Some Additional Cases of Bicyclic Antiautomorphisms of Mendelsohn Triple Systems

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Abstract

A cyclic triple, (a, b, c), is defined to be the set $\{(a, b), (b, c), (c, a)\}$ of ordered pairs. A Mendelsohn triple system of order v, or MTS(v), is a pair (M, β), where M is a set of v points and β is a collection of cyclic triples, each containing pairwise distinct points of M such that every ordered pair of distinct points of M exists in exactly one cyclic triple of β. An antiautomorphism of a Mendelsohn triple system (M, β) is a permutation of M which maps β to β^{-1}, where $\beta^{-1} = \{(c, b, a) | (a, b, c) \in \beta\}$. Necessary conditions for the existence of an MTS(v) admitting an antiautomorphism consisting of two cycles of lengths M and N, where $1 < M \leq N$, have been shown, and for the cases of $N = M$ and $N = 2M$, sufficiency has been shown. We show sufficiency for the cases in which $M = 13$ and $N = 78, 390$, and 702.

Keywords: antiautomorphism, bicyclic, Mendelsohn triple system.

1 Introduction

A *Steiner triple system of order v*, or STS(v), is a pair (S, β), where S is a set of v points and β is a collection of 3-element subsets of S, called *blocks*, such that any two distinct points of S are contained together in exactly one block of β. Kirkman [8] proved that an STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6) or $v = 0$.

An *automorphism* of a Steiner triple system (S,β) is a permutation of S which maps β to itself. An automorphism, α, of an STS(v) is said to be *cyclic* if the permutation defined by α consists of a single cycle of length v. Peltesohn [10] showed that an STS(v) admitting a cyclic automorphism exists if and only if $v \equiv 1$ or 3 (mod 6) with $v \neq 9$.

An automorphism, α, of an STS(v) is called *bicyclic* if the permutation defined by α consists of two cycles. Calahan-Zijlstra and Gardner [2] showed that there exists an STS(v) which admits a bicyclic automorphism consisting of cycles of lengths M and N, where $1 < M \leq N$, if and only if $M \equiv 1$ or 3 (mod 6), $M \neq 9$, $M|N$, and $M + N \equiv 1$ or 3 (mod 6).

A *cyclic triple*, (a, b, c), is defined to be the set $\{(a, b), (b, c), (c, a)\}$ of ordered pairs. A *Mendelsohn triple system of order v*, MTS(v), is a pair (M, β), where M is a set of v points and β is a collection of cyclic triples of pairwise distinct points of M, called *triples*, such that any ordered pair of distinct points of M is contained in precisely one element of β. Mendelsohn [9] has shown that necessary and sufficient conditions for the existence of an MTS(v) are that $v \equiv 0$ or 1 (mod 3) with $v \neq 6$.

For an MTS(v), (M, β), let $\beta^{-1} = \{(c, b, a)|(a, b, c) \in \beta\}$. Then (M, β^{-1}) is also an MTS(v) and is called the *converse* of (M, β). If an MTS(v) is isomorphic to its converse, it is said to be *self-converse*. An *automorphism* of (M, β) is a permutation of M which maps β to itself, and an *antiautomorphism* of (M, β) is a permutation of M which maps β to β^{-1}. It follows, then, that an MTS(v) is self-converse if and only if it admits an antiautomorphism.

Consider an STS(v), (S, β'). Let $\beta = \{(a, b, c), (c, b, a)|\{a, b, c\} \in \beta'\}$. Then the pair (S, β) is called the *corresponding* MTS(v), and the identity mapping on S is both an automorphism and an antiautomorphism since $\beta = \beta^{-1}$ in this case. Therefore, for $v \equiv 1$ or 3 (mod 6), there exists a self-converse MTS(v). As such, a self-converse MTS(v) can be constructed from any STS(v).

An antiautomorphism, α, on an MTS(v) is called *cyclic* if the permutation defined by α consists of a single cycle of length d and $v - d$ fixed points. Carnes [3] has shown that there exists an MTS(v) admitting an automorphism consisting of a single cycle of length v, where v is even, and 0 fixed points if and only if $v \equiv 0$ or 4 (mod 12). For v odd, Peltesohn's results apply.

An antiautomorphism, α, on an MTS(v) is said to be *bicyclic* if the permutation defined by α consists of two cycles. Carnes, Dye, and Reed [5] have shown that there exists an MTS(v) admitting a bicyclic antiautomorphism consisting of two cycles of equal length and 0 fixed points if and only if $v \equiv 0$ or 16 (mod 24). Carnes and Andrus [1] have shown that there exists an MTS(v) admitting a bicyclic antiautomorphism having cycles of lengths M and N, where $v = M + N$ and $N = 2M$, if and only if M is odd or $M \equiv 0$ or 4 (mod 12).

We now consider bicyclic antiautomorphisms with cycles of lengths M and N, with $1 < M$ and $N > 2M$.

2 Preliminaries

We let the cycle of length M be $(0_0, 1_0, 2_0, \ldots, (M-1)_0)$ and the cycle of length N be $(0_1, 1_1, 2_1, \ldots, (N-1)_1)$. If $\Delta = \{0, 1, 2, \ldots, (K-1)\}$, where $K \in \{M, N\}$, then the *orbit* of the triple $(a_i, b_j, c_k) \in \beta$, $i, j, k \in \{0, 1\}$, is given by $\{((a+t)_i, (b+t)_j, (c+t)_k) | t \in \Delta \text{ and } t \text{ even}\} \cup \{((c+t)_k, (b+t)_j, (a+t)_i) | t \in \Delta \text{ and } t \text{ odd}\}$. All additions within the triples shall be performed modulo K. Note that the orbits form a partition of β. Also, a collection of triples β' such that one triple of each orbit is contained in β' and the union of the orbits of the triples in β' forms β is called a collection of *base triples* of an MTS(v) under α.

Since all points are contained in one of two cycles, any triple (a_i, b_j, c_k), $i, j, k \in \{0, 1\}$, can be classified as one of four types:

Type 1 triples are of the form (a_0, b_0, c_0).

Type 2 triples are of the form (a_1, b_1, c_1).

Type 3 triples are of the form (a_0, b_1, c_1), (a_1, b_0, c_1), or (a_1, b_1, c_0).

Type 4 triples are of the form (a_0, b_0, c_1), (a_0, b_1, c_0), or (a_1, b_0, c_0).

Carnes [4] proved that an MTS(v) which admits a bicyclic antiautomorphism with cycles of lengths M and N, where $1 < M < N$, and contains a type 4 triple exists only if M is odd and $N = 2M$. Carnes also gave the necessary conditions for the existence of an MTS(v) admitting a bicyclic antiautomorphism consisting of cycles of lengths M and N, where $1 < M$ and $N > 2M$.

Theorem 2.1 *If an MTS(v) admits a bicyclic antiautomorphism having cycles of lengths M and N, where $1 < M < N$, then $M|N$, $v = M + N \equiv 0$ or $1 \pmod 3$, and $M \equiv 0$ or $4 \pmod{12}$ if M is even or $M \equiv 1$ or $3 \pmod 6$ with $M \neq 9$ if M is odd.*

For an MTS(v), (M, β), a *cyclic subsystem* is a subset $M_0 \in M$ such that, if two elements of M_0 occur together in a triple of β, the third element of that triple must also belong to M_0. If an MTS(v) admits a bicyclic antiautomorphism consisting of two cycles of lengths M and N, where $1 < M$ and $N > 2M$, then the set of elements from the cycle of length M must form a cyclic subsystem. Therefore, we can treat the cycle of length M as an MTS(M) admitting a cyclic antiautomorphism and use the triples from such a system, as given by Carnes, to complete the constructions.

3 Sufficient Conditions

Recall that given an STS(v), (S, β'), if $\beta = \{(a, b, c), (c, b, a) | \{a, b, c\} \in \beta'\}$, then the pair (S, β) is the corresponding MTS(v), with the identity mapping being both an automorphism and an antiautomorphism. As such, the results of Calahan-Zijlstra and Gardner on bicyclic Steiner triple systems can be used to construct Mendelsohn triple systems that admit bicyclic antiautomorphisms, thereby showing sufficiency. Calahan-Zijlstra and Gardner's results yield the following two theorems.

Theorem 3.1 *There exists an MTS(v) admitting a bicyclic antiautomorphism with cycles of lengths M and N, where $v = M + N$ and $N = kM$, if $M \equiv 1 \pmod 6$, $M > 1$, and $k \equiv 2 \pmod 6$.*

Theorem 3.2 *There exists an MTS(v) admitting a bicyclic antiautomorphism with cycles of lengths M and N, where $v = M + N$ and $N = kM$, if $M \equiv 3 \pmod 6$, $M > 9$, and $k \equiv 0 \pmod 6$.*

Carnes and Snider have shown the following [7].

Theorem 3.3 *There exists an MTS(v) which admits a bicyclic antiautomorphism with cycles of lengths M and N, where $v = M+N$ and $N = kM$, if $M \equiv 4 \pmod{12}$ and $k \equiv 2 \pmod 6$.*

Additionally, Carnes and Qualls have shown the following [6].

Theorem 3.4 *There exists an MTS(v) which admits a bicyclic antiautomorphism with cycles of lengths M and N, where $M = 24$ and $N = 336$.*

We now consider the cases in which $M = 13$ and $k = 6, 30$, and 54.

Theorem 3.5 *There exists an MTS(v) which admits a bicyclic antiautomorphism with cycles of lengths M and N, where $M = 13$ and $N = 78, 390$, and 702.*

Proof. First, let $M = 13$ and $k = 6$. Then $N = 78$.

The base triples include the following:
$(0_1, 26_1, 52_1)$, $(0_1, 1_1, 2_1)$, $(0_1, 2_1, 26_1)$, $(0_1, 54_1, 27_1)$.

The following triples and their reverses are also included:
$(0_0, 0_1, 39_1)$, $(0_0, 1_1, 38_1)$, $(0_0, 2_1, 37_1)$, $(0_0, 3_1, 36_1)$, $(0_0, 4_1, 48_1)$, $(0_0, 5_1, 47_1)$, $(0_0, 6_1, 46_1)$, $(0_1, 12_1, 20_1)$, $(0_1, 13_1, 19_1)$, $(0_1, 14_1, 18_1)$, $(0_1, 22_1, 31_1)$, $(0_1, 23_1, 30_1)$, $(0_1, 15_1, 25_1)$, $(0_1, 16_1, 21_1)$, $(0_1, 17_1, 28_1)$, $(0_1, 29_1, 32_1)$.

The remaining triples in the cycle of length M are from an MTS(13) admitting a cyclic antiautomorphism.

Next, let $M = 13$ and $k = 30$. Then $N = 390$.

The base triples include the following:
$(0_1, 130_1, 260_1)$, $(0_1, 1_1, 2_1)$, $(0_1, 2_1, 130_1)$, $(0_1, 262_1, 131_1)$.

The following triples and their reverses are also included:
$(0_0, 0_1, 195_1)$, $(0_0, 1_1, 194_1)$, $(0_0, 2_1, 193_1)$, $(0_0, 3_1, 192_1)$, $(0_0, 4_1, 204_1)$,

$(0_0, 5_1, 203_1)$, $(0_0, 6_1, 202_1)$, $(0_1, 97_1, 160_1)$, $(0_1, 98_1, 157_1)$, $(0_1, 103_1, 158_1)$, $(0_1, 159_1, 187_1)$, $(0_1, 91_1, 109_1)$, $(0_1, 92_1, 108_1)$, $(0_1, 86_1, 100_1)$, $(0_1, 89_1, 101_1)$, $(0_1, 85_1, 95_1)$, $(0_1, 88_1, 96_1)$, $(0_1, 87_1, 93_1)$, $(0_1, 90_1, 94_1)$, $(0_1, 99_1, 102_1)$, $(0_1, 127_1, 188_1)$, $(0_1, 129_1, 186_1)$,
$(0_1, (64+s)_1, (126-s)_1)$ for $s = 0, 1, \ldots, 16$;
$(0_1, (132+s)_1, (185-s)_1)$ for $s = 0, 1, \ldots, 24$.

The remaining triples in the cycle of length M are from an MTS(13) admitting a cyclic antiautomorphism.

Finally, let $M = 13$ and $k = 54$. Then $N = 702$.

The base triples include the following:
$(0_1, 234_1, 468_1)$, $(0_1, 1_1, 2_1)$, $(0_1, 2_1, 234_1)$, $(0_1, 470_1, 235_1)$.

The following triples and their reverses are also included:
$(0_0, 0_1, 351_1)$, $(0_0, 1_1, 350_1)$, $(0_0, 2_1, 349_1)$, $(0_0, 3_1, 348_1)$, $(0_0, 4_1, 360_1)$, $(0_0, 5_1, 359_1)$, $(0_0, 6_1, 358_1)$, $(0_1, 289_1, 343_1)$, $(0_1, 175_1, 290_1)$, $(0_1, 176_1, 287_1)$, $(0_1, 181_1, 288_1)$, $(0_1, 160_1, 184_1)$, $(0_1, 162_1, 182_1)$, $(0_1, 161_1, 179_1)$, $(0_1, 169_1, 185_1)$, $(0_1, 166_1, 180_1)$, $(0_1, 171_1, 183_1)$, $(0_1, 168_1, 178_1)$, $(0_1, 164_1, 172_1)$, $(0_1, 159_1, 165_1)$, $(0_1, 163_1, 167_1)$, $(0_1, 170_1, 173_1)$, $(0_1, 174_1, 200_1)$, $(0_1, 177_1, 199_1)$, $(0_1, 231_1, 344_1)$, $(0_1, 233_1, 342_1)$,
$(0_1, (116+s)_1, (230-s)_1)$ for $s = 0, 1, \ldots, 29$;
$(0_1, (236+s)_1, (341-s)_1)$ for $s = 0, 1, \ldots, 50$;
$(0_1, (146+s)_1, (198-s)_1)$ for $s = 0, 1, \ldots, 12$;

The remaining triples in the cycle of length M are from an MTS(13) admitting a cyclic antiautomorphism. □

References

[1] T. R. Andrus and N. P. Carnes, *A class of antiautomorphisms of Mendelsohn triple systems with two cycles*, Australasian Journal of Combinatorics Vol. 39 (2007), 19-25.

[2] R. Calahan-Zijlstra and R. B. Gardner, *Bicyclic Steiner triple systems*, Discrete Math. 128 (1994), 35-44.

[3] N. P. Carnes, *Cyclic antiautomorphisms of Mendelsohn triple systems*, Discrete Math. 126 (1994), 29-45.

[4] N. P. Carnes, *Two cycle antiautomorphisms of Mendelsohn triple systems*, Congressus Numerantium 154 (2002), 217-220.

[5] N. P. Carnes, A. Dye, and J. F. Reed, *Bicyclic antiautomorphisms of Mendelsohn triple systems with 0 or 1 fixed points*, Ars Combinatoria 54 (2000), 179-186.

[6] N. P. Carnes, B. L. Qualls, *A note on bicyclic antiautomorphisms of Mendelsohn triple systems*, Congressus Numerantium, to appear.

[7] N. P. Carnes, L. L. Snider, *Some bicyclic antiautomorphisms of Mendelsohn triple systems*, Congressus Numerantium 226 (2016), 281-287.

[8] T.P. Kirkman, *On a problem in combinations*, Cambridge and Dublin Math. J. 2 (1847), 191-204.

[9] N. S. Mendelsohn, *A natural generalization of Steiner triple systems*, Computers in Number Theory (Academic Press, New York, 1971), 323-338.

[10] R. Peltesohn, *Eine Lösung der beiden Heffterschen Differenzeprobleme*, Compositio Math. 6 (1939), 251-257.