;& \pi(r,s+1,a-1,t) && \text{by Theorem 6,}\\
\le\;& \pi(r,s+1,a-1,t) && \text{by Lemma 14,}\\
\le\;& \pi(r,s+1,a-1,t) && \text{by Lemma 14 again,}\\
=\;& r\left\lceil \tfrac{tr+(s+1)-1}{a-1}\right\rceil+(t-1)r && \text{by Theorem 6,}\\
=\;& r\left\lceil \tfrac{tr+s-1}{a-1}\right\rceil+(t-1)r && \text{since } rt+s\equiv 3(\text{mod } a-1),\\
=\;& \pi(r,s,a-1,t) && \text{by Theorem 6.}
\end{aligned}
$$

Case 3(iii)

$rt+s\not\equiv 1$ (mod $a-1$) *and* $rt+s\not\equiv 3$ (mod $a-1$). Then

$$
\begin{aligned}
&\pi(r,s,a-1,t)-1 && \\
=\;& r\left\lceil \tfrac{tr+s-1}{a-1}\right\rceil+(t-1)r-1 && \text{by Theorem 6,}\\
=\;& r\left\lceil \tfrac{tr+(s-1)-1}{a-1}\right\rceil+(t-1)r-1 && \text{since } rt+s\not\equiv 1(\text{mod } a-1),\\
=\;& \pi(r,s-1,a-1,t) && \text{by Theorem 6,}\\
\le\;& \pi(r,s-1,a,t) && \text{by Lemma 14 since}\\
& && rt+(s-1)\not\equiv 2(\text{mod } a-1),\\
\le\;& \pi(r,s,a,t) && \text{by Lemma 4,}\\
\le\;& \pi(r,s+1,a,t) && \text{by Lemma 4 again,}\\
\le\;& \pi(r,s+1,a-1,t) && \text{by Lemma 14, since}\\
& && rt+(s+1)\not\equiv 2(\text{mod } a-1),\\
=\;& r\left\lceil \tfrac{tr+(s+1)-1}{a-1}\right\rceil+(t-1)r && \text{by Theorem 6,}\\
=\;& r\left\lceil \tfrac{tr+s-1}{a-1}\right\rceil+(t-1)r && \text{since } rt+s\not\equiv 1(\text{mod } a-1),\\
=\;& \pi(r,s,a-1,t) && \text{by Theorem 6.}
\end{aligned}
$$

□

Finally we consider the case when r is odd and a is even.

Lemma 17. *Let r,s,a,t be integers with r,t positive, $a\ge 3$ and s non-negative. Let r be odd and a,s be even. Let $(r+1)t+s\not\equiv 2$ (mod $a-2$). Then*

$$\pi(r+1,s,a-2,t)-1\le\pi(r,s,a,t)\le\pi(r+1,s,a-2,t).$$

Proof. By Lemma 2,

$$\pi(r,s,a,t)\le\pi(r+1,s,a-1,t)\le\pi(r+1,s,a-2,t).$$

To prove the other inequality, let $d=\pi(r+1,s,a-2,t)-2$. Then d is even. Let G be the $(d,d+s)$-pseudograph with two components, G_1 and G_2, where G_1 has one vertex on which are placed $\frac{d}{2}$ loops, and G_2 has one vertex on which are placed $\frac{d+s}{2}$ loops. Since r is odd and a is even, any $(r,r+a)$-factor of G is actually an $((r+1),(r+1)+(a-2))$-factor.

By Theorem 10, it follows that, for any $((r+1),(r+1)+(a-2))$-factorization into x $((r+1),(r+1)+(a-2))$-factors,

$$\frac{d+s}{(r+1)+(a-2)}\le x\le\frac{d}{r+1}.$$

By Theorem 6,

$$\begin{aligned} d &= (r+1)\left\lceil\frac{t(r+1)+s-1}{a-2}\right\rceil + (t-1)(r+1)-2 \\ &= (r+1)\left\lceil\frac{(t(r+1)+s)}{a-2}\right\rceil - (t-1)(r+1)-2 \end{aligned}$$

since r is odd and a and s are even. Therefore

$$\frac{d}{r+1} = \left\lceil\frac{t(r+1)+s}{a-2}\right\rceil + (t-1) - \frac{2}{r+1}$$

so

$$x \leq \left\lceil\frac{t(r+1)+s}{a-2}\right\rceil + (t-2).$$

We also have

$$\begin{aligned} d+s &= (r+1)\left\lceil\frac{t(r+1)+s}{a-2}\right\rceil + (t-1)(r+1)+s-2 \\ &= (r+1)\frac{(t(r+1)+s+c)}{a-2} + (t-1)(r+1)+s-2 \end{aligned}$$

where $0 \leq \frac{c}{2} \leq \frac{a-2}{2} - 1$, c is even and $(a-2) \mid t(r+1)+s+c$. Therefore

$$\begin{aligned} d+s &= \frac{(r+1)+(a-2)}{a-2}(t(r+1)+s+c) + (t-1)(r+1) + \\ &\qquad s-2-(r+1)t-s-c \\ &= \frac{(r+1)+(a-2)}{a-2}(t(r+1)+s+c) - (r+1)-2-c \end{aligned}$$

so

$$\begin{aligned} \frac{d+s}{(r+1)+(a-2)} &= \frac{t(r+1)+s+c}{a-2} - \frac{r+3+c}{(r+1)+(a-2)} \\ &= \left\lceil\frac{t(r+1)+s}{a-2}\right\rceil - \frac{(r+1)+(c+2)}{(r+1)+(a-2)}. \end{aligned}$$

But $c = a-4$ if and only if $(r+1)t+s \equiv 2 \pmod{a-2}$ so that, since $(r+1)t+s \not\equiv 2 \pmod{a-2}$,

$$x \geq \left\lceil\frac{t(r+1)+s}{a-2}\right\rceil.$$

Therefore there do not exist t values of x for which G has an $(r, r+a)$-factorization into x $(r, r+a)$-factors. Therefore $d < \pi(r,s,a,t)$ and so $\pi(r+1, s, a-2, t) - 1 \leq \pi(r,s,a,t)$. □

The case when $(r+1)t+s \equiv 2 \pmod{a-2}$, missed by Lemma 17, is covered by Lemma 18.

Lemma 18. *Let r, s, a, t be integers with r, t positive, $a \geq 3$ and $s \geq 2$. Let r be odd, a be even, and $(r+1)t+s \equiv 2 \pmod{a-2}$ (so s is even). Then*

$$\pi(r+1, s-2, a-2, t) - 1 \leq \pi(r,s,a,t) \leq \pi(r+1, s-2, a-2, t) + (r+1).$$

Proof.

$$\begin{aligned} & \pi(r+1, s-2, a-2, t) - 1 \\ \leq\ & \pi(r, s-2, a, t) && \text{by Lemma 17,} \\ \leq\ & \pi(r, s+2, a, t) && \text{by Lemma 4,} \\ \leq\ & \pi(r+1, s+2, a-2, t) && \text{by Lemma 17 again} \\ =\ & (r+1)\left\lceil\tfrac{t(r+1)+(s+2)-1}{a-2}\right\rceil + (t-1)(r+1) && \text{by Theorem 6,} \\ =\ & (r+1)\left\lceil\tfrac{t(r+1)+(s-2)-1}{a-2}\right\rceil + (t-1)(r+1) + (r+1) && \text{since } (r+1)t+s \\ & && \equiv 2 (\text{mod } a-2), \\ =\ & \pi(r+1, s+2, a-2, t) + (r+1). \end{aligned}$$

□

Theorem 19. *Suppose $r \geq 1$ is odd, $a \geq 3$ is even, and $s \geq 0$, $t \geq 1$. If $(r+1)t+s \not\equiv i \in \{1,2,3\} \pmod{a-2}$, then*

$$\pi(r+1, s, a-2, t) - 1 \leq \pi(r,s,a,t) \leq \pi(r+1, s, a-2, t).$$

If $(r+1)t+s \equiv i \in \{1,2\} \pmod{a-2}$ and $s \geq i$, then

$$\pi(r+1, s-i, a-2, t) - 1 \leq \pi(r,s,a,t) \leq \pi(r+1, s-i, a-2, t) + r + 1.$$

If $(r+1)t+s \equiv 3 \pmod{a-2}$ then

$$\pi(r+1, s, a-2, t) - 1 \leq \pi(r, s-3, a, t) + r \leq \pi(r+1, s, a-2, t)$$

and

$$\pi(r+1, s, a-2, t) - 1 \leq \pi(r, s+1, a, t) \leq \pi(r+1, s, a-2, t).$$

The bounding terms in each case are given explicitly in each case in Theorem 6. The proof of Theorem 19 follows the proof of Theorem 13, but uses Lemmas 17 and 18 instead of Lemmas 11 and 12.

Proof. We consider various cases.

Case 1:
$(r+1)t+s \equiv 2 \pmod{a-2}$. In this case s is even and the theorem follows from Lemma 18.

Case 2:
$(r+1)t+s \not\equiv 2 \pmod{a-2}$ *and s is even.* In this case we also have that $(r+1)t+s \not\equiv 1 \pmod{a-2}$ and the theorem follows from Lemma 17.

Case 3:
$(r+1)t+s \not\equiv 2 \pmod{a-2}$ *and s is odd.*

Case 3 (i)
$(r+1)t+s \equiv 1 \pmod{a-2}$. Then

$$\begin{aligned}
&\pi(r+1,s-1,a-2,t)-1 && \\
\le\ &\pi(r,s-1,a,t) && \text{by Lemma 17,}\\
\le\ &\pi(r,s,a,t) && \text{by Lemma 4,}\\
\le\ &\pi(r,s+1,a,t) && \text{by Lemma 4 again,}\\
\le\ &\pi(r+1,s-1,a-2,t)+r+1 && \text{by Lemma 18 since } (r+1)t+(s+1) \equiv 2 \pmod{a-2}.
\end{aligned}$$

Case 3(ii)
$(r+1)t+s \equiv 3 \pmod{a-2}$. Then

$$\begin{aligned}
&\pi(r+1,s,a-2,t)-1 && \\
=\ &(r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1)-1 && \text{by Theorem 6,}\\
=\ &(r+1)\left\lceil \tfrac{t(r+1)+(s-3)-1}{a-2} \right\rceil + (t-1)(r+1)-1+(r+1) && \\
=\ &\pi(r+1,s-3,a-2,t)-1+(r+1) && \text{by Theorem 6,}\\
\le\ &\pi(r,s-3,a,t)-1+(r+1) && \text{by Lemma 17 since } (r+1)t+s-3 \not\equiv 2 \pmod{a-2},\\
\le\ &\pi(r+1,s-3,a-2,t)+(r+1) && \text{by Lemma 17 again,}\\
=\ &(r+1)\left\lceil \tfrac{t(r+1)+(s-3)-1}{a-2} \right\rceil + (t-1)(r+1)+(r+1) && \text{by Theorem 6,}\\
=\ &(r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1) && \\
=\ &\pi(r+1,s,a-2,t). &&
\end{aligned}$$

Also

$$\begin{aligned}
&\pi(r+1,s,a-2,t)-1 && \\
=\ &(r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1)-1 && \text{by Theorem 6,}\\
=\ &(r+1)\left\lceil \tfrac{t(r+1)+(s+1)-1}{a-2} \right\rceil + (t-1)(r+1)-1 && \text{since } (r+1)t+s \equiv 3 \pmod{a-2},\\
=\ &\pi(r+1,s+1,a-2,t)-1 && \text{by Theorem 6,}\\
\le\ &\pi(r,s+1,a,t) && \text{by Theorem 16,}\\
\le\ &\pi(r+1,s+1,a-2,t) && \text{by Theorem 16 again since } (r+1)t+(s+1) \equiv 2 \pmod{a-2},\\
=\ &(r+1)\left\lceil \tfrac{t(r+1)+(s+1)-1}{a-2} \right\rceil + (t-1)(r+1) && \text{by Theorem 6,}\\
=\ &(r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1) && \text{since } (r+1)t+s \equiv 3 \pmod{a-2},\\
=\ &\pi(r+1,s,a-2,t). &&
\end{aligned}$$

Case 3(iii)
$(r+1)t+s \not\equiv 1 \pmod{a-2}$ *and* $(r+1)t+s \not\equiv 3 \pmod{a-2}$. Then

$$\begin{aligned}
&\pi(r+1,s,a-2,t)-1 && \\
=\ &(r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1)-1 && \text{by Theorem 6,}\\
=\ &(r+1)\left\lceil \tfrac{t(r+1)+(s-1)-1}{a-2} \right\rceil + (t-1)(r+1)-1 && \text{since } (r+1)t+s \not\equiv 1 \pmod{a-2},\\
=\ &\pi(r+1,s-1,a-2,t)-1 && \text{by Theorem 6,}\\
\le\ &\pi(r,s-1,a,t) && \text{by Lemma 17 since } (r+1)t+(s-1) \not\equiv 2 \pmod{a-2},\\
\le\ &\pi(r,s,a,t) && \text{by Lemma 4,}\\
\le\ &\pi(r,s+1,a,t) && \text{by Lemma 4 again,}\\
\le\ &\pi(r+1,s+1,a-2,t) && \text{by Lemma 17, since } (r+1)t+(s+1) \not\equiv 2 \pmod{a-2},
\end{aligned}$$

$$
\begin{aligned}
&= (r+1)\left\lceil \tfrac{t(r+1)+(s+1)-1}{a-2} \right\rceil + (t-1)(r+1) && \text{by Theorem 6,} \\
&= (r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1) && \text{since } (r+1)t+s \not\equiv 1 \\
& && (\text{mod } a-2), \\
&= \pi(r+1, s, a-2, t).
\end{aligned}
$$

□

3 Exact Evaluation of the pseudograph threshold number $\pi(r, s, a, t)$

3.1

In this section we refine the results in Section 2 and obtain exact evaluations of $\pi(r, s, a, t)$ in the cases when r and a are not both even. We recall from Theorem 6 that when r and a are both even and a is positive then

$$\pi(r, s, a, t) = N(r, s, a, t) = r\left\lceil \frac{tr+s-1}{a} \right\rceil + (t-1)r.$$

For the reader's convenience, we repeat Theorems 7 and 8, giving the exact evaluations of $\pi(r, s, a, t)$ in all cases.

The exact evaluations when $a \geq 2$ are given in Theorem 7 below.

Theorem 7. *Let r, s, a and t be integers with r, t positive, $a \geq 2$ and s non-negative.*

(1) If r and a are both even, then

$$\pi(r, s, a, t) = N(r, s, a, t).$$

(2) If r and a are both odd then

$$\pi(r, s, a, t) = \begin{cases} N(r+1, s, a-1, t) - 1 & \text{if } (r+1)t+s \not\equiv 2 \ (\text{mod}\, a-1), \\ N(r+1, s, a-1, t) - (r+1) - 1 & \text{if } (r+1)t+s \equiv 2 \ (\text{mod}\, a-1). \end{cases}$$

(3) If r is odd and a is even, $a \geq 4$, then

$$\pi(r, s, a, t) = \begin{cases} N(r+1, s, a-2, t) - 1 & \text{if } (r+1)t+s \not\equiv 2, 3 \ (\text{mod}\, a-2), \\ N(r+1, s, a-2, t) - (r+1) - 1 & \text{if } (r+1)t+s \equiv 2, 3 \ (\text{mod}\, a-2). \end{cases}$$

(4) If r is even and a is odd, then

$$\pi(r, s, a, t) = \begin{cases} N(r, s, a-1, t) & \text{if } rt+s \not\equiv 2 \ (\text{mod}\, a-1), \\ N(r, s, a-1, t) - r & \text{if } rt+s \equiv 2 \ (\text{mod}\, a-1). \end{cases}$$

For $a = 0$ or 1 or $a = 2$, r odd, we give the results in Theorem 8. Note that we use the notation $\pi(r, s, a, t) = \infty$ when there is no finite threshold number for the given value of r, s, a and t. If $a = 2$ then $\pi(r, s, a, t)$ is given by Theorem 6 if r is even, but if r is odd is given below in Theorem 8.

Theorem 8. *Let r, s and t be integers with r and t positive and s non-negative. Then*

$$\pi(r, s, 0, t) = \infty$$

and

$$\pi(r, s, 1, t) = \begin{cases} 2 & \text{if } r = 2, s = 0 \text{ and } t = 1, \\ 1 & \text{if } r = 1, s - 0 \text{ and } t = 1, \\ & \text{otherwise,} \end{cases}$$

and if r is odd then

$$\pi(r, s, 2, t) = \begin{cases} \infty & \text{if } r \geq 1,\ s > 1 \text{ or } t > 1, \\ 1 & \text{if } r = 1,\ s \in \{0, 1\} \text{ and } t = 1. \end{cases}$$

The recent discovery of the mistake mentioned before in the paper by Hilton [14] had the consequence that a lot more careful argument is now needed to establish Theorem 7 than was needed to prove the corresponding (incorrect) theorem of Hilton and Rajkumar [15].

Section 3 is very long and so we have divided it into sub-sections. Some of the results needed for Theorem 7 are common to more than one of the cases (2), (3) and (4), but some are particular to just one of the cases. Section 3.2 includes all the results needed for Case (2) of Theorem 7, which is given separately as Lemma 24. Sections 3.2 and 3.3 include all the results needed for Case (3) of Theorem 7, which is given separately as Theorem 26. Sections 3.2, 3.3 and 3.4 include all the results needed for Case (4) of Theorem 7, which is given separately as Lemma 29.

Section 3.5 contains evaluations of $\pi(r, s, 0, t)$, $\pi(r, s, 1, t)$ and $\pi(r, s, 2, t)$, cases which are not covered by Theorem 7.

3.2 Everything related to the case r and a both odd.

We start by lowering the upper bound on $\pi(r, s, a, t)$ in the case $(r+1)t + s \equiv 1(\text{mod } a-1)$ in Theorem 13, raising the lower bound in the case $rt + s \equiv 1(\text{mod } a-1)$ in Theorem 16 and lowering the upper bound in the case $(r+1)t + s \equiv 1(\text{mod } a-2)$ in Theorem 19.

Lemma 20. *Let r, s, a and t be integers with r, t positive, $a \geq 3$ and s non-negative.*

1. *If both a and r are odd and $(r+1)t+s \equiv 1(mod\ a-1)$ then* $N(r+1, s, a-1, t) - 1 \leq \pi(r, s, a, t) \leq N(r+1, s, a-1, t)$.
2. *If r is odd and a is even and $(r+1)t+s \equiv 1(mod\ a-2)$ then* $N(r+1, s, a-2, t) - 1 \leq \pi(r, s, a, t) \leq N(r+1, s, a-2, t)$.
3. *If r is even and a is odd and $rt+s \equiv 1(mod\ a-1)$ then* $N(r, s, a-1, t) - 1 \leq \pi(r, s, a, t) \leq N(r, s, a-1, t)$.

Proof. The proofs of (1), (2) and (3) are very similar.

Proof of (1). By Lemma 2, if $\rho \leq r \leq r+a \leq \rho+\alpha$ then $\pi(r, s, a, t) \geq \pi(\rho, s, \alpha, t)$. Since $r \leq r+1 \leq (r+1)+(a-1) \leq r+a$, it follows that

$$\pi(r, s, a, t) \leq \pi(r+1, s, a-1, t).$$

In the case when r and a are both odd and $(r+1)t+s \equiv 1(mod\ a-1)$, it follows from Theorem 13 that

$$\pi(r+1, s-1, a-1, t) - 1 \leq \pi(r, s, a, t).$$

By Theorem 6, since $r+1$ and $a-1$ are both even,

$$\pi(r+1, s-1, a-1, t) = N(r+1, s-1, a-1, t).$$

Also

$$N(r+1, s-1, a-1, t) = (r+1)\left\lceil \frac{(r+1)t+(s-1)-1}{a-1} \right\rceil + (t-1)(r+1).$$

But

$$\left\lceil \frac{(r+1)t+(s-1)-1}{a-1} \right\rceil = \left\lceil \frac{(r+1)t+(s-2)}{a-1} \right\rceil = \left\lceil \frac{(r+1)t+s-1}{a-1} \right\rceil$$

since $a-1$ divides $(r+1)t+s-1$. Therefore

$$\pi(r+1, s-1, a-1, t) = N(r+1, s, a-1, t) = \pi(r+1, s, a-1, t),$$

so

$$\pi(r+1, s, a-1, t) - 1 \leq \pi(r, s, a, t).$$

Consequently

$$\begin{aligned} N(r+1, s, a-1, t) - 1 = \pi(r+1, s, a-1, t) - 1 &\leq \pi(r, s, a, t) \\ &\leq \pi(r+1, s, a-1, t) = N(r+1, s, a-1, t). \end{aligned}$$

The proofs in cases (2) and (3) are both similar to this.

Proof of (2). Let r be odd and $a \geq 2$ be even. Using Lemma 2 again, we have that since
$r \leq r+1 \leq (r+1)+(a-2) = r+(a-1) \leq r+a$, we have in general that

$$\pi(r, s, a, t) \leq \pi(r+1, s, a-2, t).$$

In the present case, when $(r+1)t+s \equiv 1(mod\ a-2)$ we have using Theorem 16 that

$$\pi(r+1, s-1, a-2, t) - 1 \leq \pi(r, s, a, t).$$

But, as in (1),

$$N(r+1, s-1, a-2, t) = N(r+1, s, a-2, t).$$

Then by Theorem 6, since $r+1$ and $a-2$ are both even,

$$N(r+1, s, a-2, t) - 1 \leq \pi(r, s, a, t) \leq N(r+1, s, a-2, t).$$

Proof of (3). Let r be even and a be odd, and let $rt+s \equiv 1(mod\ a-1)$. Using Lemma 2 again, we have
$r \leq r+a-1 \leq r+a$, so that

$$\pi(r, s, a, t) \leq \pi(r, s, a-1, t).$$

Using Theorem 19 we have in this case that

$$\pi(r, s-1, a-1, t) - 1 \leq \pi(r, s, a, t).$$

But,

$$N(r, s-1, a-2, t) = N(r, s, a-2, t).$$

Then by Theorem 6, since r and $a-1$ are both even,

$$N(r, s, a-1, t) - 1 \leq \pi(r, s, a, t) \leq N(r, s, a-1, t).$$

□

Next we deal with the cases in Theorem 13, Theorem 19 and Theorem 16 when $(r+1)t+s \equiv 2(\text{mod } a-1)$, $(r+1)t+s \equiv 2(\text{mod } a-2)$ and $rt+s \equiv 2(\text{mod } a-1)$ respectively, making similar improvements. However in these cases the arguments are more difficult.

Lemma 21. *Let r, s, a and t be integers with r, t positive, $a \geq 3$ and s non-negative.*

1. *If both a and r are odd and $(r+1)t+s \equiv 2 \pmod{a-1}$ then* $N(r+1,s-2,a-1,t)-1 \leq \pi(r,s,a,t) \leq N(r+1,s-2,a-1,t)$.

2. *If r is odd and a is even and $(r+1)t+s \equiv 2 (mod\ a-2)$ then* $N(r+1,s-2,a-2,t)-1 \leq \pi(r,s,a,t) \leq N(r+1,s-2,a-2,t)$.

3. *If r is even and a is odd and $rt+s \equiv 2(mod\ a-1)$ then* $N(r,s-2,a-1,t)-1 \leq \pi(r,s,a,t) \leq N(r,s-2,a-1,t)$.

Remark: In Case 1, since $(r+1)t+s \equiv 2(mod\ a-1)$,

$$\begin{aligned}
N(r+1,s-2,a-1,t) &= (r+1)\left\lceil \frac{t(r+1)+(s-2)-1}{a-1} \right\rceil + (t-1)(r+1) \\
&= (r+1)\left\lceil \frac{t(r+1)+(s-2)}{a-1} \right\rceil + (t-1)(r+1) \\
&= N(r+1,s-1,a-1,t) \\
&= (r+1)\left\lceil \frac{t(r+1)+(s-2)+(a-1)}{a-1} \right\rceil - (r+1) \\
&\qquad + (t-1)(r+1) \\
&= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-1} \right\rceil - (r+1) + (t-1)(r+1) \\
&= N(r+1,s,a-1,t) - (r+1).
\end{aligned}$$

Similar equalities are true in cases 2 and 3.
In Case 2, since $(r+1)t+s \equiv 2(mod\ a-2)$,

$$\begin{aligned}
N(r+1,s-2,a-2,t) &= (r+1)\left\lceil \frac{t(r+1)+(s-2)-1}{a-2} \right\rceil + (t-1)(r+1) \\
&= (r+1)\left\lceil \frac{t(r+1)+(s-2)}{a-2} \right\rceil + (t-1)(r+1) \\
&= N(r+1,s-1,a-2,t) \\
&= (r+1)\left\lceil \frac{t(r+1)+(s-2)+(a-2)}{a-2} \right\rceil - (r+1) \\
&\qquad + (t-1)(r+1) \\
&= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-2} \right\rceil - (r+1), \\
&= N(r+1,s,a-1,t) - (r+1).
\end{aligned}$$

and in Case 3, since $rt+s \equiv 2(mod\ a-1)$,

$$\begin{aligned}
N(r,s-2,a-1,t) &= r\left\lceil \frac{tr+(s-2)-1}{a-1} \right\rceil + (t-1)r \\
&= r\left\lceil \frac{tr+(s-2)}{a-1} \right\rceil + (t-1)r \\
&= r\left\lceil \frac{tr+(s-2)+(a-1)}{a-1} \right\rceil - r + (t-1)r \\
&= r\left\lceil \frac{tr+s-1}{a-1} \right\rceil - r + (t-1)r \\
&= N(r,s,a-1,t) - r.
\end{aligned}$$

Proof of Lemma 21

Proof of (1) In all cases it follows that s is even. By Lemma 12, and Theorem 6, since $r+1$ and $a-1$ are even,

$$N(r+1,s-2,a-1,t)-1 \leq \pi(r,s,a,t).$$

We need to show here that

$$\pi(r,s,a,t) \leq N(r+1,s-2,a-1,t).$$

Let G be a $(d,d+s)$ - pseudograph with $d=d(G)=N(r+1,s,a-1,t)-(r+1)+y = N(r+1,s-2,a-1,t)+y$, where y is a non-negative integer(using the Remark before the proof). By Theorem 10, it is enough to show that the inequality $\frac{d+s}{r+a} \leq x \leq \frac{d}{r+1}$ is satisfied by at least t integer values of x. For then G will have an $(r+1,(r+1)+(a-1)) = (r+1,r+a)$ - factorization with x factors for t different values of x.

Since

$$d = N(r+1,s,a-1,t)-(r+1)+y$$

it follows that

$$\begin{aligned}
d &= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1) - (r+1) + y \\
&= (r+1)\frac{(t(r+1)+s+a-3)}{a-1} + (t-2)(r+1) + y
\end{aligned}$$

so that

$$\frac{d}{r+1} = \frac{t(r+1)+s+a-3}{a-1} + (t-2) + \frac{y}{r+1}.$$

Also

$$\begin{aligned}
d+s &= (r+1)\left\lceil\frac{t(r+1)+s-1}{a-1}\right\rceil+(t-1)(r+1)-(r+1)+y+s\\
&= \frac{r+a}{a-1}(t(r+1)+s+a-3)+(t-2)(r+1)+y+s-t(r+1)\\
&\qquad\qquad -s-a+3\\
&= \frac{r+a}{a-1}(t(r+1)+s+a-3)-2(r+1)-a+3+y\\
&= \frac{r+a}{a-1}(t(r+1)+s+a-3)-(r+a)-(r-y)+1.
\end{aligned}$$

Consequently

$$\frac{d+s}{r+a}=\frac{t(r+1)+s+a-3}{a-1}-1-\frac{r-y-1}{r+a}.$$

The inequality

$$\frac{t(r+1)+s+a-3}{a-1}-1-\frac{r-y-1}{r+a}\le x\le\frac{t(r+1)+s+a-3}{a-1}+(t-2)+\frac{y}{r+1}$$

is satisfied by at least t integer values of x if $0\le y\le r-1$, in particular the following integer values of x:

$$\frac{t(r+1)+s+a-3}{a-1}+i$$

for $i=-1,0,1,2,...,t-2$.

We defer the argument for the case $y=r$. Suppose now that $y\ge r+1$. Let $y=(r+1)z+w$, where $z\ge 0$ and $0\le w\le r$. Then

$$\begin{aligned}
-1-\frac{r-y-1}{r+a} &= -1-\frac{(r+1)-2-(r+1)z-w}{r+a}\\
&= -1+\frac{(r+1)(z-1)+w+2}{r+a}\\
&\le -1+(z-1)+1\\
&= z-1
\end{aligned}$$

and

$$(t-2)+\frac{y}{r+1}\ge(t-2)+z,$$

so that it suffices to show that there are at least t integers x satisfying

$$\frac{t(r+1)+s+a-3}{a-1}+z-1\le x\le\frac{t(r+1)+s+a-3}{a-1}+(t-2)+z.$$

But the following integers satisfy this:

$$\frac{t(r+1)+s+a-3}{a-1}+i$$

for $i=z-1,z,z+1,...,z+(t-2)$, so there are t integers altogether, as required. Thus if $d=N(r+1,s,a-1,t)-(r+1)+y, y\ge 0, y\ne r$, G has an $(r,r+a)$-factorization with x factors for at least t integer values of x.

Now consider the case when $y=r$. In this case $d=N(r+1,s,a-1,t)-1$. If there is an odd number of vertices of minimum degree d in G, take a disjoint further copy of G, and denote the two copies of G by 2G. Now pair off the vertices of minimum degree in G (or in 2G if there is an odd number of such vertices in G). Let G^* denote G (or 2G) with these extra edges added. Then G^* is a $(d+1,d+s)$-pseudograph, where $d+1=N(r+1,s,a-1,t)$. By Theorem 6,

$$\pi(r+1,s,a-1,t)=N(r+1,s,a-1,t),$$

so G^* has an $(r+1,r+a)$-factorization with x factors for t different values of x. Removing the extra edges to revert to G, we see that G has an $(r,r+a)$-factorization with x factors for t different values of G.

Consequently $\pi(r,s,a,t)\le N(r+1,s-2,a-1,t)$, as asserted. □

Proof of (2): The proof of (2) is very similar to the proof of (1). We have that r is odd and a is even, and also that $(r+1)t+s\equiv 2(mod\ a-2)$. By Lemma 17 and Theorem 6, since $(r+1)$ and $(a-2)$ are even,

$$N(r+1,s-2,a-2,t)-1\le\pi(r,s,a,t).$$

We need to show here that

$$\pi(r,s,a,t)\le N(r+1,s-2,a-2,t).$$

Let G be a $(d,d+s)$-pseudograph with $d=d(G)=N(r+1,s,a-2,t)-(r+1)+y=N(r+1,s-2,a-2,t)+y$, where y is a non-negative integer. By Theorem 10, it is enough to show that the inequality $\frac{d+s}{(r+1)+(a-2)}\le x\le\frac{d}{r+1}$ is satisfied by at least t integer values of x. For then G will have an $(r+1,(r+1)+(a-2))=(r+1,r+a-1)$-factorization with x factors for t different values of x. Clearly an $(r+1,r+a-1)$-factor is an $(r,r+a)$-factor.

Since

$$d=N(r+1,s,a-2,t)-(r+1)+y$$

it follows that

$$\begin{aligned} d &= (r+1)\left\lceil\frac{t(r+1)+s-1}{a-2}\right\rceil+(t-1)(r+1)-(r+1)+y \\ &= (r+1)\frac{(t(r+1)+s+a-4)}{a-2}+(t-2)(r+1)+y \end{aligned}$$

so that

$$\frac{d}{r+1}=\frac{t(r+1)+s+a-4}{a-2}+(t-2)+\frac{y}{r+1}.$$

Also

$$\begin{aligned} d+s &= (r+1)\left\lceil\frac{t(r+1)+s-1}{a-2}\right\rceil+(t-1)(r+1)-(r+1)+y+s \\ &= \frac{r+a-1}{a-2}(t(r+1)+s+a-4)+(t-2)(r+1)+y+s \\ &\qquad -t(r+1)-s-a+4 \\ &= \frac{r+a-1}{a-2}(t(r+1)+s+a-4)-2(r+1)-a+4+y \\ &= \frac{r+a-1}{a-2}(t(r+1)+s+a-4)-(r+a+1)-(r-y)+1. \end{aligned}$$

Consequently

$$\frac{d+s}{r+a-1}=\frac{t(r+1)+s+a-4}{a-2}-1-\frac{r-y-1}{r+a-1}.$$

The inequality

$$\frac{t(r+1)+s+a-4}{a-2}-1-\frac{r-y-1}{r+a-1}\leq x\leq\frac{t(r+1)+s+a-4}{a-2}+(t-2)+\frac{y}{r+1}$$

is satisfied by at least t integer values of x if $0\leq y\leq r-1$, in particular the following integer values of x:

$$\frac{t(r+1)+s+a-4}{a-2}+i$$

for $i=-1,0,1,2,...,t-2$.

We defer the argument for the case $y=r$. Suppose now that $y\geq r+1$. Let $y=(r+1)z+w$, where $z\geq 0$ and $0\leq w\leq r$. Then

$$\begin{aligned} -1-\frac{r-y-1}{r+a-1} &= -1-\left(\frac{(r+1)-2-(r+1)z-w}{r+a-1}\right) \\ &= -1+\frac{(r+1)(z-1)+w+2}{r+a-1} \\ &\leq -1+(z-1)+1 \\ &= z-1, \end{aligned}$$

and

$$(t-2)+\frac{y}{r+1}\geq(t-2)+z,$$

so that it suffices to show that there are at least t integers x satisfying

$$\frac{t(r+1)+s+a-4}{a-2}+z-1\leq x\leq\frac{t(r+1)+s+a-4}{a-2}+(t-2)+z.$$

But the following integers satisfy this:

$$\frac{t(r+1)+s+a-4}{a-2}+i$$

for $i=-1,0,1,2,...,(t-2)$, so there are t integers altogether, as required.

Thus if $d=N(r+1,s,a-2,t)-(r+1)+y, y\geq 0, y\neq r$, G has an $(r,r+a)$-factorization with x factors for at least t integer values of x.

Now consider the case when $y=r$. In this case, $d=N(r+1,s,a-2,t)-1$. If there is an odd number of vertices of minimum degree d in G, take a disjoint further copy of G, and denote the two copies of G by 2G. Now pair off the vertices of minimum degree in G (or in 2G if there is an odd number of such vertices in G). Let G^* denote G (or 2G) with these extra edges added. Then G^* is a $(d+1,d+s)$-pseudograph, where $d+1=N(r+1,s,a-2,t)$, so G^* has an $(r+1,r+a-1)=(r+1,(r+1)+(a-2))$-factorization with x factors for t different values of x. Removing the extra edges to revert to G, we find that G has an $(r,r+a)$-factorization with x factors for t different values of G.

Consequently $\pi(r,s,a,t)\leq N(r+1,s-2,a-2,t)$, as asserted. □

Proof of (3) The proof of (3) is like that of (1) and (2), but simpler.

We have that r is even and a is odd, and also that $rt+s\equiv 2(\text{mod } a-1)$. By Lemma 18 and Theorem 6, since r and $(a-1)$ are even,

$$N(r,s-2,a-1,t)-1\leq\pi(r,s,a,t).$$

We need to show here that

$$\pi(r,s,a,t)\leq N(r,s-2,a-1,t).$$

Let G be a $(d,d+s)$ - pseudograph with $d=d(G)=N(r,s,a-1,t)-r+y=N(r,s-2,a-1,t)+y$, where y is a non-negative integer. By Theorem 10, it is enough to show that the inequality $\frac{d+s}{r+(a-1)}\leq x\leq\frac{d}{r}$ is satisfied by at least t integer values of x. For then G will have an $(r,r+(a-1))$-factorization with x factors for t different values of x. Clearly an $(r,r+a-1)$-factorization is an $(r,r+a)$-factorization.

Since

$$d=N(r,s,a-1,t)-(r+1)+y$$

it follows that

$$\begin{aligned} d &= r\left\lceil\frac{tr+s-1}{a-1}\right\rceil+(t-1)r-r+y \\ &= r\frac{(tr+s+a-2)}{a-1}+(t-2)r+y \end{aligned}$$

so that

$$\frac{d}{r}=\frac{tr+s+a-2}{a-1}+(t-2)+\frac{y}{r}.$$

Also

$$\begin{aligned} d+s &= r\left\lceil\frac{tr+s-1}{a-2}\right\rceil+(t-1)r-r+y+s \\ &= \frac{r+a-1}{a-1}(tr+s+a-2)+(t-2)r+y+s-tr-s-a+2 \\ &= \frac{r+a-1}{a-1}(tr+s+a-2)-2r-a+2+y \\ &= \frac{r+a-1}{a-1}(tr+s+a-2)-(r+a-1)-(r-y)+1. \end{aligned}$$

Consequently

$$\frac{d+s}{r+a-1}=\frac{tr+s+a-2}{a-1}-1-\frac{r-y-1}{r+a-1}.$$

The inequality

$$\frac{tr+s+a-2}{a-1}-1-\frac{r-y-1}{r+a-1}\le x\le\frac{tr+s+a-2}{a-1}+(t-2)+\frac{y}{r}$$

is satisfied by at least t integer values of x if $0\le y$. For if $y=rz+w$, where $z\ge 0$ and $0\le w\le r-1$. Then

$$\begin{aligned} -1-\frac{r-y-1}{a-1} &= -1-\frac{r-rz-w-1}{r+a-1} \\ &= -1+\frac{(z-1)r+w+1}{r+a-1} \\ &\le -1+(z-1)+1 \\ &= z-1, \end{aligned}$$

and

$$(t-2)+\frac{y}{r}=t-2+z+\frac{w}{r}\ge(t-2)+z,$$

so that the integer values of x include the t integers:

$$\frac{tr+s+a-2}{a-1}+i$$

for $i=z-1,z,z+1,...,z+(t-2)$, so there are t integers altogether, as required.

Thus if $d=N(r,s,a-1,t)-r+y, y\ge 0$, G has an $(r,r+a)$-factorization with x factors for at least t integer values of x, as required.

This concludes the proof of Lemma 21. □

Finally we need to deal with the cases in Theorem 13, Theorem 19 and Theorem 16 when $(r+1)t+s\equiv 3(\text{mod } a-1)$, $(r+1)t+s\equiv 3(\text{mod } a-2)$ and $rt+s\equiv 3(\text{mod } a-1)$, respectively, making similar improvements.

Lemma 22. *Let r,s,a and t be integers with r,t positive, $a\ge 3$ and s non-negative.*

1. *If both a and r are odd and $(r+1)t+s\equiv 3$ (mod $a-1$) then $N(r+1,s,a-1,t)-1\le\pi(r,s,a,t)\le N(r+1,s,a-1,t)$.*
2. *If r is odd and a is even and $(r+1)t+s\equiv 3$ (mod $a-2$) then $N(r+1,s,a-2,t)-1\le\pi(r,s,a,t)\le N(r+1,s,a-2,t)$.*
3. *If r is even and a is odd and $rt+s\equiv 3$ (mod $a-1$) then $N(r,s,a-1,t)-1\le\pi(r,s,a,t)\le N(r,s,a-1,t)$.*

Remark 1: In the case when both r and a are odd, between the bounding numbers in the case when $(r+1)t+s\equiv 2$ (mod $a-1$) and in the case when $(r+1)t+s\equiv 3(\text{mod } a-1)$, there is a gap, or jump, of $r+1$. For if $(r+1)t+s\equiv 2(\text{mod } a-1)$ then

$$\begin{aligned} &N(r+1,s,a-1,t)-N(r+1,s-2,a-1,t) \\ &= (r+1)\left\lceil\frac{t(r+1)+s-1}{a-1}\right\rceil-(r+1)\left\lceil\frac{t(r+1)+(s-2)-1}{a-1}\right\rceil \\ &= (r+1)\left\{\left\lceil\frac{t(r+1)+s-1}{a-1}\right\rceil-\left\lceil\frac{t(r+1)+s-3}{a-1}\right\rceil\right\} \\ &= r+1. \end{aligned}$$

There is a similar gap in the other cases.

Remark 2: Lemma 22(2) is subsumed by Theorem 26 and Lemma 22(3) is subsumed by Theorem 30.
In each case in Lemma 22, s is odd. (The possibility that s is even in other cases was considered in Lemma 11.) We prove next Lemma 22(1).

Lemma 23. Let r,s,a,t be integers with r,t positive, $a\ge 3$ and s non-negative. Let a and r be odd and $(r+1)t+s\equiv 3(\text{mod } a-1)$. Then

$$N(r+1,s,a-1,t)-1\le\pi(r,s,a,t)\le N(r+1,s,a-1,t).$$

Proof. Recall that in Theorem 13 we showed that

$$\pi(r, s+1, a, t) \leq N(r+1, s, a-1, t).$$

But since $\pi(r, s, a, t) \leq \pi(r, s+1, a, t)$ it follows that

$$\pi(r, s, a, t) \leq N(r+1, s, a-1, t).$$

We need to show that $N(r+1, s, a-1, t) - 1 \leq \pi(r, s, a, t)$. We do this by exhibiting a $(d, d+s)$-graph G of degree $d = N(r+1, s, a-1, t) - 2$ which does not have an $(r, r+a)$-factorization with x factors for t values of x. Note that d is even and $d+s$ is odd. We let G have two components, one, G_1, has one vertex and $\frac{d}{2}$ loops. The other, G_2, has two vertices, v_1 and v_2, between which there is an edge, and each of v_1 and v_2 is incident with $\frac{d+s-1}{2}$ loops. In any $(r, r+a)$- factorization of G, all but at most two of the factors must be $(r+1, r+a-1)$-factors, but the remaining factors must be $(r+1, r+a)$-factors. If there are x factors altogether in an $(r, r+a)$-factorization of G, then $(r+a-1)+(x-1)(r+a) \geq d(v_1) = d+s$ so that

$$x(r+a) \geq d+s+1$$
$$x \geq \frac{d+s+1}{r+a}.$$

Similarly all but one of the factors would have minimum degree $r+1$ and one might have minimum degree r. Therefore $(x-1)(r+1)+r \leq d$, so that $x(r+1) \leq d+1$, so $x \leq \frac{d+1}{r+1}$. Therefore x must satisfies the double inequality

$$\frac{d+s+1}{r+a} \leq x \leq \frac{d+1}{r+1}.$$

Since $r+1$ and $a-1$ are even and $d = N(r+1, s, a-1, t) - 2$ it follows that

$$d = (r+1)\left\lceil \frac{(r+1)t+s-1}{a-1} \right\rceil + (t-1)(r+1) - 2$$

so that

$$d+1 = (r+1)\left\lceil \frac{(r+1)t+s-1}{a-1} \right\rceil + (t-1)(r+1) - 1$$

so that, as $(r+1)t+s \equiv 3(\text{mod } a-1)$

$$\frac{d+1}{r+1} = \frac{(r+1)t+s+a-4}{a-1} + (t-1) - \frac{1}{r+1}.$$

Also

$$\begin{aligned}
d+s+1 &= \frac{(r+1)((r+1)t+s+a-4)}{a-1} + (t-1)(r+1) - 1 + s \\
&= \frac{(r+1)+(a-1)}{a-1}(t(r+1)+s+a-4) + (t-1)(r+1) - 1 \\
&\quad +s - (t(r+1)+s+(a-1)-4) \\
&= \frac{r+a}{a-1}(t(r+1)+s+a-4) - (r+1) - 1 - a + 4 \\
&= \frac{r+a}{a-1}(t(r+1)+s+a-4) - r - a + 2
\end{aligned}$$

so

$$\frac{d+s+1}{r+a} = \frac{1}{a-1}((r+1)t+s+a-4) - 1 + \frac{2}{r+a}.$$

Therefore if G has x $(r, r+a)$-factors then x satisfies

$$\frac{1}{a-1}((r+1)t+s+a-4) - 1 + \frac{2}{r+a} \leq x \leq \frac{1}{a-1}((r+1)t+s+a-4) + (t-1) - \frac{1}{r+1}.$$

The positive integers x which satisfy this double inequality are

$$\frac{1}{a-1}((r+1)t+s+a-4) + i$$

for $i = 0, 1, 2, ..., t-2$ so there are $t-1$ integers. So G does not have an $(r, r+a)$- factorization with x factors for t values of x. Therefore

$$N(r+1, s, a-1, t) - 1 \leq \pi(r, s, a, t).$$

□

We next determine the value of $\pi(r, s, a, t)$ when r is odd, $a \geq 3$, provided that $(r+1)t+s \not\equiv 3(\text{mod } a-1)$ when a is odd, or $(r+1)t+s \not\equiv 3(\text{mod } a-2)$ when a is even.

Lemma 24. *Let r, s, a and t be integers with r, t positive, $a \geq 3$ and s non-negative.*

1. If r and a are both odd, then

$$\pi(r, s, a, t) = \begin{cases} N(r+1, s, a-1, t) - 1 & \text{if } (r+1)t+s \not\equiv 2 \\ & \text{or } 3(\text{mod } a-1), \\ N(r+1, s, a-1, t) - (r+1) - 1 & \text{if } (r+1)t+s \equiv 2 \\ & (\text{mod } a-1). \end{cases}$$

2. If r is odd and a is even, then

$$\pi(r,s,a,t) = \begin{cases} N(r+1,s,a-2,t)-1 & if\ (r+1)t+s \not\equiv 2 \\ & or\ 3(mod\ a-2), \\ N(r+1,s,a-2,t)-(r+1)-1 & if\ (r+1)t+s \equiv 2 \\ & (mod\ a-2). \end{cases}$$

Proof. First suppose that r and a are both odd. Then by Theorem 13 (and the Remark after Lemma 21, and Theorem 6),

$$N(r+1,s,a-1,t)-1 \le \pi(r,s,a,t) \le N(r+1,s,a-1,t)$$

if $(r+1)t+s \not\equiv 1,2,3(\text{mod } a-1)$; by Lemma 20 this also holds if $(r+1)t+s \equiv 1(\text{mod } a-1)$. By Lemma 21 (and the Remark following Lemma 21), if $(r+1)t+s \equiv 2(\text{mod } a-1)$,

$$N(r+1,s,a-1,t)-(r+1)-1 \le \pi(r,s,a,t) \le N(r+1,s,a-1,t)-(r+1).$$

To prove Lemma 24(1), it suffices to show that every $(d,d+s)$- pseudograph of degree $d = N(r+1,s,a-1,t)-1$ if $(r+1)t+s \not\equiv 2,3(\text{mod } a-1)$, or every $(d,d+s)$-psuedograph of degree $d = N(r+1,s,a-1,t)-(r+1)-1$ if $(r+1)t+s \equiv 2(\text{mod } a-1)$, has an $(r,r+a)$-factorization with x factors for at least t values of x. So let G be a $(d,d+s)$- pseudograph of degree $N(r+1,s,a-1,t)-1$ if $(r+1)t+s \not\equiv 2,3(\text{mod } a-1)$ or of degree $N(r+1,s,a-1,t)-(r+1)-1$ if $(r+1)t+s \equiv 2(\text{mod } a-1)$. In each case d is odd.

Take two copies of G, say G' and G'', and if a vertex $v' \in V(G')$ has minimum degree d, join it by an edge to the corresponding vertex $v'' \in V(G'')$. Let the pseudograph so formed be denoted by G^*.

If $(r+1)t+s \not\equiv 2,3\,(\text{mod } a-1)$ we note that G^* has minimum degree $N(r+1,s,a-1,t)$, and is a $(d,d+s)$ pseudograph. Therefore G^* has an $((r+1),(r+1)+(a-1)) = (r+1,r+a)$ factorization with x factors for at least t different values of x. Therefore G has an $(r,r+a)$ factorization with x factors for t values of x.

If $(r+1)t+s \equiv 2(\text{mod } a-1)$ then we note that G^* is a $(d+1,(d+1)+(s-1))$-pseudograph, so, putting $d^* = d+1$ $s^* = s-1$, G^* is a (d^*,d^*+s^*)-pseudograph with $(r+1)t+s^* \equiv 1(\text{mod } a-1)$. Then

$$\begin{aligned} N(r+1,s,a-1,t)-(r+1) &= (r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-1} \right\rceil + \\ &\qquad (t-1)(r+1)-(r+1) \\ &= (r+1)\left\lceil \tfrac{t(r+1)+(s^*)-1}{a-1} \right\rceil + (t-1)(r+1) \\ &= N(r+1,s^*,a-1,t) \end{aligned}$$

so G^* has an $(r+1,r+a)$-factorization with x factors for t values of x, and so G has an $(r,r+a)$-factorization with x factors for t values of x.

The argument if r is odd and a is even is more or less the same. Throughout $a-1$ is replaced by $a-2$ and instead of Theorem 13 we use Theorem 19. In detail it is as follows:

Suppose that r is odd and a is even. Then by Theorem 19

$$N(r+1,s,a-2,t)-1 \le \pi(r,s,a,t) \le N(r+1,s,a-2,t)$$

if $(r+1)t+s \not\equiv 1,2,3(mod\ a-2)$. By Lemma 20(2) this also holds if $(r+1)t+s \equiv 1(mod\ a-2)$, By Lemma 21, if $(r+1)t+s \equiv 2(mod\ a-2)$, then

$$N(r+1,s,a-2,t)-(r+1)-1 \le \pi(r,s,a,t) \le N(r+1,s,a-2,t)-(r+1)$$

(using the Remark after Lemma 21).

To prove Lemma 24(2), it suffices to show that every $(d,d+s)$- pseudograph of degree $d = N(r+1,s,a-1,t)-1$ if $(r+1)t+s \not\equiv 2$ *or* $(\text{mod } a-2)$, or every $(d,d+s)$-pseudograph of degree $d = N(r+1,s,a-1,t)-(r+1)-1$ if $(r+1)t+s \equiv 2(\text{mod } a-2)$, has an $(r,r+a)$-factorization with x factors for at least t values of x. So let G be a $(d,d+s)$-pseudograph of degree $N(r+1,s,a-2,t)-1$ if $(r+1)t+s \not\equiv 2,3(\text{mod } a-2)$ or of degree $N(r+1,s,a-1,t)-(r+1)-1$ if $(r+1)t+s \equiv 2(\text{mod } a-2)$. In each case d is odd.

Take two copies of G, say G' and G'', and if a vertex $v' \in V(G')$ has minimum degree d, join it by an edge to the corresponding vertex $v'' \in V(G'')$. Let the pseudograph so formed be denoted by G^*.

If $(r+1)t+s \not\equiv 2,3(\text{mod } a-2)$ we note that G^* has minimum degree $N(r+1,s,a-2,t)$, and so is a $(d,d+s)$ pseudograph. Since $r+1$ and $a-2$ are both even, G^* has an $((r+1),(r+1)+(a-2)) = (r+1,r+a-1)$ factorization with x factors for at least t different values of x. Therefore G has an $(r,r+a-1)$ factorization with x factors for t values of x.

If $(r+1)t+s \equiv 2\,(\text{mod } a-2)$ then we note that G^* is a $(d+1,(d+1)+(s-1))$-pseudograph, so, putting $d^* = d+1$ and $s^* = s-1$, G^* is a (d^*,d^*+s^*)-pseudograph with $(r+1)t+s^* \equiv 1\,(\text{mod } a-2)$. Then

$$\begin{aligned} n(r+1,s,a-2,t)-(r+1) &= (r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-2} \right\rceil \\ &+ (t-1)(r+1)-(r+1) \\ &= (r+1)\left\lceil \tfrac{t(r+1)+(s^*)-1}{a-2} \right\rceil + (t-1)(r+1) \\ &= N(r+1,s^*,a-2,t) \end{aligned}$$

so G^* has an $(r+1,r+a-1)$-factorization with x factors for t values of x, and so G has an $(r,r+a)$-factorization with x factors for t values of x.

□

Now we determine the value of $\pi(r,s,a,t)$ in every case with $a \geq 3$ when r and a are both odd.

Lemma 25. *Let r,s,a and t be integers with r,t positive, $a \geq 3$ and s non-negative. If r and a are both odd, then*

$$\pi(r,s,a,t) = \begin{cases} N(r+1,s,a-1,t)-1 & \text{if } (r+1)t+s \not\equiv 2 \\ & (\text{mod } a-1), \\ N(r+1,s,a-1,t)-(r+1)-1 & \text{if } (r+1)t+s \equiv 2 \\ & (\text{mod } a-1). \end{cases}$$

Proof. If $(r+1)t+s \not\equiv 3(\text{mod } a-1)$, this is part of Lemma 24. So we need only consider the case when $(r+1)t+s \equiv 3(\text{mod } a-1)$. By Lemma 23,

$$N(r+1,s,a-1,t)-1 \leq \pi(r,s,a,t) \leq N(r+1,s,a-1,t).$$

The argument given in the proof of Lemma 24(1) for the case when $(r+1)t+s \not\equiv 2,3(\text{mod } a-1)$ applies verbatim (except that now we no longer disallow the case when $(r+1)t+s \equiv 3(\text{mod } a-1)$. □

3.3 Everything related to the case r odd and a even that is not in Section 3.2

Theorem 26. *Let r,s,a and t be integers with r,t positive, $a \geq 3$ and s non-negative. Let r be odd and a be even, and let $(r+1)t+s \equiv 3(\text{mod } a-2)$. Then*

$$\pi(r,s,a,t) = N(r+1,s,a-2,t)-(r+1)-1.$$

Proof. We first show that

$$\pi(r,s,a,t) \geq N(r+1,s,a-2,t)-(r+1)-1.$$

We do this by exhibiting a $(d,d+s)$-graph G of degree $d = N(r+1,s,a-2,t)-(r+1)-2$ which does not have an $(r,r+a)$-factorization with x factors for t integer values of x. Note that d is even and $d+s$ is odd.

Let G consist of two components G_1 and G_2, where G_1 has one vertex w on which are placed $\frac{d}{2}$ loops, and G_2 has two vertices, v_1 and v_2, which are joined by a single edge, and on each of which are placed $\frac{1}{2}(d+s-1)$ loops.

In any $(r,r+a)$ - factorization of G into x factors, since the vertex w of minimum degree d has $\frac{d}{2}$ loops on it, and since r is odd, it must be that $x(r+1) \leq d$. Similarly since the vertex v_1 of maximum degree $d+s$ has one edge and $\frac{1}{2}(d+s-1)$ loops on it, and since $r+a$ is odd, it must be that $(r+a)+(x-1)(r+a-1) \geq d+s$, so that $x(r+a-1) \geq d+s-1$. Consequently x satisfies

$$\frac{d+s-1}{r+a-1} \leq x \leq \frac{d}{r+1}.$$

Since $r+1$ and $a-2$ are both even, and $d = N(r+1,s,a-2,t)-(r+1)-2$, it follows that

$$d = (r+1)\left\lceil\frac{(r+1)t+s-1}{a-2}\right\rceil + (t-1)(r+1)-(r+1)-2,$$

so that

$$d = \frac{r+1}{a-2}((r+1)t+s+a-5)+(t-2)(r+1)-2$$

so

$$\frac{d}{r+1} = \frac{1}{a-2}((r+1)t+s+a-5)+(t-2)-\frac{2}{r+1}.$$

Also

$$\begin{aligned} d+s-1 &= \frac{r+1}{a-2}((r+1)t+s+a-5)+(t-2)(r+1)+s-3 \\ &= \frac{(r+1)+(a-2)}{a-2}((r+1)t+s+a-5)+(t-2)(r+1) \\ &\quad +s-3-(r+1)t-s-a+5 \\ &= \frac{r+a-1}{a-2}((r+1)t+s+a-5)-2(r+1)-a+2 \\ &= \frac{r+a-1}{a-2}((r+1)t+s+a-5)-2(r+a-1)+a-2. \end{aligned}$$

so that

$$\frac{d+s-1}{r+a-1} = \frac{1}{a-2}((r+1)t+s+a-5)-2+\frac{a-2}{r+a-1}.$$

Therefore if G has an $(r,r+a)$ - factorization with x factors, then

$$\frac{t(r+1)+s+a-5}{a-2}-2+\frac{a-2}{r+a-1} \leq x \leq \frac{1}{a-2}((r+1)t+s+a-5)+(t-2)-\frac{2}{r+1}.$$

The integer values of x satisfying this double inequality are

$$\frac{1}{a-2}((r+1)t+s+a-5)+i$$

for $i = -1, 0, 1, ..., t-3$, so altogether the double inequality is satisfied by exactly $t-1$ integers. Consequently

$$d \geq N(r+1, s, a-2, t) - (r+1) - 1.$$

Next we shall show that

$$d \leq N(r+1, s, a-2, t) - (r+1) - 1.$$

Let G be a $(d, d+s)$-pseudograph with

$$d = N(r+1, s, a-2, t) - (r+1) - 1 + y$$

for some $y \geq 0$. Take two copies, G_1 and G_2, of G and join each vertex of lowest degree d in G_1 to the corresponding vertex in G_2. Call the graph so formed G^*. Then G^* has minimum degree $d+1$ and maximum degree $d+s$. An $(r+1, r+a-1)$ factor in G^* corresponds (by leaving out the extra edges joining G_1 to G_2) to an $(r, r+a-1)$-factor in G. The minimum degree of G^* is $d+1$ and we have

$$\begin{aligned} d+1 &= (r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1) - (r+1) + y \\ &= \frac{r+1}{a-2}((r+1)t + s + a - 5) + (t-2)(r+1) + y \end{aligned}$$

since $(r+1)t + s \equiv 3 (\mathrm{mod}\ a-2)$. Consequently

$$\frac{d+1}{r+1} = \frac{(r+1)t + a + s - 5}{a-2} + (t-2) + \frac{y}{r+1}.$$

We also have that

$$\begin{aligned} d+s &= (r+1)\frac{t(r+1)+s+a-5}{a-2} + (t-2)(r+1) - 1 + y + s \\ &= \frac{(r+1)+(a-2)}{a-2}((r+1)t + s + a - 5) + (t-2)(r+1) \\ &\quad -1 + y + s - (r+1)t - s - a + 5 \\ &= \frac{(r+a-1)}{a-2}((r+1)t + s + a - 5) - 2(r+1) + y - a + 4 \\ &= \frac{(r+a-1)}{a-2}((r+1)t + s + a - 5) - 2(r+a-1) + y + a \end{aligned}$$

so that

$$\frac{d+s}{r+a-1} = \frac{(r+1)t + s + a - 5}{a-2} - 2 + \frac{y+a}{r+a-1}.$$

Since $(r+1)$ and $(r+a-1)$ are even, G^* has an $(r+1, r+a-1)$-factorization with x factors if x satisfies the double inequality

$$\frac{t(r+1)+s+a-5}{a-2} - 2 + \frac{y+a}{r+a-1} \leq x \leq \frac{(r+1)t+s+a-5}{a-2} + (t-2) + \frac{y}{r+1}.$$

For $0 \leq y \leq r-1$, the integer values of x satisfying this inequality include

$$\frac{(r+1)t+s+a-5}{a-2} + i$$

for $i = -1, 0, 1, ..., t-2$, so there are t such values of x, so G^* does have an $(r+1, r+a-1)$-factorization with x factors for t values of x.

For $y = r$ we cannot make this deduction without a special argument, which we make later below.

For $r+1 \leq y \leq 2r+1$ we have

$$2 > \frac{r+1+a}{r+a-1} > 1$$

and

$$1 < \frac{2r+1}{r+1} < 2$$

so the integer values of x satisfying the equality above include

$$\frac{(r+1)t+s+a-5}{a-2} + i$$

for $i = 0, 1, ..., t-2, t-1$ so there are t such values of x.

Now let $y = p(r+1) + z$ where $p \geq 2$ and $0 \leq z \leq r$. We have

$$\frac{p(r+1)+a}{r+a-1} < p$$

and

$$p < \frac{p(r+1)+r}{r+1} < p+1,$$

so the integer values of x satisfying the inequality include

$$\frac{(r+1)t+s+a-5}{a-2} + i$$

for $i = p-2, p-1, ..., p+(t-2)$, so there are at least t such values of x.

Now let us consider the case when $y = r$. First select an independent set S_1 of edges of G_1 such that each vertex of G_1 of maximum degree $d+s$ is incident with exactly one edge (and each edge of S_1 is incident with a vertex of degree $d+s$). Let S_2 be the corresponding set of edges in G_2

and let $S = S_1 \cup S_2$. Let S_1^* be the set of edges of S_1 which are incident with a vertex of degree $d+1$, and let S_2^* be the set of corresponding edges in G_2. Let $S^* = S_1^* \cup S_2^*$. Let $G^{**} = G \backslash S$. Then the maximum degree of G^{**} is at most $d+s-1$ and the minimum degree is at least d; each vertex of degree d is incident with an edge of S^*. We join each vertex in G_1 which has degree d in G^{**} to the corresponding vertex in G_2. Let G^{***} be G^{**} with this further set of edges added in. Then the maximum degree of G^{***} is $d+s-1$ and the minimum degree is $d+1$. For G^{***} to have an $(r+1, r+a-1)$-factorization with x factors for t different values of x, it is necessary that

$$\frac{d+s-1}{r+a-1} \leq x \leq \frac{d+1}{r+1},$$

i.e.,

$$\frac{(r+1)t+s+a-5}{a-2} - 2 + \frac{y+a-1}{r+a-1} \leq x \leq \frac{(r+1)t+s+a-5}{a-2} + (t-2) + \frac{y}{r+1}.$$

In the case when $y = r$, the integers x which satisfy this double inequality include

$$\frac{(r+1)t+s+a-5}{a-2} + i$$

for $i = -1, 0, 1, 2, ..., t-2$ so there are at least t such values of x, so G^{***} does indeed have an $(r+1, r+a-1)$-factorization with x factors for t different values of x. For each such x, choose an $(r+1, r+a-1)$-factorization with x factors. Assign the edges of S to one of the $(r+1, r+a-1)$-factors; then we obtain an $(r+1, r+a)$-factor F. The vertices of $S_1^* \cup S_2^*$ now have degree at least $r+2$ in F. Remove all the extra edges joining vertices in G_1 to vertices in G_2. This produces an $(r, r+a)$-factorization of G with x factors. Thus in the case $y = r$, G also has an $(r, r+a)$-factorization with x factors, for t values of x.

This completes the proof that when r is odd and a is even,

$$\pi(r, s, a, t) = N(r+1, s, a-2, t) - (r+1) - 1.$$

□

Now we determine the value of $\pi(r, s, a, t)$ in every case with $a \geq 3$, a even and r odd.

Theorem 27. *Let r, s, a, t be integers with r, t positive, $a \geq 3$, a even, r odd and s non-negative. Then*

$$\pi(r, s, a, t) = \begin{cases} N(r+1, s, a-2, t) - 1 & \textit{if } (r+1)t+s \not\equiv 2 \textit{ or } 3 \\ & (\textit{mod } a-2), \\ N(r+1, s, a-2, t) - (r+1) - 1 & \textit{if } (r+1)t+s \equiv 2 \\ & (\textit{mod } a-2). \end{cases}$$

Proof. If $(r+1)t+s \not\equiv 2$ *or* $3(\text{mod } a-2)$ then this is part of Lemma 24(2). If $(r+1)t+s \equiv 2(\text{mod } a-2)$ this is also part of Lemma 24(2). If $(r+1)t+s \equiv 3(\textit{mod } a-2)$ this is Theorem 26. □

3.4 Everything related to the case r even and a odd that is not in Sections 3.2 or 3.3.

It remains to consider the case when r is even and a is odd. First we consider the case when $rt+s \not\equiv 3$ and s is even.

Lemma 28. *Let r, s, a and t be integers with r even and positive, t positive, $a \geq 3$ and odd and $s \geq 1$ even. Then*

$$\pi(r, s, a, t) = \begin{cases} N(r, s, a-1, t) & \textit{if } rt+s \not\equiv 2 \ (\textit{mod } a-1), \\ N(r, s, a-1, t) - r & \textit{if } rt+s \equiv 2 \ (\textit{mod } a-1). \end{cases}$$

Proof. First suppose that $rt+s \not\equiv 2(\text{mod } a-1)$. Let $d = N(r, s, a-1, t) - 1$, so that d is odd. Let F be the $(d, d+s)$ - pseudograph with two components, G_1 consisting of an edge uv with $\frac{d-1}{2}$ loops on u and $\frac{d-1}{2}$ loops on v, and G_2 consisting of an edge wx with $\frac{1}{2}(d+s-1)$ loops on w and $\frac{1}{2}(d+s-1)$ loops on x. Recall that a loop contributes two to the degree of the vertex it is on. In any $(r, r+a)$ - factorization of F, all but at most two of the factors consist entirely of loops and so are $(r, r+a-1)$-factors. If there are x factors in an $(r, r+a)$-factorization of F, then all but one of the factors would have maximum degree at most $(r+a-1)$, and one might have degree as high as $r+a$. Therefore $(r+a)+(x-1)(r+(a-1)) \geq d(w) = d+s$, so that

$$x \geq \frac{d+s-1}{r+a-1}.$$

Similarly all but one of the factors would have minimum degree at least r, and one would have minimum degree at least $r+1$. Therefore $r+1+(x-1)r \leq d$ so that

$$x \leq \frac{d-1}{r}.$$

Since r and $a-1$ are even, and $d = N(r, s, a-1, t) - 1$ it follows that

$$d = r\left\lceil\frac{rt+s-1}{a-1}\right\rceil + (t-1)r - 1,$$

so that

$$\begin{aligned}\frac{d-1}{r} &= \left\lceil\frac{tr+s-1}{a-1}\right\rceil + (t-1) - \frac{2}{r} \\ &= \frac{tr+s-1+c}{a-1} + (t-1) - \frac{2}{r}\end{aligned}$$

for some odd c, $1 \leq c \leq a-2$. But if $c = a-2$ then $(a-1) \mid (rt+s-1+a-2) = rt+s+a-3$, so that $rt+s \equiv 2(\text{mod } a-1)$, which is not allowed in this case. Thus the odd number c satisfies $0 \leq c \leq a-4$, and $(a-1) \mid (rt+s-1+c)$.

We also have that

$$d+s = r\left\lceil\frac{tr+s-1}{a-1}\right\rceil + r(t-1)+s-1$$

so that

$$\begin{aligned}d+s-1 &= r\frac{tr+s-1+c}{a-1} + r(t-1)+s-2 \\ &= \frac{r+a-1}{a-1}(tr+s-1+c) - r - c - 1.\end{aligned}$$

Therefore

$$\frac{d+s-1}{r+a-1} = \frac{tr+s-1+c}{a-1} - \frac{r+c+1}{r+a-1}.$$

Therefore if F has an $(r, r+a)$-factorization with x factors, then

$$\frac{tr+s-1+c}{a-1} - \frac{r+c+1}{r+a-1} \leq x \leq \frac{rt+s-1+c}{a-1} + (t-1) - \frac{2}{r}.$$

This inequality is satisfied by the following integer values of x:

$$\frac{tr+s-1+c}{a-1} + i$$

for $i = 0, 1, ..., t-2$. As there are only $t-1$ such values of x, it follows that

$$\pi(r,s,a,t) \geq N(r,s,a-1,t).$$

But by Lemma 2 if $rt+s \not\equiv 2(\text{mod } a-1)$ it follows that

$$\pi(r,s,a,t) \leq N(r,s,a-1,t).$$

Therefore

$$\pi(r,s,a,t) = N(r,s,a-1,t).$$

Now suppose that $rt+s \equiv 2(\text{mod}; a-1)$. We let $d = N(r,s,a-1,t) - r - 1$. With this value of d we proceed as in the case above when

$rt+s \not\equiv 2(\text{mod } a-1)$ and note that if F is the $(d, d+s)$-pseudograph with the two components G_1 and G_2 as above, and if there are x factors in an $(r, r+a)$-factorization of F, then

$$\frac{d+s-1}{r+a-1} \leq x \leq \frac{d-1}{r}.$$

Since r and $a-1$ are even and $d = N(r,s,a,t) - r - 1$ it follows that

$$d = r\left\lceil\frac{tr+s-1}{a-1}\right\rceil + r(t-1) - r - 1$$

so that

$$\begin{aligned}\tfrac{d-1}{r} &= \left\lceil\frac{tr+s-1}{a-1}\right\rceil + (t-2) - \frac{2}{r} \\ &= \frac{tr+s+a-3}{a-1} + (t-2) - \frac{2}{r}.\end{aligned}$$

We also have that

$$d+s = \frac{tr+s+a-3}{a-1}r + (t-2)r - 1 + s,$$

so that

$$\begin{aligned}d+s-1 &= \frac{r+a-1}{a-1}(rt+s+a-3) + (t-2)r - 2 + s - tr - s - a + 3, \\ &= \frac{r+a-1}{a-1} + (rt+s+a-3) - 2r + 1,\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d+s-1}{r+a-1} &= \frac{tr+s+a-3}{a-1} - \frac{2r+1}{r+a-1} \\ &= \frac{tr+s+a-3}{a-1} - 1 - \frac{r-a}{r+a-1}.\end{aligned}$$

So if F has an $(r, r+a)$-factorization with x factors, then

$$\frac{tr+s+a-3}{a-1} - 1 - \frac{r+a}{r+a-1} \leq x \leq \frac{tr+a-3}{t-2} + (t-2) - \frac{2}{r}.$$

This double inequality is satisfied by the following integer values of x:

$$\frac{tr+s+a-3}{a-1} + i$$

for $i = -1, 0, 1, 2, ..., t-3$. As there are only at most $t-1$ such integers, it follows that

$$\pi(r,s,a,t) \geq N(r,s,a,t) - r.$$

But by Lemma 21(3)

$$\begin{aligned}\pi(r,s,a,t) &\le N(r,s-2,a-1,t)\\ &= N(r,s,a-1,t)-r.\end{aligned}$$

Therefore

$$\pi(r,s,a,t)=N(r,s,a-1,t)-r.$$

□

The result corresponding to Lemma 28 but for the case when s is odd is:

Lemma 29. *Let r, s, a and t be integers with r even and positive, t positive, $a \geq 3$ and odd, and $s \geq 1$ odd. Then*

$$\pi(r,s,a,t)=N(r,s,a-1,t).$$

Proof. By Theorem 16

$$\pi(r,s,a,t)\le \pi(r,s,a-1,t)$$

when $rt+s \not\equiv 1,3(\text{mod } a-1)$ (s is odd here in Lemma 29). By Lemma 20(3) and Theorem 6 this is also true when $rt+s\equiv 1(\text{mod}; a-1)$. In the case when $rt+s\equiv 3(\text{mod } a-1)$ we showed in Theorem 16 that

$$\pi(r,s+1,a,t)\le \pi(r,s,a-1,t).$$

But since

$$\pi(r,s,a,t)\le \pi(r,s+1,a-1,t)$$

by Lemma 4, it follows that

$$\pi(r,s,a,t)\le \pi(r,s,a-1,t).$$

Thus this holds in every case when $r\geq 2$ even, $a\geq 3$ odd, $t\geq 1$ and $s\geq 1$ odd.

We need to show that $\pi(r,s,a-1,t)\le \pi(r,s,a,t)$. To this end let $d=N(r,s,a-1,t)-1$, so that d is odd. Let F be the $(d,d+s)$-pseudograph with two components, G_1 consisting of an edge uv with $\frac{d-1}{2}$ loops on u and $\frac{d-1}{2}$ loops on v, and G_2 consisting of single vertex w on which are placed $\frac{1}{2}(d+s)$ loops. Since a loop contributes two to the degree of the vertex it is on, u and v have degree d, and w has degree $d+s$. Thus F is a $(d,d+s)$-pseudograph.

Since a and d are odd and r is even, and since all but one of the edges of F are loops, in any $(r,r+a)$-factorization of F, all but one of the factors would be $(r,r+(a-1))$-factors. If there are x factors in an $(r,r+a)$-factorization of F, then all but one of the factors would have maximum degree at most $(r+a-1)$, and one might have degree as high as $r+a$. Therefore $(r+a)+(x-1)(r+a-1)\geq d(w)=d+s$ so that $x(r+a-1)\geq d+s-1$. But $d+s-1$ is odd and $r+a-1$ is even, so $x(r+a-1)\geq d+s$, and so

$$x\geq \frac{d+s}{r+a-1}.$$

Each $(r,r+a)$-factor has minimum degree at least r, so $xr\le d$. But since r is even and d is odd, we have $xr\le d-1$, so that

$$x\le \frac{d-1}{r}.$$

Since r and $a-1$ are even, and $d=N(r,s,a-1,t)-1$ it follows that

$$d=r\left\lceil\frac{tr+s-1}{a-1}\right\rceil+r(t-1)-1$$

so that

$$\begin{aligned}\frac{d-1}{r} &= \left\lceil\frac{tr+s-1}{a-1}\right\rceil+(t-1)-\frac{2}{r}\\ &= \frac{tr+s-1+c}{a-1}+(t-1)-\frac{2}{r}\end{aligned}$$

for some even c, $0\le c\le a-3$.

We also have that

$$d+s=r\left\lceil\frac{tr+s-1}{a-1}\right\rceil+r(t-1)+s-1$$

so that

$$\begin{aligned}d+s &= r\frac{tr+s-1+c}{a-1}+r(t-1)+s-1\\ &= \frac{r+a-1}{a-1}(tr+s-1+c)+r(t-1)+s-(tr+s-1+c)\\ &= \tfrac{r+a-1}{a-1}(tr+s-1+c)-r-c.\end{aligned}$$

Therefore

$$\frac{d+s-1}{r+a-1}=\frac{tr+s-1+c}{a-1}-\frac{r+c}{r+a-1}.$$

If F has an $(r,r+a)$-factorization with x factors, then

$$\frac{tr+s-1+c}{a-1}-\frac{r+c}{r+a-1}\le x\le \frac{tr+s-1+c}{a-1}+(t-1)-\frac{2}{r}.$$

This double inequality is satisfied by the following integer values of x

$$\frac{tr+s-1+c}{a-1}+i$$

for $i = 0, 1, ..., t-2$. As there are only $t-1$ such integers, it follows that

$$\pi(r, s, a, t) \geq \pi(r, s, a-1, t).$$

Therefore in this case

$$\pi(r, s, a, t) = \pi(r, s, a-1, t).$$

□

We now have the evaluation of $\pi(r, s, a, t)$ when r is even and a is odd.

Theorem 30. *Let r, s, a and t be integers with r even and positive, t positive, $a \geq 2$ and odd and s non-negative. Then*

$$\pi(r, s, a, t) = \begin{cases} N(r, s, a-1, t) & \text{if } rt+s \not\equiv 2 \ (mod\ a-1), \\ N(r, s, a-1, t) - r & \text{if } rt+s \equiv 2 \ (mod\ a-1). \end{cases}$$

Proof. This follows from Lemma 28 if s is even and from Lemma 29 if s is odd.

□

Proof of Theorem 7.
(1) If r and a are both even, this is Theorem 6.
(2) If r and a are both odd, this is Lemma 25.
(3) If r is odd and a is even, this is Theorem 27.
(4) If r is even and a is odd, this is Theorem 30. □

3.5 Determination of $\pi(r, s, a, t)$ in the cases when $a = 0$ or 1, or when $a = 2$ and r is odd.

Theorem 7 settles the value of the pseudograph threshold number $\pi(r, s, a, t)$ when $a \geq 3$ or $a = 2$ and r is even. Recall that we noted earlier that $\pi(r, s, 0, t) = \infty$ means that for no integer d_0 is it true that whenever $d \geq d_0$ then any pseudograph of minimum degree d has an $(r, r+1)$ - factorization with x $(r, r+1)$ - factors for t different values of x (if $t > 0$ and $s > 0$). With two exceptions, it is generally true that $\pi(r, s, a, t) = \infty$ if $a = 0$ or 1.

Proof of Theorem 8

Case 1: Suppose $a = 0$.

Clearly we cannot r-factorize any irregular graph. Therefore $\pi(r, s, 0, t) = \infty$ if $s \neq 0$ and $t \geq 1$. For the case when $s = 0$, if $r \geq 2$ we note that we cannot r-factorize any graph of degree $pr+1$ for any positve integer p. Therefore $\pi(r, 0, 0, t) = \infty$. For the case when $r = 1$, it is well-known that there are regular graphs of degree $d \geq 2$ which cannot be 1-factorized. Therefore, again, $\pi(r, 0, 0, t) = \infty$; in particular $\pi(1, 0, 0, 1) = \infty$.

Case 2: Suppose $a = 1$.

Suppose that $r \geq 3$. If d is even and $d \equiv 2 \pmod{r+1}$ if r is odd, or $d \equiv 2 \pmod{r}$ if r is even, and if a pseudograph G contains a component with one vertex on which are placed $\frac{d}{2}$ loops, then clearly G has no $(r, r+1)$-factorization. Therefore $\pi(r, s, 1, t) = \infty$ when $r \geq 3$; in particular $\pi(r, 0, 1, 1) = \infty$ for $r \geq 3$.

Now suppose that $r = 2$. If G has a component consisting of one vertex and $\frac{d}{2}$ loops when d is even, then G has an $(r, r+1)$factorization with x factors only if $x = \frac{d}{2}$. Therefore $\pi(2, s, 1, t) = \infty$ unless $t = 1$; in particular $\pi(2, 0, 1, 2) = \infty$.

So now suppose that $a = 1$, $r = 2$, s odd and $t = 1$. Consider the case when d is odd and G contains two components, C_1, with two vertices u and v joined by an edge and with $\frac{d-1}{2}$ loops on each vertex, and C_2 with one vertex and $\frac{d+s}{2}$ loops. Then G has no $(r, r+1)$factorization (since any $(r, r+1)$factorization of C_2 has $\frac{d+s}{2} > \frac{d-1}{2}$ factors). Therefore $\pi(2, s, 1, 1) = \infty$ if s is odd. In particular $\pi(2, 1, 1, 1) = \infty$. Since $\pi(r, s+1, a, t) \geq \pi(r, s, a, t)$ it follows now that $\pi(2, s, 1, 1) = \infty$ for each $s \geq 1$. Therefore $\pi(2, s, 1, t) \geq \infty$ whenever $s \geq 1$.

We now show that $\pi(2, 0, 1, 1) = 2$. Let G be a regular pseudograph of degree $d \geq 2$. If d is even, then, by Petersen's Theorem, G has a 2-factorization, i.e. in this case an $(r, r+1)$factorization with $r = 2$.

So we need to consider the case when d is odd. We may pair off the vertices of G, and join each such pair by an edge. Call the graph obtained by adding these extra edges G^+. Each component of G^+ is Eulerian, and so contains an Eulerian circuit. Going one way round each such circuit, we may orient the edges of that circuit. If the vertices of G are $u_1, ..., u_n$ construct a bipartite graph B on vertices $v_1, ..., v_n$ and $w_1, ..., w_n$ and join vertex v_i to vertex w_j for each edge u_iu_j of G when the direction goes from u_i to u_j in the Eulerian circuit of G^+.

Notice that in G a vertex u_i has degree d, and in B the corresponding vertices v_i and w_i have degrees $\frac{d+1}{2}$ and $\frac{d-1}{2}$ in some order. By a theorem of de Werra [23], we may colour the edges of B with $\frac{d-1}{2}$ colours in such a way that each colour occurs on exactly one edge incident with a vertex of degree $\frac{d-1}{2}$, but at a vertex of degree $\frac{d+1}{2}$, some colour will occur on two edges and the rest on one edge each. Colouring the edges of G with the colours of the corresponding edges of B yields a $(2,3)$factorization of G.

Thus every regular pseudograph of degree at least two has a $(2,3)$ factorization. Therefore $\pi(2,0,1,1) = 2$. (I believe this to be a new result).

Now suppose that $r = 1$. If G is a pseudograph consisting of one vertex and $d/2$ loops, then any $(r, r+1)$ - factorization of G in the case $r = 1$ must in fact be an $(r+1)$factorization. By the same argument as in the case above when $r = 2$, we have $\pi(1,s,1,t) = \infty$ unless $t = 1$. In particular $\pi(1,1,1,2) = \infty$.

Now suppose that $a = 1$, $r = 1$ and $t = 1$. If G contains C_1 with one vertex u and $\frac{d}{2}$ loops on it, if d is even, and C_2 with two vertices joined by one edge with $\frac{d+s-1}{2}$ loops on each vertex if d is even and s is odd, then G has no $(r, r+1)$-factorization since any $(r, r+1)$-factorization of C_1 has $\frac{d}{2}$ factors, and of C_2 has $\frac{d+s-1}{2} + 1 = \frac{d+s+1}{2} > \frac{d}{2}$ factors. Therefore $\pi(1,s,1,1) = \infty$ unless $s = 0$.

Finally, we show that $\pi(1,0,1,1) = 1$. Let G be a regular pseudograph of degree $d \geq 1$. If $d = 1$ then G is a regular pseudograph of degree 1, so is its own $(r, r+1)$factorization. If d is even, then by Theorem 1.7 (Petersen [32]), G can be 2 -factorized and thus G has an $(r, r+1)$factorization. If d is odd, $d \geq 3$, then we may pair off the vertices, and join each pair of vertices by an edge. The graph G^+ obtained this way is regular of degree $d+1$, which is even. Now 2-factorize G^+ , and then remove the extra edges. What remains is an $(r, r+1)$factorization of G. Thus $\pi(1,0,1,1) = 1$.

It remains to consider the case when $r = 0$. However it is clear that no pseudograph containing at least one loop can have a $(0,1)$factorization. Thus $\pi(0,0,1,1) = \infty$.

Case 3: Suppose $a = 2$.

Recall that the case when r is even is covered by Theorem 6. We need to consider the case when r is odd. Let G be a graph with one vertex, v, and $\frac{1}{2}x(r+1)+1$ loops incident with v. Any $(r, r+2)$-factor consists of $\frac{1}{2}(r+1)$ loops on v, and so x $(r, r+2)$-factors utilize $\frac{1}{2}x(r+1)$ loops, leaving one loop over not in any factor. Therefore, provided $r+1 \geq 1$, i.e. $r \geq 3$ (as r is odd), G has no $(r, r+2)$-factorization. Therefore $\pi(r,s,2,t) = \infty$ if r is odd and $r \geq 3$. So we may suppose that $r = 1$, so we are investigating $(1,3)$-factorizations.

Let G be a pseudograph with two components, G_1 of degree d and G_2 of degree $d+s$. Suppose s is even. If d is even, let G_1 consist of one vertex with $\frac{d}{2}$ loops on it, and let G_2 consist of one vertex with $\frac{d+s}{2}$ loops on it. Any $(1,3)$-factor of G contains exactly one loop from G_1 and one loop from G_2. Therefore G cannot have a $(1,3)$-factorization unless $s = 0$ or $t = 1$.

Now suppose that $r = 1$, $s = 0$ and $t = 1$. We showed above that $\pi(2,0,1,1) = 2$. Therefore every regular pseudograph of degree at least 2 has a $(2,3)$-factorization, and so has a $(1,3)$-factorization. But a regular graph of degree 1 has a 1-factorization, and so has a $(1,3)$-factorization. Therefore $\pi(1,0,2,1) = 1$.

Next suppose that $r = 1$ and s is odd. Let G now be a 2-component graph, one component being G_1 with one vertex and $\frac{d}{2}$ loops (d being even), and G_2 containing an edge v_3v_4 with $\frac{d+s-1}{2}$ loops on each of v_3 and v_4 . Any $(1,3)$-factorization of G_1 has exactly $\frac{d}{2}$ factors (each being a loop), so any $(1,3)$-factorization of G has exactly $\frac{d}{2}$ factors (so $t = 1$). In any $(1,3)$-factorization of G, each factor has a loop from G_1, and one factor contains the edge v_3v_4 and a loop on each of v_3 and v_4, the other factors containing a loop on G_1 and two loops from G_2, one on v_3 the other on v_4. Therefore $s = 1$ and $t = 1$.

Now let G be an arbitrary $(d, d+s)$-pseudograph. Pair off the vertices odd degree in G, and join each such pair with an edge. Call the graph so formed G^+. Then form a bipartite graph B^+, and subsequently a bipartite graph B as described in the proof above that $\pi(2,0,1,1) = 2$.

Suppose that d is even. Then each extra edge u_iv_j joined two vertices which had degree $d+2$ in G^+. In B one of v_i, w_i has degree $\frac{d}{2}$, the other has degree $\frac{d}{2}+1$. Give the edges of B an equitable edge colouring with $\frac{d}{2}$ colours, and let each edge of G be coloured with the colour of the corresponding edge in B. In G, each vertex will be incident with two or three edges of the same colour. Then G will have a $(2,3)$-factorization, and so will have a $(1,3)$-factorization.

Next suppose that d is odd. Then in G^+ each extra edge u_iu_j joined two vertices of degree $d+1$. In B, one of v_i, w_i has degree $\frac{d-1}{2}$, the other $\frac{d+1}{2}$. Vertices u_i of G^+ not incident with an extra edge have degree $d+1$, and in B both v_i and w_i have degree $\frac{d+1}{2}$. Give the edges of B an equitable edge-colouring with $\frac{d+1}{2}$ colours. Then the corresponding edge-colouring of G is a $(1,2)$-factorization, and so is a $(1,3)$-factorization.

Bearing in mind that a regular graph of degree 1 has a 1-factorization, and hence a $(1,3)$-factorization, it follows that $\pi(1;1,2,1) = 1$.

□

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