Ternary Strings and the Pell Numbers

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ABSTRACT

For $n \geq 1$, let a_n count the number of strings $s_1 s_2 s_3 \ldots s_n$, where (i) $s_1 = 0$; (ii) $s_i \in \{0, 1, 2\}$, for $2 \leq i \leq n$; and (iii) $|s_i - s_{i-1}| \leq 1$, for $2 \leq i \leq n$. Then $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, $a_4 = 12$, and $a_5 = 29$. In general, for $n \geq 3$, $a_n = 2a_{n-1} + a_{n-2}$, and a_n equals P_n, the nth *Pell* number.

For these P_n strings of length n, we count (i) the number of occurrences of each of the symbols $0, 1, 2$; (ii) the number of times each of the symbols $0, 1, 2$ occur in an even or odd position; (iii) the number of levels, rises, and descents that occur within the strings; (iv) the number of runs that occur within the strings; (v) the sum of all the strings considered as base 3 integers; (vi) the numbers of inversions and coinversions that occur within the strings; and, (vii) the sum of the major indices for the strings.

1. Determining a_n.

For $1 \leq n \leq 4$, we list the a_n strings $s_1 s_2 s_3 \ldots s_n$, where (i) $s_1 = 0$, (ii) $s_i \in \{0, 1, 2\}$, for $2 \leq i \leq n$; and, $|s_i - s_{i-1}| \leq 1$, for $2 \leq i \leq n$.

$(n = 1)$: 0

$(n = 2)$: 00, 01

$(n = 3)$: 000, 001, 010, 011, 012

$(n = 4)$: 0000, 0001, 0010, 0011, 0012, 0100,
0101, 0110, 0111, 0112, 0121, 0122

This is comparable to an example provided to the Online Encyclopedia of Integer Sequences [5] on June 2, 2004, by Herbert Kociemba. However, the alphabet used there was $\{1, 2, 3\}$ instead of $\{0, 1, 2\}$.

CONGRESSUS NUMERANTIUM 232 (2019), pp.67-100

To determine a general formula for a_n, in terms of n, we shall make use of the following auxiliary variables: $a_n^{(0)}$ will count the strings of length n that end with 0; $a_n^{(1)}$ will count those strings of length n that end with 1; and, $a_n^{(2)}$ will account for the strings of length n that end with 2.

Then, for $n \geq 3$, we find that

$$\begin{aligned} a_n &= a_n^{(0)} + a_n^{(1)} + a_n^{(2)} \\ &= 2a_{n-1}^{(0)} + 3a_{n-1}^{(1)} + 2a_{n-1}^{(2)}, \end{aligned}$$

where, for example, the coefficient 2 in front of $a_{n-1}^{(0)}$ arises because when a string of length $n-1$ ends in a 0, then we can only append another 0 or a 1 at the end of the string. Likewise, the coefficient 2 in front of $a_{n-1}^{(2)}$ arises here because we only append a 1 or another 2 at the end of each of these strings. However, the coefficient for $a_{n-1}^{(1)}$ is 3, for here it is possible to append any of the symbols $0, 1, 2$ at the end of the strings of length $n-1$. Consequently, we now realize the following, which will prove useful throughout this material - namely, for $n \geq 2$,

$$a_n^{(1)} = a_{n-1}.$$

This leads us to the following recurrence relation

$$\begin{aligned} a_n &= 2a_{n-1}^{(0)} + 3a_{n-1}^{(1)} + 2a_{n-1}^{(2)} \\ &= 2[a_{n-1}^{(0)} + a_{n-1}^{(1)} + a_{n-1}^{(2)}] + a_{n-1}^{(1)} \\ &= 2a_{n-1} + a_{n-2}. \end{aligned}$$

Following the methods presented in Chapter 7 of [1] and Chapter 10 of [2], upon substituting Ar^n for a_n, with $A \neq 0$ and $r \neq 0$, in the above recurrence relation, we have

$$\begin{aligned} &a_n = 2a_{n-1} + a_{n-2} \Longrightarrow Ar^n = 2Ar^{n-1} + Ar^{r-2} \Longrightarrow \\ &r^2 - 2r - 1 = 0 \Longrightarrow r = 1 + \sqrt{2} \text{ and } r = 1 - \sqrt{2}. \end{aligned}$$

So $1+\sqrt{2}$ and $1-\sqrt{2}$ are the characteristic roots for the recurrence relation. These are generally designated by $\gamma = 1+\sqrt{2}$ and $\delta = 1-\sqrt{2}$, and, in the literature, $\gamma = 1+\sqrt{2}$ is often referred to as the *silver ratio*. Consequently, for $n \geq 1$,

$$a_n = c_1\gamma^n + c_2\delta^n.$$

From the recurrence relation we can backtrack to derive the value of a_0 as $a_2 - 2a_1 = 2 - 2 = 0$. Then using the initial conditions $a_0 = 0$

and $a_1 = 1$, we learn that $c_1 = \frac{1}{\gamma-\delta} = \frac{1}{2\sqrt{2}}$ and that $c_2 = -c_1$, so for $n \geq 0$,

$$a_n = \frac{1}{\gamma-\delta}(\gamma^n) - \frac{1}{\gamma-\delta}(\delta^n) = \frac{\gamma^n - \delta^n}{\gamma-\delta} = \frac{\gamma^n - \delta^n}{2\sqrt{2}} = P_n,$$

where P_n denotes the nth *Pell number.* [The expression $\frac{\gamma^n-\delta^n}{\gamma-\delta}$ is referred to as the *Binet form* for the nth Pell number.]

We mentioned earlier that for $n \geq 2$, $a_n^{(1)} = a_{n-1}$. This is also true for $n = 1$. To determine $a_n^{(0)}$, we now observe that

$$\begin{aligned} &a_n^{(0)} = a_{n-1}^{(0)} + a_{n-1}^{(1)} = a_{n-1}^{(0)} + a_{n-2}, \quad \text{so} \\ &a_n^{(0)} - a_{n-1}^{(0))} = P_{n-2} = \frac{\gamma^{n-2} - \delta^{n-2}}{\gamma-\delta} = \frac{\gamma^{n-2} - \delta^{n-2}}{2\sqrt{2}}. \end{aligned}$$

The solution for this first-order nonhomogeneous recurrence relation has the form $a_n^{(0)} = c + A\gamma^n + B\delta^n$, where the constant c denotes the homogeneous part of the solution and $A\gamma^n + B\delta^n$ the particular part. Upon substituting $A\gamma^n$ for $a_n^{(0)}$ into the recurrence relation for $a_n^{(0)}$ we find that $A\gamma^n - A\gamma^{n-1} = \frac{1}{2\sqrt{2}}\gamma^{n-2}$, from which it follows that $A = \frac{-1+\sqrt{2}}{4}$. A similar calculation provides $B = \frac{-1-\sqrt{2}}{4}$.

From $1 = a_1^{(0)} = c + \left(\frac{-1+\sqrt{2}}{4}\right)\gamma + \left(\frac{-1-\sqrt{2}}{4}\right)\delta = c + \frac{1}{4}(-1+2) + \frac{1}{4}(-1+2) = c + \frac{1}{2}$, we have $c = \frac{1}{2}$ and, consequently,

$$\begin{aligned} a_n^{(0)} &= \frac{1}{2} + \left(\frac{-1+\sqrt{2}}{4}\right)\gamma^n + \left(\frac{-1-\sqrt{2}}{4}\right)\delta^n \\ &= \frac{1}{2} + P_n - \frac{1}{2}Q_n, \quad n \geq 1. \end{aligned}$$

[Here Q_n denotes the nth *Pell-Lucas* number, whose *Binet form* is given by $\frac{1}{2}(\gamma^n + \delta^n)$.]

Lastly, we have $a_n^{(2)} = a_n - a_n^{(0)} - a_n^{(1)} = P_n - [\frac{1}{2} + P_n - \frac{1}{2}Q_n] - P_{n-1} = \frac{1}{2}Q_n - P_{n-1} - \frac{1}{2}$.

We summarize the preceding in the following.

Theorem 1. For $n \geq 1$,

$$\begin{aligned} a_n^{(0)} &= \frac{1}{2} + P_n - \frac{1}{2}Q_n \\ a_n^{(1)} &= a_{n-1} = P_{n-1} \\ a_n^{(2)} &= \frac{1}{2}Q_n - P_{n-1} - \frac{1}{2}. \end{aligned}$$

2. Determining the Numbers of 0's, 1's, and 2's in the a_n Strings

2.1 For $n \geq 1$, let z_n count the number of 0's that occur in the $a_n\,(= P_n)$ strings of length n. Then, for $i = 0, 1, 2$, let $z_n^{(i)}$ count the number of 0's that occur in the $a_n^{(i)}$ strings of length n that end with i. We observe that

$$\begin{aligned} z_n^{(1)} &= z_{n-1} \\ z_n^{(2)} &= z_{n-1}^{(1)} + z_{n-1}^{(2)}, \quad \text{and} \\ z_n^{(0)} &= z_{n-1} - z_{n-1}^{(2)} + a_{n-1}^{(0)} + a_{n-1}^{(1)}, \end{aligned}$$

where the expression $z_{n-1}^{(2)}$ in the third equation is due to the fact that 0 cannot occur in position n if 2 occurs in position $n-1$. The remaining terms in the third equation account for the 0 that can be appended to each of the $a_{n-1}^{(0)}$ strings of length $n-1$ that end in 0 and for the 0 that can be appended to each of the $a_{n-1}^{(1)}$ strings of length $n-1$ that end in 1. From these results we are led to

$$\begin{aligned} z_n &= z_n^{(0)} + z_n^{(1)} + z_n^{(2)} \\ &= z_{n-1} - z_{n-1}^{(2)} + a_{n-1}^{(0)} + a_{n-1}^{(1)} + z_{n-1} + z_{n-1}^{(1)} + z_{n-1}^{(2)} \\ &= 2z_{n-1} + z_{n-1}^{(1)} + a_{n-1}^{(0)} + a_{n-1}^{(1)} \\ &= 2z_{n-1} + z_{n-2} + \left[\frac{1}{2} + \frac{\gamma^{n-1} - \delta^{n-1}}{2\sqrt{2}} - \frac{1}{4}(\gamma^{n-1} + \delta^{n-1})\right] + \frac{\gamma^{n-2} - \delta^{n-2}}{2\sqrt{2}} \end{aligned}$$

and this implies that the form of the solution is

$$z_n = c_1\gamma^n + c_2\delta^n + A + Bn\gamma^n + Cn\delta^n.$$

To determine A we substitute $z_n = A$ into the recurrence relation $z_n = z_{n-1} + z_{n-2} + \frac{1}{2}$ and find that $A = -\frac{1}{4}$. We then substitute

$z_n = Bn\gamma^n$ into the recurrence relation $z_n = 2z_{n-1} + z_{n-2} + \frac{1}{2\sqrt{2}}\gamma^{n-1} - \frac{1}{4}\gamma^{n-1} + \frac{1}{2\sqrt{2}}\gamma^{n-2}$. Since $\gamma^2 = 2\gamma + 1$, this results in the equation $B(2\gamma + 2) = \frac{1}{2\sqrt{2}}\gamma - \frac{1}{4}\gamma + \frac{1}{2\sqrt{2}}$, and so $B = \frac{1}{16}\sqrt{2}$. A comparable calculation reveals that $C = -\frac{1}{16}\sqrt{2}$. Consequently, we now have

$$z_n = c_1\gamma^n + c_2\delta^n - \frac{1}{4} + \frac{1}{16}\sqrt{2}n\gamma^n - \frac{1}{16}\sqrt{2}n\delta^n.$$

Then from the initial conditions $z_1 = 1$ and $z_2 = 3$, it follows that $c_1 = \frac{1}{8} + \frac{3\sqrt{2}}{16}$ and $c_2 = \frac{1}{8} - \frac{3\sqrt{2}}{16}$, and we have the following.

Theorem 2. For $n \geq 1$,

$$\begin{aligned} z_n &= \left(\frac{1}{8} + \frac{3\sqrt{2}}{16}\right)\gamma^n + \left(\frac{1}{8} - \frac{3\sqrt{2}}{16}\right)\delta^n - \frac{1}{4} + \frac{1}{16}\sqrt{2}n\gamma^n - \frac{1}{16}\sqrt{2}n\delta^n \\ &= \frac{1}{4}Q_n + \frac{3}{4}P_n - \frac{1}{4} + \frac{1}{4}nP_n. \end{aligned}$$

For the z_n zeros that occur among the $a_n\,(= P_n)$ strings of length n, we now turn our attention to the number that occur in even positions and the number that occur in odd positions. For $n \geq 1$, we let ze_n count those zeros that occur in even positions among the a_n strings. Then for $i = 0, 1, 2$, let $ze_n^{(i)}$ count the number of zeros that occur in even positions for the $a_n^{(i)}$ strings that end with i. We find that

$$\begin{aligned} ze_n = {} & 2ze_{n-1}^{(0)} + \frac{1}{2}\left[1 + (-1)^n\right] a_{n-1}^{(0)} \\ & + 3ze_{n-1}^{(1)} + \frac{1}{2}\left[1 + (-1)^n\right] a_{n-1}^{(1)} \\ & + 2ze_{n-1}^{(2)}, \end{aligned}$$

where, for example, the term $\frac{1}{2}\left[1 + (-1)^n\right] a_{n-1}^{(0)}$ accounts for the zeros that are appended at the (even) nth position at the end of each of the $a_{n-1}^{(0)}$ strings of length $n-1$ ending in 0.

Consequently,

$$ze_n = 2ze_{n-1} + ze_{n-1}^{(1)} + \frac{1}{2}\left[1 + (-1)^n\right](a_{n-1}^{(0)} + a_{n-1}^{(1)}),$$

and, since $ze^{(1)}_{n-1} = ze_{n-2}$, this leads us to the recurrence relation

$$\begin{aligned} ze_n &= 2ze_{n-1} + ze_{n-2} + \frac{1}{2}[1+(-1)^n](a^{(0)}_{n-1} + a^{(1)}_{n-1}) \\ &= 2ze_{n-1} + ze_{n-2} + \frac{1}{2}[1+(-1)^n]\left[\frac{1}{2} + P_{n-1} - \frac{1}{2}Q_{n-1} + P_{n-2}\right] \\ &= 2ze_{n-1} + ze_{n-2} + \frac{1}{2}[1+(-1)^n]\left[\frac{1}{2} + \frac{1}{2}Q_{n-1}\right], \text{ for} \end{aligned}$$

$$\begin{aligned} P_{n-1} + P_{n-2} &= \frac{1}{2\sqrt{2}}(\gamma^{n-1} - \delta^{n-1}) + \frac{1}{2}(\gamma^{n-2} - \delta^{n-2}) \\ &= \frac{1}{2\sqrt{2}}\gamma^{n-2}(\gamma+1) - \frac{1}{2\sqrt{2}}\delta^{n-2}(\delta+1) \\ &= \gamma^{n-2}\left(\frac{1}{2}\gamma\right) - \delta^{n-2}\left(-\frac{1}{2}\delta\right) = \frac{1}{2}(\gamma^{n-1} + \delta^{n-1}) = Q_{n-1}. \end{aligned}$$

So

$$ze_n = 2ze_{n-1} + ze_{n-2} + \frac{1}{4} + \frac{1}{4}Q_{n-1} + \frac{1}{4}(-1)^n + \frac{1}{4}(-1)^n Q_{n-1},$$

for which the solution has the form

$$ze_n = c_1\gamma^n + c_2\delta^n + A + B(-1)^n + Cn\gamma^n + Dn\delta^n + E(-\gamma)^n + F(-\delta)^n.$$

To determine A, we substitute $ze_n = A$ into the recurrence relation $ze_n = 2ze_{n-1} + ze_{n-2} + \frac{1}{4}$ and find that $A = -\frac{1}{8}$. To find B we substitute $ze_n = B(-1)^n$ into the relation $ze_n = 2ze_{n-1} + ze_{n-2} + \frac{1}{4}(-1)^n$ and learn that $B = \frac{1}{8}$. Upon substituting $ze_n = Cn\gamma^n$ into the relation $ze_n = 2ze_{n-1} + ze_{n-2} + \frac{1}{8}\gamma^{n-1}$, after dividing through by γ^{n-2}, we arrive at the equation

$$Cn\gamma^2 = 2C(n-1)\gamma + C(n-2) + \frac{1}{8}\gamma.$$

Comparing the constants on each side of this equation we have $0 = -2C\gamma - 2C + \frac{1}{8}\gamma$, from which it follows that $C = \frac{1}{32}\sqrt{2}$. Then a similar calculation provides $D = -\frac{1}{32}\sqrt{2}$. Continuing, for the coefficient E, we substitute $ze_n = E(-\gamma)^n$ into the relation $ze_n = 2ze_{n-1} + ze_{n-2} + \frac{1}{8}(-1)^n\gamma^{n-1}$, or, $ze_n = 2ze_{n-1} + ze_{n-2} - \frac{1}{8}(-\gamma)^{n-1}$. Dividing through by $(-\gamma)^{n-2}$, we arrive at $E\gamma^2 = -2E\gamma + E + \frac{1}{8}\gamma$, from which it follows that $E = \frac{1}{32}$. Finally, a similar calculation reveals that F is also $\frac{1}{32}$.

From the initial conditions $ze_1 = 0$ and $ze_2 = 1$, we learn that $c_1 = -\frac{1}{32} + \frac{1}{16}\sqrt{2}$ and that $c_2 = -\frac{1}{32} - \frac{1}{16}\sqrt{2}$. Consequently,

$$\begin{aligned} ze_n = &\left(-\frac{1}{32} + \frac{1}{16}\sqrt{2}\right)\gamma^n + \left(-\frac{1}{32} - \frac{1}{16}\sqrt{2}\right)\delta^n \\ &- \frac{1}{8} + \frac{1}{8}(-1)^n + \frac{1}{32}\sqrt{2}n\gamma^n - \frac{1}{32}\sqrt{2}n\delta^n + \frac{1}{32}(-\gamma)^n + \frac{1}{32}(-\delta)^n. \end{aligned}$$

Simplifying the above leads to the following:

Theorem 3. For $n \geq 1$,

$$\begin{aligned} ze_n &= \frac{1}{8}[(-1)^n - 1] + \frac{1}{16}[(-1)^n - 1]Q_n + \frac{1}{8}(n+2)P_n \\ &= \begin{cases} -\frac{1}{4} - \frac{1}{8}Q_n + \frac{1}{8}(n+2)P_n, & n \text{ odd} \\ \frac{1}{8}(n+2)P_n, & n \text{ even.} \end{cases} \end{aligned}$$

If we now let zo_n count the number of zeros that occur among the odd positions for the a_n strings, since $z_n = ze_n + zo_n$, it follows that

$$\begin{aligned} zo_n &= z_n - ze_n \\ &= \begin{cases} \frac{3}{8}Q_n + \frac{1}{2}P_n + \frac{1}{8}nP_n, & n \text{ odd} \\ -\frac{1}{4} + \frac{1}{4}Q_n + \frac{1}{2}P_n + \frac{1}{8}nP_n, & n \text{ even.} \end{cases} \end{aligned}$$

2.2 Now we consider the case for the ones that appear among the a_n strings. For $n \geq 1$, let w_n count the number of ones that appear among the $a_n (= P_n)$ strings of length n. Then, for $i = 0, 1, 2$, let $w^{(i)}_n$ count the number of ones that occur in the $a^{(i)}_n$ strings of length n that end with i. We observe that

$$\begin{aligned} w^{(1)}_n &= w_{n-1} + a^{(1)}_n \\ w^{(2)}_n &= w^{(1)}_{n-1} + w^{(2)}_{n-1}, \quad \text{and} \\ w^{(0)}_n &= w_{n-1} - w^{(2)}_{n-1}. \end{aligned}$$

Then

$$\begin{aligned} w_n &= w^{(0)}_n + w^{(1)}_n + w^{(2)}_n = 2w_{n-1} + w^{(1)}_{n-1} + a^{(1)}_n \\ &= 2w_{n-1} + w_{n-2} + a^{(1)}_{n-1} + a^{(1)}_n = 2w_{n-1} + w_{n-2} + P_{n-2} + P_{n-1} \\ &= 2w_{n-1} + w_{n-2} + Q_{n-1} = 2w_{n-1} + w_{n-2} + \frac{1}{2}\left(\gamma^{n-1} + \delta^{n-1}\right), \end{aligned}$$

and here the form of the solution is given as

$$w_n = c_1\gamma^n + c_2\delta^n + An\gamma^n + Bn\delta^n.$$

Continuing as we did to determine the solution for z_n, we find that $A = \frac{1}{8}\sqrt{2}$, while $B = -\frac{1}{8}\sqrt{2}$. Then from the initial conditions $w_1 = 0$ and $w_1 = 1$, we learn that $c_1 = -\frac{1}{8}\sqrt{2}$ and that $c_2 = \frac{1}{8}\sqrt{2}$. This leads to the following result.

Theorem 4. For $n \geq 1$,

$$w_n = -\frac{1}{8}\sqrt{2}\gamma^n + \frac{1}{8}\sqrt{2}\delta^n + \frac{1}{8}\sqrt{2}n\gamma^n - \frac{1}{8}\sqrt{2}n\delta^n$$
$$= -\frac{1}{2}P_n + \frac{1}{2}nP_n = \frac{1}{2}(n-1)P_n.$$

Table 1 provides the values of w_n for $1 \leq n \leq 20$.

n	w_n	n	w_n	n	w_n	n	w_n
1	0	6	175	11	28705	16	3531240
2	1	7	507	12	76230	17	9093512
3	5	8	1428	13	200766	18	23325785
4	18	9	3940	14	525083	19	59625981
5	58	10	10701	15	1365175	20	151947066

Table 1

The results in Table 1 suggest the following.

Theorem 5.
(a) For $n \geq 1, w_{3n}$ is divisible by 5.
(b) For $n \geq 0, w_{3n+1}$ is divisible by 3.
(c) For $n \geq 0, w_{4n+2}$ and w_{4n+3} are odd, and w_{4n+4} and w_{4n+5} are even.

Proof: (a) First we observe that for $n \geq 1, P_{3n}$ is divisible by 5. This follows inductively since $P_3 = 5$ and because $P_{3(n+1)} = P_{3n+3} = 2P_{3n+2} + P_{3n+1} = 2(2P_{3n+1} + P_{3n}) + P_{3n+1} = 5P_{3n+1} + 2P_{3n}$.

(i) When n is odd, then $3n-1$ is even and $\frac{1}{2}(3n-1)$ is an integer. Then since $w_{3n} = \frac{1}{2}(n-1)P_{3n}$, the result follows because 5 divides P_{3n}.

(ii) Note that for $n \geq 0, n$ and P_n have the same parity. This follows from the recursive definition $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$. So if n is even, then P_{3n} is even, and since 5 divides P_{3n}, it follows that 10 divides P_{3n}, because $gcd(2,5) = 1$. Consequently, $P_{3n} = 10k$, for some $k \in \mathbf{Z}^+$. The result then follows as $w_{3n} = \frac{1}{2}(3n-1)P_{3n} = \frac{1}{2}(3n-1)(10k) = 5k(3n-1)$.

(b) For $n \geq 0$, $w_{3n+1} = \frac{1}{2}((3n+1)-1)P_{3n+1} = \frac{1}{2}(3n)P_{3n+1}$.

(i) If n is even, then $n = 2k$ for some integer $k \geq 0$. Therefore, $w_{3n+1} = \frac{1}{2}(3)(2k)P_{3n+1} = 3kP_{3n+1}$, and the result follows.

(ii) For the case where n is odd, $n = 2k+1$ for some integer $k \geq 0$, and $w_{3n+1} = \frac{1}{2}(3(2k+1))P_{3n+1} = 3kP_{3n+1} + \frac{1}{2}(3)P_{3n+1}$. Since n is odd, $3n+1$ is even and so is P_{3n+1}. Consequently, w_{3n+1} is divisible by 3.

(c) Note that $2P_nQ_n = 2\,\frac{\gamma^n-\delta^n}{\gamma-\delta}\left(\frac{1}{2}\right)(\gamma^n+\delta^n) = \frac{\gamma^{2n}-\delta^{2n}}{\gamma-\delta} = P_{2n}$. Also, recall that P_n is odd when n is odd, and that Q_n is odd for all $n \geq 0$.

(i) For $n \geq 0$, $((2+4n)-1)$ is odd and $\frac{1}{2}P_{4n+2} = \frac{1}{2}(2P_{2n+1}Q_{2n+1}) = P_{2n+1}Q_{2n+1}$ is also odd. Consequently, $w_{4n+2} = \frac{1}{2}((2+4n)-1)P_{4n+2}$ is odd.

(ii) For $n \geq 0, w_{4n+3} = \frac{1}{2}((4n+3)-1)P_{4n+3} = (2n+1)P_{4n+3}$, which is odd.

(iii) For $n \geq 0, w_{4n+4} = \frac{1}{2}((4n+4)-1)P_{4n+4} = (4n+3)\frac{1}{2}(2P_{2n+2}Q_{2n+2}) = (4n+3)P_{2n+2}Q_{2n+2}$. The result follows because P_{2n+2} is even.

(iv) Finally, for $n \geq 0, w_{4n+5} = \frac{1}{2}((4n+5)-1)P_{4n+5} = 2(n+1)P_{4n+5}$, which is even.

Now let we_n count the number of ones that occur in even positions for the a_n strings. As in the case for the zeros, for $i = 1, 2, 3$, let $we_n^{(i)}$ count the number of ones that occur in even positions for the $a_n^{(i)}$ strings that end with i. We find that

$$\begin{aligned} we_n &= 2we_{n-1}^{(0)} + \frac{1}{2}\left[1+(-1)^n\right]a_{n-1}^{(0)} \\ &+ 3we_{n-1}^{(1)} + \frac{1}{2}\left[1+(-1)^n\right]a_{n-1}^{(1)} \\ &+ 2we_{n-1}^{(2)} + \frac{1}{2}\left[1+(-1)^n\right]a_{n-1}^{(2)}, \text{ where} \\ we_{n-1}^{(1)} &= we_{n-2} + \frac{1}{2}\left[1+(-1)^{n-1}\right]a_{n-1}^{(1)} \\ &= we_{n-2} + \frac{1}{2}\left[1+(-1)^{n-1}\right]a_{n-2}. \end{aligned}$$

Consequently,

$$\begin{aligned} we_n &= 2we_{n-1} + we_{n-2} + \frac{1}{2}\left[1+(-1)^n\right]a_{n-1} + \frac{1}{2}\left[1+(-1)^{n-1}\right]a_{n-2} \\ &= 2we_{n-1} + we_{n-2} + \frac{1}{2}\left[P_{n-1}+P_{n-2}\right] + \frac{1}{2}(-1)^n\left[P_{n-1}-P_{n-2}\right] \\ &= 2we_{n-1} + we_{n-2} + \frac{1}{2}Q_{n-1} + \frac{1}{2}(-1)^nQ_{n-2} \\ &= 2we_{n-1} + we_{n-2} + \frac{1}{2}Q_{n-1} + \frac{1}{2}(-1)^{n-2}Q_{n-2}. \end{aligned}$$

If we now let sum_n equal the sum of the na_n entries in the a_n strings, then we arrive at the following result.

To determine A we substitute $we_n = An\gamma^n$ into the recurrence relation $we_n = 2we_{n-1} + we_{n-2} + \frac{1}{4}\gamma^{n-1}$. This results in $A(2\gamma+2) = \frac{1}{4}\gamma$ from which it follows that $A = \frac{\sqrt{2}}{16}$, and then a similar calculation yields that $B = -\frac{\sqrt{2}}{16}$. Turning to C, we now substitute $we_n = C(-\gamma)^n$ into the recurrence relation $we_n = 2we_{n-1} + we_{n-2} + \frac{1}{4}(-\gamma)^{n-2}$. This results in $C\gamma^2 = -2C\gamma + C + \frac{1}{4}$, and so $C = -\frac{1}{16} + \frac{1}{16}\sqrt{2}$, and in a similar way we learn that $D = -\frac{1}{16} - \frac{1}{16}\sqrt{2}$. Finally, from the initial conditions $we_1 = 0$ and $we_2 = 1$, it follows that $c_1 = \frac{1}{16} - \frac{1}{16}\sqrt{2}$ and $c_2 = \frac{1}{16} + \frac{1}{16}\sqrt{2}$. This leads to the following.

Theorem 6. For $n \geq 1$,

$$\begin{aligned} we_n = &\left(\frac{1}{16} - \frac{1}{16}\sqrt{2}\right)\gamma^n + \left(\frac{1}{16} + \frac{1}{16}\sqrt{2}\right)\delta^n + \frac{\sqrt{2}}{16}n\gamma^n - \frac{\sqrt{2}}{16}n\delta^n \\ &+ \left(-\frac{1}{16} + \frac{1}{16}\sqrt{2}\right)(-\gamma)^n + \left(-\frac{1}{16} - \frac{1}{16}\sqrt{2}\right)(-\delta)^n \\ = &\frac{1}{4}(n-1)P_n + \frac{1}{8}Q_n + \frac{1}{4}(-1)^n P_n - \frac{1}{8}(-1)^n Q_n \\ = &\begin{cases} \frac{1}{4}(n-2)P_n + \frac{1}{4}Q_n, & n \text{ odd} \\ \frac{1}{4}nP_n, & n \text{ even.} \end{cases} \end{aligned}$$

If we now let wo_n count the number of ones that occur among the odd positions for the a_n strings, since $w_n = we_n + wo_n$, it follows that

$$\begin{aligned} wo_n &= w_n - we_n \\ &= \begin{cases} \frac{1}{4}(nP_n - Q_n), & n \text{ odd} \\ \frac{1}{4}(n-2)P_n, & n \text{ even.} \end{cases} \end{aligned}$$

2.3 If we let t_n count the number of twos that appear among the a_n strings of length n, then as $z_n + w_n + t_n = na_n$, this next result follows.

Theorem 7. For $n \geq 1$,

$$\begin{aligned} t_n &= nP_n - z_n - w_n \\ &= nP_n - \left(\frac{1}{4} + \frac{3}{4}P_n - \frac{1}{4} + \frac{1}{4}nP_n\right) - \frac{1}{2}(n-1)P_n \\ &= \frac{1}{4}\left[(n-1)P_n - Q_n + 1\right]. \end{aligned}$$

Here the form of the solution is given by

$$we_n = c_1\gamma^n + c_2\delta^n + An\gamma^n + Bn\delta^n + C(-\gamma)^n + D(-\delta)^n.$$

Corollary 8. For $n \geq 1$,

$$\begin{aligned} sum_n &= w_n + 2t_n = \frac{1}{2}(n-1)P_n + 2\left(\frac{1}{4}\right)\left[(n-1)P_n - Q_n + 1\right] \\ &= (n-1)P_n - \frac{1}{2}(Q_n - 1). \end{aligned}$$

Now let te_n count the number of times a two appears in an even position among the a_n strings. Then, as we did for ze_n and we_n, we find that

$$\begin{aligned} te_n &= 2te_{n-1} + te_{n-2} + \frac{1}{2}\left[a_{n-1}^{(1)} + a_{n-1}^{(2)}\right] \\ &= 2te_{n-1} + te_{n-2} + \frac{1}{4}\left[Q_{n-1} - 1\right] + \frac{1}{4}(-1)^n\left[Q_{n-1} - 1\right] \\ &= 2te_{n-1} + te_{n-2} + \frac{1}{8}\left(\gamma^{n-1} + \delta^{n-1}\right) - \frac{1}{4} \\ &\quad - \frac{1}{8}\left((-\gamma)^{n-1} + (-\delta)^{n-1}\right) - \frac{1}{4}(-1)^n \end{aligned}$$

So the form of the solution is given by

$$te_n = c_1\gamma^n + c_2\delta^n + A + B(-1)^n + Cn\gamma^n + Dn\delta^n + E(-\gamma)^n + F(-\delta)^n.$$

As we have done previously, we determine that $A = \frac{1}{8}, B = -\frac{1}{8}, C = \frac{1}{32}\sqrt{2}, D = -\frac{1}{32}\sqrt{2}, E = \frac{1}{32}$, and $F = \frac{1}{32}$. Then from the initial conditions $te_1 = 0$ and $te_2 = 0$, we learn that $c_1 = -\frac{1}{32} - \frac{1}{16}\sqrt{2}$ and that $c_2 = -\frac{1}{32} + \frac{1}{16}\sqrt{2}$. Consequently, we arrive at the following result.

Theorem 9. For $n \geq 1$,

$$\begin{aligned} te_n &= \frac{1}{8} - \frac{1}{8}(-1)^n - \frac{1}{16}Q_n + \frac{1}{16}(-1)^n Q_n + \frac{1}{8}(n-2)P_n \\ &= \begin{cases} \frac{1}{8}(n-2)P_n, & n \text{ even} \\ \frac{1}{4} - \frac{1}{8}Q_n + \frac{1}{8}(n-2)P_n, & n \text{ odd.} \end{cases} \end{aligned}$$

Note that for a string of length n there are $\lfloor \frac{n}{2} \rfloor$ even positions in the string. Consequently, the preceding theorem can also be obtained by observing that $te_n = \lfloor \frac{n}{2} \rfloor a_n - ze_n - we_n = \lfloor \frac{n}{2} \rfloor P_n - ze_n - we_n$.

If we now let to_n count the number of twos that occur among the odd positions for the a_n strings, since $t_n = te_n + to_n$, it follows that

$$
\begin{aligned}
to_n &= t_n - te_n \\
&= \begin{cases} \frac{1}{8}(nP_n - Q_n), & n \text{ odd} \\ \frac{1}{8}(nP_n - 2Q_n + 2), & n \text{ even.} \end{cases}
\end{aligned}
$$

2.4 In this final subsection we want to derive summation formulas for the number of zeros, ones, and twos that appear among all the strings of length n for $l \le n \le u$. In order to do so, we need the results in the following lemma.

Lemma 10. For $n \ge 1$,

$$(a)\ \sum_{i=1}^{n} P_i = \frac{Q_{n+1}-1}{2} \qquad (b)\ \sum_{i=1}^{n} Q_i = P_{n+1} - 1$$

$$(c)\ \sum_{i=1}^{n} iP_i = \frac{n}{2}Q_{n+1} - \frac{1}{2}P_{n+1} + \frac{1}{2}$$

Proof:

$$
\begin{aligned}
(a)\ \sum_{i=1}^{n} P_i &= \sum_{i=1}^{n} \frac{\gamma^i - \delta^i}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}\left(\sum_{i=1}^{n}\gamma^i - \sum_{i=1}^{n}\delta^i\right) \\
&= \frac{1}{2\sqrt{2}}\left[\left(\frac{\gamma^{n+1}-1}{\gamma-1} - 1\right) - \left(\frac{\delta^{n+1}-1}{\delta-1} - 1\right)\right] \\
&= \frac{1}{2\sqrt{2}}\left(\frac{\gamma^{n+1}+\delta^{n+1}}{\sqrt{2}} - \sqrt{2}\right) \\
&= \frac{1}{4}(\gamma^{n+1}+\delta^{n+1}) - \frac{1}{2} = \frac{Q_{n+1}-1}{2}.
\end{aligned}
$$

$$
\begin{aligned}
(b)\ &\sum_{i=1}^{n} Q_i = \sum_{i=0}^{n} Q_i - Q_0 = \sum_{i=0}^{n} \frac{1}{2}(\gamma^i + \delta^i) - 1 \\
&= \frac{1}{2}\left[\frac{\gamma^{n+1}-1}{\gamma-1} + \frac{\delta^{n+1}-1}{\delta-1}\right] - 1 = \frac{1}{2}\left[\frac{\gamma^{n+1}-1}{\sqrt{2}} - \frac{\delta^{n+1}-1}{\sqrt{2}}\right] - 1 \\
&= \frac{1}{2\sqrt{2}}(\gamma^{n+1} - \delta^{n+1}) - 1 = P_{n+1} - 1
\end{aligned}
$$

(c) The proof here follows that given on P. 508 of the text by Thomas Koshy [3].

$$
\begin{aligned}
\sum_{i+1}^{n} iP_i &= P_1 + 2P_2 + 3P_3 + \cdots + nP_n \\
&= \sum_{i=1}^{n} P_i + \sum_{i=2}^{n} P_i + \sum_{i=3}^{n} P_i + \cdots + \sum_{i=n}^{n} P_i
\end{aligned}
$$

If we now let $A_n = \sum_{i=1}^{n} P_i$, then

$$
\begin{aligned}
&\sum_{i=1}^{n} P_i + \sum_{i=2}^{n} P_i + \sum_{i=3}^{n} P_i + \cdots + \sum_{i=n}^{n} P_i \\
&= A_n + (A_n - A_1) + (A_n - A_2) + \cdots + (A_n - A_{n-1}) \\
&= nA_n - \sum_{i=1}^{n-1} A_i = n\left(\frac{1}{2}\right)(Q_{n+1}-1) - \sum_{i=1}^{n-1}\left[\frac{Q_{i+1}-1}{2}\right] \\
&= \frac{n}{2}(Q_{n+1}-1) - \frac{1}{2}\left[\sum_{i=1}^{n-1} Q_{i+1} - (n-1)\right] \\
&= \frac{n}{2}(Q_{n+1}-1) - \frac{1}{2}[(P_{n+1}-1) - 1 - (n-1)] \\
&= \frac{n}{2}Q_{n+1} - \frac{1}{2}P_{n+1} + \frac{1}{2}
\end{aligned}
$$

These results lead us to the following.

Theorem 11. For $l \le i \le u$,

$$(a) \sum_{i=l}^{u} z_i = \frac{1}{8}[(u+3)Q_{u+1} - (l+2)Q_l + P_{u+1} - P_l - 2(u-l+1)]$$

$$(b) \sum_{i=l}^{u} w_i = \frac{1}{4}[(u-1)Q_{u+1} - (l-2)Q_l - (P_{u+1} - P_l)]$$

$$(c) \sum_{i=l}^{u} t_i = \frac{1}{8}[(u-1)Q_{u+1} - (l-2)Q_l - 3P_{u+1} + 3P_l + 2(u-l+1)].$$

Proof: We prove part (a). The proofs for parts (b) and (c) are similar.

$$\begin{aligned}
\sum_{i=l}^{u} z_i &= \sum_{i=l}^{u}\left(\frac{1}{4}Q_i + \frac{3}{4}P_i - \frac{1}{4} + \frac{1}{4}iP_i\right)\\
&= \frac{1}{4}\left[P_{u+1} - 1 - (P_{(l-1)+1} - 1)\right]\\
&+ \frac{3}{4}\left[\frac{1}{2}(Q_{u+1} - 1) - \frac{1}{2}(Q_{(l-1)+1} - 1)\right] - \frac{1}{4}(u - l + 1)\\
&+ \frac{1}{4}\left[\frac{u}{2}Q_{u+1} - \frac{1}{2}P_{u+1} + \frac{1}{2} - \left(\frac{l-1}{2}Q_{(l-1)+1} - \frac{1}{2}P_{(l-1)+1} + \frac{1}{2}\right)\right]\\
&= \frac{1}{4}(P_{u+1} - P_l) + \frac{3}{4}\left[\frac{1}{2}(Q_{u+1} - Q_l)\right] - \frac{1}{4}(u - l + 1)\\
&+ \frac{1}{4}\left(\frac{u}{2}Q_{u+1} - \frac{l-1}{2}Q_l - \frac{1}{2}P_{u+1} + \frac{1}{2}P_l\right)\\
&= \frac{1}{8}\left[(u+3)Q_{u+1} - (l+2)Q_l + P_{u+1} - P_l - 2(u - l + 1)\right]
\end{aligned}$$

3. Levels, Rises, and Descents

3.1 For $n \geq 1$, let lev_n count the number of *levels* $(00, 11, 22)$ that occur among the $a_n (= P_n)$ strings of length n. We find, for example that, $lev_1 = 0$, $lev_2 = 1$, $lev_3 = 4$ (two in 000, one in 001, and one in 011), $lev_4 = 15$, and $lev_5 = 48$. Then for $i \in \{0, 1, 2\}$, let $lev_n^{(i)}$ count the number of levels that occur among the a_n strings that end with i. We find for $n \geq 2$ that

$$\begin{aligned}
lev_n &= \left[lev_{n-1}^{(0)} + lev_{n-1}^{(1)} + a_{n-1}^{(0)}\right]\\
&+ \left[lev_{n-1}^{(0)} + lev_{n-1}^{(1)} + lev_{n-1}^{(2)} + a_{n-1}^{(1)}\right] + \left[lev_{n-1}^{(1)} + lev_{n-1}^{(2)} + a_{n-1}^{(2)}\right]\\
&= 2\, lev_{n-1} + a_{n-1} + lev_{n-1}^{(1)},
\end{aligned}$$

where $lev_n^{(1)} = lev_{n-1} + a_{n-1}^{(1)} = lev_{n-1} + a_{n-2}$. So

$$\begin{aligned}
lev_n &= 2\, lev_{n-1} + lev_{n-2} + a_{n-1} + a_{n-3}\\
&= 2\, lev_{n-1} + lev_{n-2} + \frac{\gamma^{n-1} - \delta^{n-1}}{\gamma - \delta} + \frac{\gamma^{n-3} - \delta^{n-3}}{\gamma - \delta}
\end{aligned}$$

for which the solution has the form $c_1\gamma^n + c_2\delta^n + An\gamma^n + Bn\delta^n$.

To determine A we substitute $lev_n = An\gamma^n$ into the relation $lev_n = 2\, lev_{n-1} + lev_{n-2} + \frac{1}{2\sqrt{2}}\gamma^{n-1} + \frac{1}{2\sqrt{2}}\gamma^{n-3}$ and find that $A = \frac{1}{2} - \frac{1}{4}\sqrt{2}$. Then

a similar calculation provides $B = \frac{1}{2} + \frac{1}{4}\sqrt{2}$. So $lev_n = c_1\gamma^n + c_2\delta^n + \left(\frac{1}{2} - \frac{1}{4}\sqrt{2}\right)n\gamma^n + \left(\frac{1}{2} + \frac{1}{4}\sqrt{2}\right)n\delta^n$. From the initial conditions $lev_1 = 0$ and $lev_2 = 1$, we learn that $c_1 = -\frac{1}{2} + \frac{1}{4}\sqrt{2}$ and $c_2 = -\frac{1}{2} - \frac{1}{4}\sqrt{2}$. Consequently, we have the following result.

Theorem 12. For $n \geq 1$,

$$\begin{aligned}
lev_n &= \left(-\frac{1}{2} + \frac{1}{4}\sqrt{2}\right)\gamma^n + \left(-\frac{1}{2} - \frac{1}{4}\sqrt{2}\right)\delta^n\\
&+ \left(\frac{1}{2} - \frac{1}{4}\sqrt{2}\right)n\gamma^n + \left(\frac{1}{2} + \frac{1}{4}\sqrt{2}\right)n\delta^n\\
&= -Q_n + P_n + nQ_n - nP_n = (n-1)(Q_n - P_n).
\end{aligned}$$

Note that when n is odd, $(n-1)$ is even and both P_n and Q_n are odd. As a result lev_n is divisible by 4 for n odd.

3.2 For $n \geq 1$, now let $rise_n$ count the number of *rises* (01 or 12) that occur among the a_n strings of length n. We find, for example, that $rise_1 = 0$, $rise_2 = 1$, $rise_3 = 5$ (one in 001, one in 010, one in 011, and two in 012), and $rise_4 = 16$. Then for $i \in \{0, 1, 2\}$, let $rise_n^{(i)}$ count the number of rises that occur among the a_n strings of length n that end with i. Similar to the way we dealt with the levels, we now find that, for $n \geq 2$,

$$\begin{aligned}
rise_n &= \left[rise_{n-1}^{(0)} + rise_{n-1}^{(1)}\right]\\
&+ \left[rise_{n-1}^{(0)} + rise_{n-1}^{(1)} + rise_{n-1}^{(2)} + a_{n-1}^{(0)}\right]\\
&+ \left[rise_{n-1}^{(1)} + rise_{n-1}^{(2)} + a_{n-1}^{(1)}\right]\\
&= 2\, rise_{n-1} + rise_{n-1}^{(1)} + a_{n-1}^{(0)} + a_{n-1}^{(1)}\\
&= 2\, rise_{n-1} + rise_{n-2} + a_{n-2}^{(0)} + a_{n-1}^{(0)} + a_{n-1}^{(1)}\\
&= 2\, rise_{n-1} + rise_{n-2} + \left(\frac{1}{2} + P_{n-2} - \frac{1}{2}Q_{n-2}\right)\\
&+ \left(\frac{1}{2} + P_{n-1} - \frac{1}{2}Q_{n-1}\right) + P_{n-2},
\end{aligned}$$

which simplifies to $rise_n = 2\, rise_{n-1} + rise_{n-2} + 1 + 2\,P_{n-2}$, since $Q_{n-1} - P_{n-1} = P_{n-2}$. So the form of the solution is

$$rise_n = c_1\gamma^n + c_2\delta^n + A + Bn\gamma^n + Cn\delta^n.$$

To determine A we substitute $rise_n = A$ into the relation $rise_n = 2\,rise_{n-1} + rise_{n-2} + 1$ and find that $A = -\frac{1}{2}$. Continuing for B, we substitute $rise_n = Bn\gamma^n$ into the relation $rise_n = 2\,rise_{n-1} + rise_{n-2} + 2\left(\frac{\gamma^{n-2}}{2\sqrt{2}}\right)$, which reduces to $2B\gamma + 2B = \frac{1}{\sqrt{2}}$, and so $B = -\frac{1}{4} + \frac{1}{4}\sqrt{2}$. Then a comparable calculation yields $C = -\frac{1}{4} - \frac{1}{4}\sqrt{2}$.

From the initial conditions $rise_1 = 0$ and $rise_2 = 1$, we learn that $c_1 = \frac{1}{4} - \frac{1}{8}\sqrt{2}$ and $c_2 = \frac{1}{4} + \frac{1}{8}\sqrt{2}$. Therefore,

$$\begin{aligned} rise_n = &\left(\frac{1}{4} - \frac{1}{8}\sqrt{2}\right)\gamma^n + \left(\frac{1}{4} + \frac{1}{8}\sqrt{2}\right)\delta^n - \frac{1}{2} \\ &+ \left(-\frac{1}{4} + \frac{1}{4}\sqrt{2}\right) n\gamma^n + \left(-\frac{1}{4} - \frac{1}{4}\sqrt{2}\right) n\delta^n, \end{aligned}$$

and this simplifies to the next result.

Theorem 13. For $n \geq 1$,

$$rise_n = \frac{1}{2}Q_n - \frac{1}{2}P_n - \frac{1}{2} - \frac{1}{2}nQ_n + nP_n = \frac{1}{2}\left(P_{n-1} + nQ_{n-1} - 1\right).$$

Now let dri_n count the number of double rises - that is, the substrings '012' - that occur among the a_n strings of length n. We find, for example, that $dri_1 = 0$, $dri_2 = 0$, $dri_3 = 1$, $dri_4 = 3$, and $dri_5 = 9$. As we've done similarly in the past, for $i \in \{0, 1, 2\}$, we let $dri^{(i)}$ count the number of double rises that occur for the a_n strings that end with i. In addition, now we'll let $a_n^{(01)}$ count the number of strings of length n that end with 01. Noting that $a_n^{(01)} = a_{n-1}^{(0)}$ and that $dri_n^{(1)} = dri_{n-1}$, we are led to

$$\begin{aligned} dri_n &= 2\,dri_{n-1}^{(0)} + 3\,dri_{n-1}^{(1)} + a_{n-1}^{(01)} + 2\,dri_{n-1}^{(2)} \\ &= 2\,dri_{n-1} + dri_{n-1}^{(1)} + a_{n-1}^{(01)} \\ &= 2\,dri_{n-1} + dri_{n-2} + a_{n-2}^{(0)} \\ &= 2\,dri_{n-1} + dri_{n-2} + \frac{1}{2} + P_{n-2} - \frac{1}{2}Q_{n-2}. \end{aligned}$$

Upon solving this recurrence relation, with the initial conditions $dri_1 = 0$ and $dri_2 = 0$, we reach the following result.

Theorem 14. For $n \geq 1$,

$$dri_n = -\frac{1}{4} + \left(\frac{3}{4}n - \frac{1}{4}\right)P_n + \left(-\frac{1}{2}n + \frac{1}{4}\right)Q_n.$$

3.3 At this point we consider the descents (10 or 21) that occur among the a_n strings of length n. If we let $desc_n$ count these length two substrings, we now realize that since

$$desc_n + lev_n + rise_n = (n-1)a_n,$$

upon simplifying, we obtain the next result.

Theorem 15. For $n \geq 1$,

$$desc_n = \frac{1}{2}\left[(n-1)Q_{n-1} - P_n + 1\right].$$

Finally, we let $ddesc_n$ count the number of double descents - that is, the substrings '210' - that occur among the a_n strings of length n. We let $ddesc^{(i)}$ count the number of double descents that occur for the a_n strings that end with i, for $i \in \{0, 1, 2\}$. We also let $a_n^{(21)}$ count the number of strings of length n that end with 21, and note that $a_n^{(21)} = a_{n-1}^{(2)}$. This leads us to

$$\begin{aligned} ddesc_n &= 2\,ddesc_{n1}^{(0)} + a_{n-1}^{(21)} + 3\,ddesc_{n-1}^{(1)} + 2\,ddesc_{n-1}^{(2)} \\ &= 2\,ddesc_{n-1} + ddesc_{n-1}^{(1)} + a_{n-2}^{(2)} \\ &= 2\,ddesc_{n-1} + ddesc_{n-2} + \frac{1}{2}Q_{n-2} - P_{n-3} - \frac{1}{2}. \end{aligned}$$

With the initial conditions $ddesc_1 = 0$ and $ddesc_2 = 0$, we find the following solution for the above recurrence relation.

Theorem 16. For $n \geq 1$,

$$ddesc_n = \frac{1}{4}\left[(3n-5)P_n + (3-2n)Q_n + 1\right].$$

4. Runs and Isolated Entries

4.1 For $n \geq 1$, let $runs_n$ count the number of runs that occur among the a_n strings of length n. In general, a *run* is a consecutive list of identical entries that are preceded and followed by different entries or no entries at all. For example, $runs_1 = 1$ and $runs_2 = 3$. When $n = 3$, there are 11 runs: one for 000; two for 001 - 00 and 1; three for 010 - 0, 1, and 0; two for 011 - 0 and 11; and, three for 012 - 0, 1, and 2. Similar to what we've done previously, we let $runs_n^{(i)}$ count the number of runs that occur for the strings of length n that end with an i, for $i \in \{0, 1, 2\}$. We find that for $n \geq 3$,

$$
\begin{aligned}
runs_n = &\left[runs^{(0)}_{n-1} + runs^{(1)}_{n-1} + a^{(1)}_{n-1}\right]\\
&+ \left[runs^{(0)}_{n-1} + runs^{(1)}_{n-1} + runs^{(2)}_{n-1} + a^{(0)}_{n-1} + a^{(2)}_{n-1}\right]\\
&+ \left[runs^{(1)}_{n-1} + runs^{(2)}_{n-1} + a^{(1)}_{n-1}\right]\\
&= 2\,runs_{n-1} + runs^{(1)}_{n-1} + a_{n-1} + a^{(1)}_{n-1}\\
&= 2\,runs_{n-1} + runs_{n-2} + a^{(0)}_{n-2} + a^{(2)}_{n-2} + a_{n-1} + a^{(1)}_{n-1}\\
&= 2\,runs_{n-1} + runs_{n-2} + P_{n-1} + 2\,P_{n-2} - P_{n-3}.
\end{aligned}
$$

So the form of the solution is given by

$$runs_n = c_1\gamma^n + c_2\delta^n + An\gamma^n + Bn\delta^n.$$

To determine A, we substitute $runs_n = An\gamma^n$ into the relation $runs_n = 2\,runs_{n-1} + runs_{n-2} + \frac{1}{2\sqrt{2}}\gamma^{n-1} + \frac{1}{\sqrt{2}}\gamma^{n-2} - \frac{1}{2\sqrt{2}}\gamma^{n-3}$. This simplifies to $A(2\gamma^2 + 2\gamma) = \frac{1}{2\sqrt{2}}\gamma^2 + \frac{1}{\sqrt{2}}\gamma - \frac{1}{2\sqrt{2}}$, from which it follows that $A = -\frac{1}{2} + \frac{\sqrt{2}}{2}$. A comparable calculation provides $B = -\frac{1}{2} - \frac{\sqrt{2}}{2}$. Then from the initial conditions $runs_1 = 1$ and $runs_2 = 3$ we find that $c_1 = \frac{1}{2} - \frac{1}{4}\sqrt{2}$ and $c_2 = \frac{1}{2} + \frac{1}{4}\sqrt{2}$. This now provides the following result.

Theorem 17. For $n \geq 1$,

$$
\begin{aligned}
runs_n = &\left(\frac{1}{2} - \frac{1}{4}\sqrt{2}\right)\gamma^n + \left(\frac{1}{2} + \frac{1}{4}\sqrt{2}\right)\delta^n\\
&+ \left(-\frac{1}{2} + \frac{\sqrt{2}}{2}\right)n\gamma^n + \left(-\frac{1}{2} - \frac{\sqrt{2}}{2}\right)n\delta^n\\
= &\ (1-n)Q_n + (2n-1)P_n.
\end{aligned}
$$

4.2 Now we turn our attention to *isolated entries*, or runs of length 1. Here, for a specified entry x, where $x \in \{0, 1, 2\}$, there is no occurrence of x to the left or right of the specified x. For example, for the string 0012110 there are three isolated entries - namely, the first 1 (from the left), the 2 and the final 0.

For $n \geq 1$, let iso_n count the number of isolated entries (or runs of length 1) that occur among the a_n strings. Then let $iso^{(0)}_n$ count the number of isolated entries for the strings of length n ending with a 0. We define $iso^{(1)}_n$ and $iso^{(2)}_n$ similarly. When $n = 3$, for example, we find that $iso_3 = 8$, $iso^{(0)}_3 = 3$, $iso^{(1)}_3 = 2$, and $iso^{(2)}_3 = 3$.

In addition, we now define $a^{(i0)}_n$ to be the number of strings of length n which end with an isolated 0. Similarly, we define $a^{(i1)}_n$ and $a^{(i2)}_n$ for strings of length n ending with an isolated 1 or 2, respectively. Lastly, we use $a^{(iso)}_n$ to count the number of strings of length n that end with an isolated entry.

For $n \geq 2$,

$$
\begin{aligned}
iso_n = &\left[iso^{(0)}_{n-1} - a^{(i0)}_{n-1} + iso^{(1)}_{n-1} + a^{(1)}_{n-1}\right]\\
&+ \left[iso^{(0)}_{n-1} + a^{(0)}_{n-1} + iso^{(1)}_{n-1} - a^{(i1)}_{n-1} + iso^{(2)}_{n-1} + a^{(2)}_{n-1}\right]\\
&+ \left[iso^{(1)}_{n-1} + a^{(1)}_{n-1} + iso^{(2)}_{n-1} - a^{(i2)}_{n-1}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
a^{(i0)}_n &= a^{(1)}_{n-1},\quad a^{(i1)}_n = a^{(0)}_{n-1} + a^{(2)}_{n-1},\quad a^{(i2)}_n = a^{(1)}_{n-1},\ \text{and}\\
a^{(iso)}_n &= a^{(i0)}_n + a^{(i1)}_n + a^{(i2)}_n = a^{(1)}_{n-1} + a^{(0)}_{n-1} + a^{(2)}_{n-1} + a^{(1)}_{n-1}\\
&= a_{n-1} + a^{(1)}_{n-1} = a_{n-1} + a_{n-2}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
iso_n &= 2\,iso_{n-1} + iso^{(1)}_{n-1} - a^{(i)}_{n-1} + 2\,a^{(1)}_{n-1} + a^{(2)}_{n-1} + a^{(0)}_{n-1}\\
&= 2\,iso_{n-1} + iso^{(1)}_{n-1} - a_{n-2} - a_{n-3} + 2\,a_{n-2} + a^{(2)}_{n-1} + a^{(0)}_{n-1}\\
&= 2\,iso_{n-1} + iso^{(1)}_{n-1} + a_{n-2} - a_{n-3} + a^{(2)}_{n-1} + a^{(0)}_{n-1},\ \text{and}\\
iso^{(1)}_n &= iso_{n-1} + a_{n-1} - a^{(1)}_{n-1} - a^{(i1)}_{n-1}\\
&= iso_{n-1} + a_{n-1} - a^{(1)}_{n-1} - a^{(0)}_{n-2} - a^{(2)}_{n-2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
iso_n = &\ 2\,iso_{n-1} + (iso_{n-2} + a_{n-2} - a^{(1)}_{n-2} - a^{(0)}_{n-3} - a^{(2)}_{n-3})\\
&+ a_{n-2} - a_{n-3} + a^{(2)}_{n-1} + a^{(0)}_{n-1}\\
= &\ 2\,iso_{n-1} + iso_{n-2} + (a^{(2)}_{n-1} + a^{(0)}_{n-1})\\
&+ (2\,a_{n-2} - a^{(1)}_{n-2}) - (a_{n-3} + a^{(0)}_{n-3} + a^{(2)}_{n-3})\\
= &\ 2\,iso_{n-1} + iso_{n-2} + Q_{n-2} + (P_{n-2} + Q_{n-3}) - (2\,P_{n-3} - P_{n-4}).
\end{aligned}
$$

Since $P_n - P_{n-1} = Q_{n-1}$ and $Q_n = 2\,Q_{n-1} + Q_{n-2}$,

$$
\begin{aligned}
\text{for } n \geq 3,\ iso_n &= 2\,iso_{n-1} + iso_{n-2} + 4\,Q_{n-3}\\
&= 2\,iso_{n-1} + iso_{n-2} + 2\left(\gamma^{n-3} + \delta^{n-3}\right),
\end{aligned}
$$

for which the form of the solution is $iso_n = c_1\gamma^n + c_2\delta^n + An\gamma^n + Bn\delta^n$.

To determine A we substitute $iso_n = An\gamma^n$ into the relation $iso_n = 2\,iso_{n-1} + iso_{n-2} + 2\gamma^{n-3}$. This simplifies to $A\gamma^2 + A\gamma - 1 = 0$ and leads to $A = -2 + \frac{3}{2}\sqrt{2}$. Then a similar calculation leads to $B = -2 - \frac{3}{2}\sqrt{2}$. Using the initial conditions $iso_3 = 8$ and $iso_4 = 22$, we learn that $c_1 = 3 - 2\sqrt{2}$ and $c_2 = 3 + 2\sqrt{2}$. [Note that the (same) values for c_1 and c_2 are obtained if we use the initial conditions $iso_2 = 2$ and $iso_3 = 8$.] As a result we have the following.

Theorem 18.

$$\begin{aligned} iso_1 &= 1, \\ iso_n &= \left(3 - 2\sqrt{2}\right)\gamma^n + \left(3 + 2\sqrt{2}\right)\delta^n \\ &\quad + \left(-2 + \frac{3\sqrt{2}}{2}\right) n\gamma^n + \left(-2 - \frac{3\sqrt{2}}{2}\right) n\delta^n \\ &= 6Q_n - 8P_n - 4nQ_n + 6nP_n, \quad n \geq 2. \end{aligned}$$

Table 2 provides the values of iso_n and $iso_n\,(mod\,8)$ for $1 \leq n \leq 14$.

n	iso_n	$iso_n\,(mod\,8)$	n	iso_n	$iso_n\,(mod\,8)$
1	1	1	8	1318	6
2	2	2	9	3520	0
3	8	0	10	9314	2
4	22	6	11	24456	0
5	64	0	12	63798	6
6	178	2	13	165504	0
7	488	0	14	427282	2

Table 2

The results in Table 2 suggest the following.

Theorem 19.
(a) For n odd, $n \geq 3$, $iso_n \equiv 0\,(mod\,8)$.
(b) For n even, $n \not\equiv 0 \mod 4$, $iso_n \equiv 2\,(mod\,8)$.
(c) For n even, $n \equiv 0 \mod 4$, $iso_n \equiv 6\,(mod\,8)$.

In order to prove Theorem 19, we examine the results in the following table.

n	$6\,Q_n\,(mod\,8)$	$4\,nQ_n\,(mod\,8)$	$6\,nP_n\,(mod\,8)$	$iso_n\,(mod\,8)$
2	2	0	0	2
3	2	4	2	0
4	6	0	0	6
5	6	4	6	0
6	2	0	0	2
7	2	4	2	0
8	6	0	0	6
9	6	4	6	0
10	2	0	0	2
11	2	4	2	0
12	6	0	0	6

Table 3

Since $iso_n = 6\,Q_n - 8\,P_n - 4\,nQ_n + 6\,nP_n$ and $8\,P_n \equiv 0\,(mod\,8)$, the proof will follow by establishing the patterns suggested in columns 2, 3, and 4 of Table 3. To do so, let us start by recalling that for all n, Q_n is odd, and n and P_n have the same parity.

(Column 2): Assume the pattern in column 2 is valid for $n = 2, 3, 4, \ldots k-1$ and now consider $n = k$.

(i) If k is even and divisible by 4, then $6\,Q_{k-1} \equiv 2\,(mod\,8)$, $6\,Q_{k-2} \equiv 2\,(mod\,8)$, and $6\,Q_k = 12\,Q_{k-1} + 6\,Q_{k-2} \equiv 2(2) + 2 \equiv 6\,(mod\,8)$.

(ii) Now suppose that k is even but not divisible by 4. Then $6\,Q_{k-1} \equiv 6\,(mod\,8)$ and $6\,Q_{k-2} \equiv 6\,(mod\,8)$, so $6\,Q_k = 12\,Q_{k-1} + 6\,Q_{k-2} \equiv 2(6) + 6\,(mod\,8) \equiv 2\,(mod\,8)$.

(iii) Turning now to the case where k is odd, suppose that $k = 4t+1$ for some positive integer t. Then $6\,Q_{k-1} \equiv 6\,(mod\,8)$ and $6\,Q_{k-2} \equiv 2\,(mod\,8)$, so $6\,Q_k = 12\,Q_{k-1} + Q_{k-2} \equiv 2(6) + 2\,(mod\,8) \equiv 6\,(mod\,8)$.

(iv) Finally, suppose k is odd and $k = 4t - 1$ for some positive integer t. Then $6\,Q_{k-1} \equiv 2\,(mod\,8)$ and $6\,Q_{k-2} \equiv 6\,(mod\,8)$, and $6\,Q_k = 12\,Q_{k-1} + Q_{k-2} \equiv 2(2) + 6\,(mod\,8) \equiv 2\,(mod\,8)$.

This now establishes the pattern suggested in column 2.

(Column 3): When n is even, then nQ_n is even and $4\,nQ_n \equiv 0\,(mod\,8)$. For n odd, nQ_n is odd, since Q_n is odd for all $n \geq 1$. So $nQ_n = 2m+1$ for some integer $m \geq 0$ and $4\,nQ_n = 4(2m+1) \equiv 4\,(mod\,8)$. Consequently, the pattern suggested in column 3 continues for all $n \geq 2$.

(Column 4): (i) When n is even, then P_n is also even, and $6\,nP_n \equiv 0\,(mod\,8)$.

(ii) When n is odd, we first observe that not only is P_n odd, but $P_n \equiv 1\,(mod\,4)$. We see that this is true for $P_1 = 1$, $P_3 = 5$, $P_5 = 29$, and $P_7 = 169$. So assuming the result true for $1, 3, 5, \ldots, n-2$, we then find that $P_n = 2\,P_{n-1} + P_{n-2}$, where P_{n-1} is even. Consequently, $2\,P_{n-1} \equiv 0\,(mod\,4)$ and $P_n = 2\,P_{n-1} + P_{n-2} \equiv 1\,(mod\,4)$.

Now we consider two cases. (a) Suppose that $n = 4r+1$, where $r \geq 0$. Then $6\,nP_n = 6(4r+1)(4\,M+1) = 6(16\,rM + 4\,M + 4\,r + 1) = 96\,rM + 24\,(M+r) + 6 \equiv 6\,(mod\,8)$. (b) When $n = 4r+3$, where $r \geq 0$, then $6\,nP_n = 6(4r+3)(4\,M+1) = 6(16rM + 12\,M + 4\,r + 3) = 96\,rM + 24(3\,M + r) + 18 \equiv 2\,(mod\,8)$.

Adding across the rows for columns 2, 3, and 4 in Table 3, we now find that the pattern suggested in column 5 continues for all $n \geq 2$, and this establishes the results in Theorem 19.

5. The Sum of the a_n Strings Considered as Base 3 Integers

For $n \geq 1$, let val_n denote the sum of the a_n strings considered as base 3 integers. We find that $val_1 = 0$ and $val_2 = 1$. For $n = 3$, we have $000_3 = 0$, $001_3 = 1$, $010_3 = 3$, $011_3 = 4$, and $012_3 = 5$, so $val_3 = 0+1+3+4+5 = 13$. When $n = 4$, we find that $val_4 = 104$.

To determine a formula for val_n let us first introduce the following auxiliary variables. Similar to what we've done in previous sections, for $i \in \{0,1,2\}$, let $val_n^{(i)}$ denote the sum of the $a_n^{(i)}$ strings (that end in i) considered as base 3 integers.

For $n \geq 3$, we find that

$$\begin{aligned} val_n = &\left[3\,val_{n-1}^{(0)} + 3\,val_{n-1}^{(1)}\right] \\ &+ \left[3\,val_{n-1}^{(0)} + 3\,val_{n-1}^{(1)} + 3\,val_{n-1}^{(2)} + a_{n-1}^{(0)} + a_{n-1}^{(1)} + a_{n-1}^{(2)}\right] \\ &+ \left[3\,val_{n-1}^{(1)} + 3\,val_{n-1}^{(2)} + 2\,a_{n-1}^{(1)} + 2\,a_{n-1}^{(2)}\right], \end{aligned}$$

where, for example, the last bracketed sum accounts for when we append a 2 to each of the strings of length $n-1$ that end with a 1 or a 2. This then leads us to

$$\begin{aligned} val_n &= 2\,(3\,val_{n-1}) + 3\,val_{n-1}^{(1)} + a_{n-1} + 2\,a_{n-1}^{(1)} + 2\,a_{n-1}^{(2)} \\ &= 6\,val_{n-1} + 3\,(3\,val_{n-2} + a_{n-2}) + a_{n-1} + 2\,a_{n-2} \\ &\quad + 2\left(\frac{1}{2}Q_{n-1} - P_{n-2} - \frac{1}{2}\right) \\ &= 6\,val_{n-1} + 9\,val_{n-2} + 5\,a_{n-2} + P_{n-1} + (Q_{n-1} - 2\,P_{n-2} - 1) \\ &= 6\,val_{n-1} + 9\,val_{n-2} + 3\,P_{n-2} + P_n - 1, \end{aligned}$$

since $P_{n-1} + Q_{n-1} = P_n$.

From the homogeneous recurrence relation $val_n = 6val_{n-1} + 9val_{n-2}$ we obtain the characteristic equation $r^2 - 6\,r + 9 = 0$ and the characteristic roots $r = 3 + 3\sqrt{2} = 3(1+\sqrt{2}) = 3\gamma$ and $r = 3 - 3\sqrt{2} = 3(1-\sqrt{2}) = 3\delta$. Therefore, the form of the solution is given as $val_n = c_1(3+3\sqrt{2})^n + c_2(3-3\sqrt{2})^n + A + B\gamma^n + C\delta^n$.

To determine the nonhomogeneous part of the solution, we first substitute $val_n = A$ into the relation $val_n = 6\,val_{n-1} + 9\,val_{n-2} - 1$. This gives us the equation $A = 6\,A + 9\,A - 1$, so $A = \frac{1}{14}$. To find B, we substitute $val_n = B\gamma^n$ into the relation $val_n = 6\,val_{n-1} + 9\,val_{n-2} + \frac{3}{2\sqrt{2}}\gamma^{n-2} + \frac{1}{2\sqrt{2}}\gamma^n$. This leads to $B(\gamma^2 - 6\,\gamma - 9) = \frac{3}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\gamma^2$ and $B = -\frac{1}{8}\sqrt{2}$. Then a similar calculation provides $C = \frac{1}{8}\sqrt{2}$. So

$$val_n = c_1(3+3\sqrt{2})^n + c_2(3-3\sqrt{2})^n + \frac{1}{14} + \left(-\frac{1}{8}\sqrt{2}\right)\gamma^n + \left(\frac{1}{8}\sqrt{2}\right)\delta^n.$$

From the initial conditions $val_1 = 0$ and $val_2 = 1$ it subsequently follows that $c_1 = -\frac{1}{28} + \frac{3}{56}\sqrt{2}$ and $c_2 = -\frac{1}{28} - \frac{3}{56}\sqrt{2}$. This now provides the following result.

Theorem 20. For $n \geq 1$,

$$\begin{aligned} val_n &= \left(-\frac{1}{28} + \frac{3}{56}\sqrt{2}\right)(3+3\sqrt{2})^n + \left(-\frac{1}{28} - \frac{3}{56}\sqrt{2}\right)(3-3\sqrt{2})^n \\ &\quad + \frac{1}{14} + \left(-\frac{1}{8}\sqrt{2}\right)\gamma^n + \left(\frac{1}{8}\sqrt{2}\right)\delta^n \\ &= \left(-\frac{1}{28} + \frac{3}{56}\sqrt{2}\right)(3\gamma)^n + \left(-\frac{1}{28} - \frac{3}{56}\sqrt{2}\right)(3\delta)^n \\ &\quad + \frac{1}{14} + \left(-\frac{1}{8}\sqrt{2}\right)\gamma^n + \left(\frac{1}{8}\sqrt{2}\right)\delta^n \\ &= -\frac{1}{14}(3^n)Q_n + \frac{3}{14}(3^n)P_n + \frac{1}{14} - \frac{1}{2}P_n. \end{aligned}$$

6. Inversions

In this and the next two sections we consider our alphabet $\{0,1,2\}$ as totally ordered, with $0 < 1 < 2$. Then given a string $s_1 s_2 \ldots s_n$, found among the a_n strings of length n, an *inversion* is said to occur whenever $s_i > s_j$ for $1 \leq i < j \leq n$. If we let inv_n count the number of inversions that occur among the a_n strings of length n, we find, for example, that $inv_1 = 0$, $inv_2 = 0$. When $n = 3$ we have $inv_3 = 1$ and this accounts for the unique inversion 10 that occurs in the string 010. For $n = 4$, we find that $inv_4 = 7$: one inversion in 0010 - namely, 10; two inversions in 0100 - one for the 1 in position 2 and the 0 in position

3, and the other for the 1 in position 2 and the 0 in position 4; one inversion in 0101 - namely, 10; two inversions in 0110 - one for the 1 in position 2 and the 0 in position 4, and the other for the 1 in position 3 and the 0 in position 4; and, finally, one in 0121 from the 2 in positon 3 and the 1 in position 4. We also find that $inv_5 = 36$ and $inv_6 = 153$.

In order to determine a formula for inv_n, we once again use auxiliary variables and, for $i \in \{0,1,2\}$ we let $inv_n^{(i)}$ count the number of inversions that occur among the $a_n^{(i)}$ strings that end with an i. Then for $n \geq 2$, we find that

$$\begin{aligned} inv_n = &\left[inv_{n-1}^{(0)} + inv_{n-1}^{(1)} + t_{n-1}^{(0)} + w_{n-1}^{(0)} + t_{n-1}^{(1)} + w_{n-1}^{(1)}\right] \\ &+ \left[inv_{n-1} + t_{n-1}\right] \\ &+ \left[inv_{n-1}^{(1)} + inv_{n-1}^{(2)}\right]. \end{aligned}$$

Here the first bracketed summand arises from when we append 0 to strings of length $n-1$ that end with a 0 or a 1. Within this summand, the term $w_{n-1}^{(1)}$, for example, accounts for all the 1's that appear within the $a_{n-1}^{(1)}$ strings that end with a 1. For each of these 1's provides an inversion with the 0 appended at position n. The last bracketed summand accounts for the inversions when we append 2 to the strings of length $n-1$ that end with a 1 or a 2. In this case no new inversions arise.

In order to solve the above recurrence relation for inv_n, we will need the following comments and two lemmas. Let us start by first observing that

$$\begin{aligned} inv_{n-1}^{(1)} &= inv_{n-2} + t_{n-2}, \\ w_n^{(1)} &= w_{n-1} + a_{n-1}, \quad \text{and} \\ t_{n-1}^{(1)} &= t_{n-2}, \end{aligned}$$

where $w_n^{(1)}$ counts the number of 1's that appear among the a_n strings that end with a 1, and $t_{n-1}^{(1)}$ counts the number of 2's that appear among the a_{n-1} strings that end with a 1. To continue we need to determine formulas for $w_n^{(0)}$ and $t_n^{(0)}$, which we turn to next.

For $w_n^{(0)}$, we consider the following:

$$\begin{aligned} w_n^{(0)} &= w_{n-1}^{(0)} + a_{n-1} \\ &= w_{n-1}^{(0)} + w_{n-2} + a_{n-2} \\ &= w_{n-1}^{(0)} + \frac{1}{2}[(n-2)-1]P_{n-2} + P_{n-2} \\ &= w_{n-1}^{(0)} + \frac{1}{2}(n-1)\frac{1}{2\sqrt{2}}\left(\gamma^{n-2} - \delta^{n-2}\right) \\ &= w_{n-1}^{(0)} + \frac{1}{4\sqrt{2}}n\gamma^{n-2} - \frac{1}{4\sqrt{2}}n\delta^{n-2} - \frac{1}{4\sqrt{2}}\gamma^{n-2} + \frac{1}{4\sqrt{2}}\delta^{n-2} \end{aligned}$$

The form of the solution for this first-order nonhomogeneous recurrence relation is given by $w_n^{(0)} = c_1 + A\gamma^n + B\delta^n + Cn\gamma^n + Dn\delta^n$. To determine A and C, we substitute $w_n^{(0)} = A\gamma^n + Cn\gamma^n$ into the recurrence relation $w_n^{(0)} = w_{n-1}^{(0)} + \frac{1}{4\sqrt{2}}n\gamma^{n-2} - \frac{1}{4\sqrt{2}}\gamma^{n-2}$. Upon dividing through by γ^{n-2} we arrive at $A\gamma^2 + Cn\gamma^2 = A\gamma + C(n-1)\gamma + \frac{1}{4\sqrt{2}}n - \frac{1}{4\sqrt{2}}$. By comparing constants and the coefficients on n we obtain the equations $A\gamma^2 = A\gamma - C\gamma - \frac{1}{4\sqrt{2}}$ and $C\gamma^2 = C\gamma + \frac{1}{4\sqrt{2}}$. Solving these equations simultaneously, we learn that $A = -\frac{1}{16}\sqrt{2}$ and $C = -\frac{1}{8} + \frac{1}{8}\sqrt{2}$. Then a similar calculation reveals that $B = \frac{1}{16}\sqrt{2}$ and $D = -\frac{1}{8} + \frac{1}{8}\sqrt{2}$. From the initial condition $w_1^{(0)} = 0$ we have $c_1 = 0$, and so now this leads to the following.

Lemma 21. For $n \geq 1$,

$$\begin{aligned} w_n^{(0)} &= \left(-\frac{1}{16}\sqrt{2}\right)\gamma^n + \left(\frac{1}{16}\sqrt{2}\right)\delta^n \\ &\quad + \left(-\frac{1}{8} + \frac{1}{8}\sqrt{2}\right)n\gamma^n + \left(-\frac{1}{8} - \frac{1}{8}\sqrt{2}\right)n\delta^n \\ &= \frac{1}{4}\left[(2n-1)P_n - nQ_n\right]. \end{aligned}$$

Turning now to $t_n^{(0)}$ we have

$$\begin{aligned} t_n^{(0)} &= t_{n-1}^{(0)} + t_{n-1}^{(1)} \\ &= t_{n-1}^{(0)} + t_{n-2} \\ &= t_{n-1}^{(0)} + \frac{1}{4}[(n-3)P_{n-2} - Q_{n-2} + 1] \\ &= t_{n-1}^{(0)} + \frac{1}{4} + \frac{1}{4}n\left[\frac{1}{2\sqrt{2}}\left(\gamma^{n-2} - \delta^{n-2}\right)\right] \\ &\quad - \frac{3}{4}\left[\frac{1}{2\sqrt{2}}\left(\gamma^{n-2} - \delta^{n-2}\right)\right] - \frac{1}{4}\left[\frac{1}{2}\left(\gamma^{n-2} + \delta^{n-2}\right)\right], \end{aligned}$$

and the solution for this first-order nonhomogeneous recurrence relation is given by $t_n^{(0)} = c_1 + An + B\gamma^n + C\delta^n + Dn\gamma^n + En\delta^n$.

To determine A, we substitute $t_n^{(0)} = An$ into the relation $t_n^{(0)} = t_{n-1}^{(0)} + \frac{1}{4}$ and obtain $An = A(n-1) + \frac{1}{4}$. From this we learn that $A = \frac{1}{4}$. Continuing with B and D, we now substitute $t_n^{(0)} = B\gamma^n + Dn\gamma^n$ into the relation $t_n^{(0)} = t_{n-1}^{(0)} + \frac{1}{8\sqrt{2}}(n-3)\gamma^{n-2} - \frac{1}{8}\gamma^{n-2}$. After dividing through by γ^{n-2}, upon comparing coefficients on n, we find that $D\gamma^2 = D\gamma + \frac{1}{8\sqrt{2}}$ and so $D = -\frac{1}{16} + \frac{1}{16}\sqrt{2}$. Comparing constants, we have $B\gamma^2 = B\gamma - D\gamma - \frac{3}{8\sqrt{2}} - \frac{1}{8}$, from which it follows that $B = -\frac{3}{32}\sqrt{2}$. Then a similar calculation provides $C = \frac{3}{32}\sqrt{2}$ and $E = -\frac{1}{16} - \frac{1}{16}\sqrt{2}$. Finally, from the initial condition $t_1^{(0)} = 0$, it follows that $c_1 = 0$ and we arrive at the following result.

Lemma 22. For $n \geq 1$,

$$\begin{aligned} t_n^{(0)} &= \frac{1}{4}n - \frac{3}{32}\sqrt{2}\gamma^n + \frac{3}{32}\sqrt{2}\delta^n \\ &\quad + \left(-\frac{1}{16} + \frac{1}{16}\sqrt{2}\right) n\gamma^n + \left(-\frac{1}{16} - \frac{1}{16}\sqrt{2}\right) n\delta^n \\ &= \frac{1}{4}n - \frac{3}{8}P_n - \frac{1}{8}nQ_n + \frac{1}{4}nP_n. \end{aligned}$$

Returning to the recurrence relation for inv_n, we can now rewrite that relation as follows:

$$\begin{aligned} inv_n &= 2inv_{n-1} + inv_{n-2} + t_{n-1} + 2t_{n-2} + w_{n-2} + a_{n-2} + w_{n-1}^{(0)} + t_{n-1}^{(0)} \\ &= 2inv_{n-1} + inv_{n-2} + \frac{1}{4}\left[((n-1)-1)P_{n-1} - Q_{n-1} + 1\right] \\ &\quad + 2\left(\frac{1}{4}\right)\left[((n-2)-1)P_{n-2} - Q_{n-2} + 1\right] + \frac{1}{2}\left[((n-2)-1)P_{n-2}\right] \\ &\quad + P_{n-2} + \frac{1}{4}\left[P_{n-1}(2n-3) - (n-1)Q_{n-1}\right] \\ &\quad + \frac{1}{4}(n-1) - \frac{3}{8}P_{n-1} - \frac{1}{8}(n-1)Q_{n-1} + \frac{1}{4}(n-1)P_{n-1}. \end{aligned}$$

This then simplifies to

$$\begin{aligned} inv_n &= 2inv_{n-1} + inv_{n-2} + \left(n - \frac{15}{8}\right)P_{n-1} + (n-2)P_{n-2} \\ &\quad + \left(-\frac{3n}{8} + \frac{1}{8}\right)Q_{n-1} - \frac{1}{2}Q_{n-2} + \frac{n}{4} + \frac{1}{2}, \end{aligned}$$

so the form of the solution is given by

$$inv_n = c_1\gamma^n + c_2\delta^n + (An + B) + (Cn^2 + Dn)\gamma^n + (En^2 + Fn)\delta^n.$$

To determine A and B, substitute $inv_n = An + B$ into the relation $inv_n = 2\,inv_{n-1} + inv_{n-2} + \frac{n}{4} + \frac{1}{2}$. Upon comparing coefficients on n, we obtain $A = 3A + \frac{1}{4}$, so $A = -\frac{1}{8}$. On comparing constants we find that $B = -4A + 3B + \frac{1}{2}$, so $B = -\frac{1}{2}$. Continuing with C and D, we substitute $inv_n = (Cn^2 + Dn)\gamma^n$ into the relation $inv_n = 2\,inv_{n-1} + inv_{n-2} + \frac{1}{2\sqrt{2}}(n - \frac{15}{8})\gamma^{n-1} + \frac{1}{2\sqrt{2}}(n-2)\gamma^{n-2} + \frac{1}{2}(-\frac{3n}{8} + \frac{1}{8})\gamma^{n-1} - \frac{1}{4}\gamma^{n-2}$. Dividing through by γ^{n-2} and then simplifying, we find that $0 = 2(-2Cn + C - D)\gamma + (-4Cn + 4C - 2D) + \frac{1}{2\sqrt{2}}(n - \frac{15}{8})\gamma + \frac{1}{2\sqrt{2}}(n-2) + \frac{1}{2}(-\frac{3n}{8} + \frac{1}{8})\gamma - \frac{1}{4}$. Comparing coefficients on n we learn that $0 = -4C\gamma - 4C + \frac{1}{2\sqrt{2}}\gamma + \frac{1}{2\sqrt{2}} - \frac{3}{16}\gamma$ and $C = \frac{5}{128}\sqrt{2}$. When we compare constants we have $0 = (2C - 2D)\gamma + (4C - 2D) - \frac{15}{16\sqrt{2}}\gamma - \frac{1}{\sqrt{2}} + \frac{1}{16}\gamma - \frac{1}{4}$. With $C = \frac{5}{128}\sqrt{2}$, it then follows that $D = -\frac{19}{128} - \frac{3}{32}\sqrt{2}$. A similar calculation provides $E = -\frac{5}{128}\sqrt{2}$ and $F = -\frac{19}{128} + \frac{3}{32}\sqrt{2}$. Finally, the initial conditions $inv_1 = 0$ and $inv_2 = 0$ provide $c_1 = \frac{1}{4} + \frac{41\sqrt{2}}{256}$ and

$c_2 = \frac{1}{4} - \frac{41\sqrt{2}}{256}$.

We now reach the following result.

Theorem 23. For $n \geq 1$,

$$\begin{aligned} inv_n &= \left(\frac{1}{4} + \frac{41\sqrt{2}}{256}\right)\gamma^n + \left(\frac{1}{4} - \frac{41\sqrt{2}}{256}\right)\delta^n + \left(-\frac{1}{8}n - \frac{1}{2}\right) \\ &\quad + \left[\frac{5\sqrt{2}}{128}n^2 + \left(-\frac{19}{128} - \frac{3\sqrt{2}}{32}\right)n\right]\gamma^n \\ &\quad + \left[-\frac{5\sqrt{2}}{128}n^2 + \left(-\frac{19}{128} + \frac{3\sqrt{2}}{32}\right)n\right]\delta^n \\ &= \left(\frac{5}{32}n^2 - \frac{3}{8}n + \frac{41}{64}\right)P_n + \left(\frac{1}{2} - \frac{19}{64}n\right)Q_n - \frac{1}{8}(n+4). \end{aligned}$$

7. Coinversions

Given a string $s_1s_2\ldots s_n$, found among the a_n strings of length n, a *coinversion* occurs whenever $s_i < s_j$ for $1 \leq i < j \leq n$. Now if we let $coinv_n$ count the number of coinversions that occur among the a_n strings of length n, we find, for example, that $coinv_1 = 0$ and *coinv*$_2 = 1$. When $n = 3$ we find that *coinv*$_3 = 8$: two coinversions in 001 - one for the 0 in position 1 and the 1 in position 3, and the other for the 0 in position 2 and the 1 in position 3; one in 010, for the 0 in position 1 and the 1 in position 2; two in 011 - one for the 0 in position 1 and the 1 in position 2, and the other for the 0 in position 1 and the 1 in position 3; and, three coinversions for 012 - one for the 0 in position 1 and the 1 in position 2, another for the 0 in position 1 and the 2 in position 3, and a third for the 1 in position 2 and the 2 in position 3. We also find that *coinv*$_4 = 37$ and *coinv*$_5 = 143$.

In order to determine a formula for $coinv_n$, as we did for inv_n, we now introduce the auxiliary variables $coinv_n^{(i)}$, for $i \in \{0, 1, 2\}$, where $coinv_n^{(i)}$ counts the number of coinversions that that occur among the $a_n^{(i)}$ strings that end with an i. As a result, for $n \geq 2$, we have

$$\begin{aligned} coinv_n &= [coinv_{n-1}^{(0)} + coinv_{n-1}^{(1)}] \\ &\quad + [coinv_{n-1} + z_{n-1}] \\ &\quad + [coinv_{n-1}^{(1)} + coinv_{n-1}^{(2)} + z_{n-1}^{(1)} + w_{n-1}^{(1)} + z_{n-1}^{(2)} + w_{n-1}^{(2)}]. \end{aligned}$$

Here, for example, the first bracketed summand arises when 0 is appended to the strings of length $n-1$, ending in 0 or 1. This results in no new coinversions. The second bracketed summand arises when we append 1 to each of the a_{n-1} strings. Now a new coinversion arises for each of the z_{n-1} 0's that occur among these strings. The third summand accounts for the coinversions that arise when 2 to appended to each of the strings of length $n-1$ that end with a 1 or a 2.

In order to determine a formula for $coinv_n$, let us first observe that

$$\begin{aligned} coinv_{n-1}^{(1)} &= coinv_{n-2} + z_{n-2}, \\ w_n^{(1)} &= w_{n-1} + a_{n-1}, \quad \text{and} \\ z_{n-1}^{(1)} &= z_{n-2}, \end{aligned}$$

where $w_n^{(1)}$ once again counts the number of 1's that appear among the a_n strings that end with a 1 (as in the case for inversions in the previous section), and $z_{n-1}^{(1)}$ counts the number of 0's that appear among the a_{n-1} strings that end with a 1. To continue we need to determine formulas for $z_n^{(2)}$ and $w_n^{(2)}$.

For $z_n^{(2)}$, we consider the following:

$$\begin{aligned} z_n^{(2)} &= z_{n-1}^{(1)} + z_{n-1}^{(2)} = z_{n-1}^{(2)} + z_{n-2} \\ &= z_{n-1}^{(2)} + \frac{1}{4}Q_{n-2} - \frac{1}{4} + \frac{1}{4}(n+1)P_{n-2} \end{aligned}$$

$$= z_{n-1}^{(2)} - \frac{1}{4} + \frac{1}{8}(\gamma^{n-2} + \delta^{n-2}) + \frac{1}{8\sqrt{2}}(n+1)(\gamma^{n-2} - \delta^{n-2})$$

From this we see that the form of the solution for this first-order nonhomogeneous recurrence relation is given by

$$z_n^{(2)} = c_1 + An + B\gamma^n + C\delta^n + Dn\gamma^n + En\delta^n.$$

To determine A we substitute $z_n^{(2)} = An$ into the relation $z_n^{(2)} = z_{n-1}^{(2)} - \frac{1}{4}$ and obtain $An = A(n-1) - \frac{1}{4}$, so $A = -\frac{1}{2}$. Continuing for B and D, we now substitute $z_n^{(2)} = B\gamma^n + Dn\gamma^n$ into the relation $z_n^{(2)} = z_{n-1}^{(2)} + \frac{1}{8}\gamma^{n-2} + \frac{1}{8\sqrt{2}}(n+1)\gamma^{n-2}$. This results in $B\gamma^n + Dn\gamma^n = B\gamma^{n-1} + D(n-1)\gamma^{n-1} + \frac{1}{8}\gamma^{n-2} + \frac{1}{8\sqrt{2}}(n+1)\gamma^{n-2}$. After dividing through by γ^{n-2}, when we compare coefficients on n, we find that $D\gamma^2 = D\gamma + \frac{1}{8\sqrt{2}}$ and so $D = -\frac{1}{16} + \frac{1}{16}\sqrt{2}$. Now upon comparing constants we arrive at $B\gamma^2 = B\gamma - D\gamma + \frac{1}{8} + \frac{1}{8\sqrt{2}}$, from which it follows that $B = \frac{1}{32}\sqrt{2}$. Then a similar calculation provides $C = -\frac{1}{32}\sqrt{2}$ and $E = -\frac{1}{16} - \frac{1}{16}\sqrt{2}$. From the initial condition $z_1^{(2)} = 0$ we learn that $c_1 = 0$. Consequently, we are now led to the following.

Lemma 24. For $n \geq 1$,

$$\begin{aligned} z_n^{(2)} &= -\frac{1}{4}n + \frac{1}{32}\sqrt{2}\gamma^n - \frac{1}{32}\sqrt{2}\delta^n \\ &\quad + \left(-\frac{1}{16} + \frac{1}{16}\sqrt{2}\right)n\gamma^n + \left(-\frac{1}{16} - \frac{1}{16}\sqrt{2}\right)n\delta^n \\ &= -\frac{1}{4}n + \frac{1}{8}P_n - \frac{1}{8}nQ_n + \frac{1}{4}nP_n. \end{aligned}$$

Turning now to $w_n^{(2)}$, the number of 1's that appear among the $a_n^{(2)}$ strings of length n that end with a 2, we consider the following:

$$\begin{aligned} w_n^{(2)} &= w_{n-1}^{(2)} + w_{n-1}^{(1)} = w_{n-1}^{(2)} + w_{n-2} + a_{n-2} \\ &= w_{n-1}^{(2)} + \frac{1}{2}[(n-2) - 1]P_{n-2} + P_{n-2} \\ &= w_{n-1}^{(2)} + \frac{1}{2}(n-1)P_{n-2} = w_{n-1}^{(2)} + \frac{1}{4\sqrt{2}}(n-1)\left(\gamma^{n-2} - \delta^{n-2}\right) \end{aligned}$$

Consequently, the form of the solution for this first-order nonhomogeneous recurrence relation is given by

$$w_n^{(2)} = c_1 + A\gamma^n + B\delta^n + Cn\gamma^n + Dn\delta^n.$$

To determine A and C, we substitute $w_n^{(2)} = A\gamma^n + Cn\gamma^n$ into the relation $w_n^{(2)} = w_{n-1}^{(2)} + \frac{1}{4\sqrt{2}}(n-1)\gamma^{n-2}$. After dividing through by γ^{n-2} we are left with $A\gamma^2 + Cn\gamma^2 = A\gamma + C(n-1)\gamma + \frac{1}{4\sqrt{2}}n - \frac{1}{4\sqrt{2}}$. At this point we compare the coefficients on n and the constants. This leads to the equations $C\gamma^2 = C\gamma + \frac{1}{4\sqrt{2}}$ and $A\gamma^2 = A\gamma - C\gamma - \frac{1}{4\sqrt{2}}$, from which it follows that $A = -\frac{1}{16}\sqrt{2}$ and $C = -\frac{1}{8} + \frac{1}{8}\sqrt{2}$. Then a comparable calculation provides $B = \frac{1}{16}\sqrt{2}$ and $D = -\frac{1}{8} - \frac{1}{8}\sqrt{2}$. Finally, the initial condition $w_1^{(2)} = 0$ implies that $c_1 = 0$, and so we now have the following:

Lemma 25. For $n \geq 1$,

$$
\begin{aligned}
w_n^{(2)} &= -\frac{1}{16}\sqrt{2}\gamma^n + \frac{1}{16}\sqrt{2}\delta^n + \left(-\frac{1}{8} + \frac{\sqrt{2}}{8}\right) n\gamma^n + \left(-\frac{1}{8} - \frac{\sqrt{2}}{8}\right) n\delta^n \\
&= -\frac{1}{4}P_n - \frac{1}{4}nQ_n + \frac{1}{2}nP_n.
\end{aligned}
$$

Returning to the recurrence relation for $coinv_n$, we can now rewrite the relation as follows:

$$
\begin{aligned}
coinv_n &= 2\,coinv_{n-1} + coinv_{n-2} + z_{n-1} + 2\,z_{n-2} + w_{n-2} + a_{n-2} \\
&\quad + z_{n-1}^{(2)} + w_{n-1}^{(2)} \\
&= 2\,coinv_{n-1} + coinv_{n-2} \\
&\quad + \frac{1}{4}Q_{n-1} + \frac{3}{4}P_{n-1} - \frac{1}{4} + \frac{1}{4}(n-1)P_{n-1} \\
&\quad + 2[\frac{1}{4}Q_{n-2} + \frac{3}{4}P_{n-2} - \frac{1}{4} + \frac{1}{4}(n-2)P_{n-2}] \\
&\quad + \frac{1}{2}[(n-2)-1]P_{n-2} + P_{n-2} \\
&\quad - \frac{1}{4}(n-1) + \frac{1}{8}P_{n-1} - \frac{1}{8}(n-1)Q_{n-1} + \frac{1}{4}(n-1)P_{n-1} \\
&\quad - \frac{1}{4}P_{n-1} - \frac{1}{4}(n-1)Q_{n-1} + \frac{1}{2}(n-1)P_{n-1}.
\end{aligned}
$$

This then simplifies to

$$
\begin{aligned}
coinv_n &= 2\,coinv_{n-1} + coinv_{n-2} + \left(n - \frac{3}{8}\right) P_{n-1} + nP_{n-2} \\
&\quad + \left(\frac{5}{8} - \frac{3}{8}n\right) Q_{n-1} + \frac{1}{2}Q_{n-2} - \left(\frac{1}{2} + \frac{1}{4}n\right),
\end{aligned}
$$

so the form of the solution is given by

$$
coinv_n = c_1\gamma^n + c_2\delta^n + A + Bn + (Cn + Dn^2)\gamma^n + (En + Fn^2)\delta^n.
$$

To determine A and B we substitute $coinv_n = A + Bn$ into the relation $coinv_n = 2\,coinv_{n-1} + coinv_{n-2} - (\frac{1}{2} + \frac{1}{4}n)$. This results in $A + Bn = 3\,A - 4\,B + 3\,Bn - \frac{1}{2} - \frac{1}{4}n$. Comparing the coefficients on n we find that $B = 3\,B - \frac{1}{4}$. Comparing the constants we have $A = 3\,A - 4\,B - \frac{1}{2}$. From these two equations we learn that $A = \frac{1}{2}$ and $B = \frac{1}{8}$. Continuing with C and D, we substitute $coinv_n = Cn\gamma^n + Dn^2\gamma^n$ into the relation $coinv_n = 2\,coinv_{n-1} + coinv_{n-2} + (n - \frac{3}{8})\left(\frac{1}{2\sqrt{2}}\right)\gamma^{n-1} + \frac{1}{2\sqrt{2}}n\gamma^{n-2} + \frac{1}{2}\left(\frac{5}{8} - \frac{3}{8}n\right)\gamma^{n-1} + \frac{1}{4}\gamma^{n-2}$. Upon dividing through by γ^{n-2} this leads to $Cn\gamma^2 + Dn^2\gamma^2 = 2[C(n-1)\gamma + D(n^2 - 2\,n + 1)\gamma] + C(n-2) + D(n^2 - 4\,n + 4) + \frac{1}{2\sqrt{2}}\left(n - \frac{3}{8}\right)\gamma + \frac{1}{2\sqrt{2}}n + \frac{1}{2}\left(\frac{5}{8} - \frac{3}{8}n\right) + \frac{1}{4}$. Comparing the coefficients on n we find that $C\gamma^2 = 2\,C\gamma - 4\,D\gamma + C - 4\,D + \frac{1}{2\sqrt{2}}\gamma + \frac{1}{2\sqrt{2}} - \frac{3}{16}\gamma$. Comparing the constants provides $0 = -2\,C\gamma + 2\,D\gamma - 2\,C + 4\,D - \frac{3}{16\sqrt{2}}\gamma + \frac{5}{16}\gamma + \frac{1}{4}$. Solving these two equations simultaneously we arrive at $C = \frac{5}{128} + \frac{3}{32}\sqrt{2}$ and $D = \frac{5}{128}\sqrt{2}$. Then a comparable calculation yields $E = \frac{5}{128} - \frac{3}{32}\sqrt{2}$ and $F = -\frac{5}{128}\sqrt{2}$. Finally, from the initial conditions $coinv_1 = 0$ and $coinv_2 = 1$, we learn that $c_1 = -\frac{1}{4} - \frac{47}{256}\sqrt{2}$ and $c_2 = -\frac{1}{4} + \frac{47}{256}\sqrt{2}$. This now leads us to the following:

Theorem 26. For $n \geq 1$,

$$
\begin{aligned}
coinv_n &= \left(-\frac{1}{4} - \frac{47}{256}\sqrt{2}\right)\gamma^n + \left(-\frac{1}{4} + \frac{47}{256}\sqrt{2}\right)\delta^n + \frac{1}{2} + \frac{1}{8}n \\
&\quad + \left(\frac{5}{128} + \frac{3}{32}\sqrt{2}\right) n\gamma^n + \frac{5}{128}\sqrt{2}n^2\gamma^n \\
&\quad + \left(\frac{5}{128} - \frac{3}{32}\sqrt{2}\right) n\delta^n - \frac{5}{128}\sqrt{2}n^2\delta^n \\
&= \frac{1}{2} + \frac{1}{8}n - \frac{1}{2}Q_n - \frac{47}{64}P_n + \frac{5}{64}nQ_n + \frac{3}{8}nP_n + \frac{5}{32}n^2P_n.
\end{aligned}
$$

8. The Sum of the Major Indices

Following the development in Chapter 6 of [4], if $s = s_1s_2s_3\ldots s_n$ is one of the a_n strings of length n, the *descent set* of s, denoted $\mathrm{Des}(s)$, is the set of all $k < n$ where $s_k > s_{k+1}$. So $\mathrm{Des}(s)$ is the set of all positions in s, starting at the left, where a descent occurs. The *major index of* s is the sum of the elements in $\mathrm{Des}(s)$.

For example, when $n = 3$, the only string where a descent occurs is 010. This descent occurs in position 2, so in this case $\mathrm{Des}(010) = \{2\}$ and the major index of 010 is 2. For $n = 4$, the string 0010 has the descent set $\{3\}$ and its major index is 3. If $n = 5$, the string 01210 has descents at positions 3 and 4, so $\mathrm{Des}(01210) = \{3, 4\}$ and its major index is $3 + 4 = 7$.

Returning to the case of $n = 4$, there are five strings where descents occur: (i) $s_1 = 0010$, where $\mathrm{Des}(s_1) = \{3\}$ and the major index is 3; (ii) $s_2 = 0100$, where $\mathrm{Des}(s_2) = \{2\}$ and the major index is 2; (iii) $s_3 = 0101$, where $\mathrm{Des}(s_3) = \{2\}$ and the major index is 2; (iv) $s_4 = 0110$, where $\mathrm{Des}(s_4) = \{3\}$ and the major index is 3; and, (v) $s_5 = 0121$, where $\mathrm{Des}(s_5) = \{3\}$ and the major index is 3. Therefore, the sum of the major indices for $n = 4$ is $3 + 2 + 2 + 3 + 3 = 13$.

For $n \geq 1$, let smi_n denote the sum of the major indices for all of the a_n strings of length n. We find that $smi_1 = 0$, $smi_2 = 0$, $smi_3 = 2$, and $smi_4 = 13$. Similar to what we've done previously, let $smi_n^{(k)}$ denote the sum of the major indices for those of the a_n strings of length n that end with a k, for $k \in \{0, 1, 2\}$.

For $n \geq 2$,

$$\begin{aligned} smi_n = &[smi_{n-1}^{(1)} + (n-1)a_{n-1}^{(1)} + smi_{n-1}^{(0)}] \\ &+ [smi_{n-1} + (n-1)a_{n-1}^{(2)}] \\ &+ [smi_{n-1}^{(1)} + smi_{n-1}^{(2)}], \end{aligned}$$

where, for example, the first bracketed summand accounts for when we append 0 to the a_{n-1} strings of length $n-1$ that end with a 0 or a 1. The summand $(n-1)a_{n-1}^{(1)}$ counts each new descent at position $n-1$ for each of the $a_{n-1}^{(1)}$ strings.

Since $smi_n^{(1)} = smi_{n-1} + (n-1)a_{n-1}^{(2)}$ and $smi_n^{(0)} = smi_{n-1}^{(1)} + (n-1)a_{n-1}^{(1)} + smi_{n-1}^{(0)}$, we rewrite the above recurrence relation for smi_n as

$$\begin{aligned} smi_n &= 2\,smi_{n-1} + smi_{n-1}^{(1)} + (n-1)a_{n-1}^{(2)} + (n-1)a_{n-1}^{(1)} \\ &= 2\,smi_{n-1} + smi_{n-2} + (n-2)a_{n-2}^{(2)} + (n-1)a_{n-1}^{(2)} + (n-1)a_{n-1}^{(1)} \\ &= 2\,smi_{n-1} + smi_{n-2} + (n-2)\left[\frac{1}{2}Q_{n-2} - P_{n-3} - \frac{1}{2}\right] \\ &+ (n-1)\left[\frac{1}{2}Q_{n-1} - P_{n-2} - \frac{1}{2}\right] + (n-1)P_{n-2} \\ &= 2\,smi_{n-1} + smi_{n-2} - (n-2)P_{n-3} \\ &+ \frac{1}{2}(n-2)Q_{n-2} + \frac{1}{2}(n-1)Q_{n-1} + \frac{1}{2}(3 - 2\,n). \end{aligned}$$

Consequently, the form of the solution for this recurrence relation is given as

$$smi_n = c_1\gamma^n + c_2\delta^n + A + Bn + Cn\gamma^n + Dn\delta^n + En^2\gamma^n + Fn^2\delta^n.$$

To determine A and B we substitute $smi_n = A + Bn$ into the relation $smi_n = 2\,smi_{n-1} + smi_{n-2} + \frac{1}{2}(3 - 2\,n)$. This results in $A + Bn = 3\,A - 4\,B + \frac{3}{2} + (3\,B - 1)n$. Comparing the coefficients on n we find that $B = 3\,B - 1$. Comparing the constants yields $A = 3\,A - 4\,B + \frac{3}{2}$. Solving these two equations simultaneously we have $A = \frac{1}{4}$ and $B = \frac{1}{2}$. Turning now to C and E, we substitute $smi_n = Cn\gamma^n + En^2\gamma^n$ into the relation $smi_n = 2\,smi_{n-1} + smi_{n-2} - (n-2)\left(\frac{1}{2\sqrt{2}}\right)\gamma^{n-3} + \frac{1}{4}(n-2)\gamma^{n-2} + \frac{1}{4}(n-1)\gamma^{n-1}$. After dividing through by γ^{n-3} and then simplifying, we arrive at $0 = 2(-C - 2\,nE + E)\gamma^2 + (-2\,C - 4\,nE + 4\,E)\gamma - (n-2)\left(\frac{1}{2\sqrt{2}}\right) + \left(\frac{n}{4} - \frac{1}{2}\right)\gamma + \left(\frac{n}{4} - \frac{1}{2}\right)\gamma^2$. Comparing the coefficients on n we find that $0 = -4\,E\gamma^2 - 4\,E\gamma - \frac{1}{2\sqrt{2}} + \frac{\gamma}{4} + \frac{\gamma^2}{4}$. Comparing the constants we have $0 = -2\,C\gamma^2 + 2\,E\gamma^2 - 2\,C\gamma + 4\,E\gamma + \frac{1}{\sqrt{2}} - \frac{\gamma}{2} - \frac{\gamma^2}{4}$. Solving simultaneously, these equations provide $C = \frac{1}{8} - \frac{1}{8}\sqrt{2}$ and $E = -\frac{1}{8} + \frac{1}{8}\sqrt{2}$. Then a similar calculation yields $D = \frac{1}{8} + \frac{1}{8}\sqrt{2}$ and $F = -\frac{1}{8} - \frac{1}{8}\sqrt{2}$. Finally, from the initial conditions $smi_1 = 0$ and $smi_2 = 0$, we learn that $c_1 = -\frac{1}{8} - \frac{1}{8}\sqrt{2}$ and $c_2 = -\frac{1}{8} + \frac{1}{8}\sqrt{2}$. This now leads to our final result.

Theorem 27. For $n \geq 1$,

$$\begin{aligned} smi_n = &\left(-\frac{1}{8} - \frac{1}{8}\sqrt{2}\right)\gamma^n + \left(-\frac{1}{8} + \frac{1}{8}\sqrt{2}\right)\delta^n + \frac{1}{4} + \frac{1}{2}n \\ &+ \left(\frac{1}{8} - \frac{1}{8}\sqrt{2}\right)n\gamma^n + \left(\frac{1}{8} + \frac{1}{8}\sqrt{2}\right)n\delta^n \\ &+ \left(-\frac{1}{8} + \frac{1}{8}\sqrt{2}\right)n^2\gamma^n + \left(-\frac{1}{8} - \frac{1}{8}\sqrt{2}\right)n^2\delta^n \\ = &\frac{1}{4} + \frac{1}{2}n + \frac{1}{2}(n^2 - n - 1)P_n - \frac{1}{4}(n^2 - n + 1)Q_n. \end{aligned}$$

9. References

[1] Brualdi, Richard A. *Introductory Combinatorics*, fifth edition. Upper Saddle River, New Jersey: Pearson Prentice Hall, 2010.

[2] Grimaldi, Ralph P. *Discrete and Combinatorial Mathematics*, fifth edition. Boston, Massachusetts: Pearson Addison Wesley, 2004.

[3] Koshy, Thomas. *Fibonacci and Lucas Numbers with Applications, Volume 1*, second edition. New York, New York: John Wiley & Sons, Inc., 2018.

[4] Loehr, Nicholas A. *Bijective Combinatorics*. Boca Raton, Florida: Chapman & Hall/CRC, 2011.

[5] Sloane, Neil James Alexander. *The On-Line Encyclopedia of Integer Sequences,*

http://www.research.att.com/~njas/sequences