The 5-cube Cut Number Problem: A Short Proof for a Basic Lemma

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Abstract

The hypercube cut number $S(d)$ is the minimum number of hyperplanes in the d-dimensional Euclidean space $\mathbb{R}^d$ that slice all the edges of the d-cube. The problem originally was posed by P. ONeil in 1971. B. Grnbaum, V. Klee, M. Saks and Z. Fredi have raised the problem in various contexts. The identity $S(d) = d$ has been well-known for $d \leq 4$ since 1986. However, it was only until the year 2000, when Sohler and Ziegler obtained a computational proof for $S(5) = 5$. Nevertheless, finding a short proof for the problem, independent of computer computations, remains to be a challenging problem. We present a short proof for the result presented by Emamy-Uribe-Tomassini in Hypercube 2002 based on Tomassini's Thesis. The proof here is substantially shorter than the original proof of 60 pages.

Keywords: Hypercube, Hyperplanes, Cube Cuts

1 Introduction

The hypercube cut number problem is to find the value of $S(d)$, that is, the minimum number of hyperplanes that slice all the edges of the d-cube in $\mathbb{R}^d$. This problem originally was announced by P. O'Neil [11] in 1971, however since then there are just a few scattered results on the problem in the literature. The only results known for the lower dimensional cubes include an example of a 5-covering for the 6-cube by M. Paterson [11], and two different proofs for $S(4) = 4$ by Emamy [2,3]. A first main computational result appeared in 2000, when Sohler and Ziegler [15] obtained a solution to the 5-cube problem via advanced parallel computing that took about two months of CPU time. For more on this problem, a very comprehensive source of references, and many other closely related problems on the hypercube cuts see Saks [12].

A short theoretical proof for the 5-cube independent of computer computations remains to be a challenging problem, that was posed as the ninth problem on the list of open problems in [13]. We present here a short proof for the result presented by Emamy-Uribe-Tomassini in Hypercube 2002 that was cited by Z. Fredi in Bezd [1] one year later. The result reported in 2002 only claimed partial theoretical proof for $S(5) = 5$, i. e., only one case out of 9 possible cases, corresponding to one edge number set out of 9 possible edge number sets for a given 3-cube, that all together will settle down the 5-cube theoretical proof. The single case proof appeared first in Tomassini's Thesis that is more than 60 pages. That proof was then improved and reduced to almost one third in Uribe-Emamy [14]. Our new result that appears here as Theorem B is the shortest proof of these type and joint with the 3-cut theorem will be the bases that shape our basic lemmas for the next step, completing the proof for the 5-cube problem.

2 The 3-Coverings for the 3-cube

A closed segment $I \subset \mathbb{R}^d$ is said to be *sliced* by a hyperplane H if $H \cap I$ is a single interior point of I. A *cut* of the d-cube is a maximal set of edges that can be sliced by a hyperplane, provided the hyperplane does not pass through any vertices neither through the center of $d-$cube. The latter center assumption can be achieved by slightly moving the cutting hyperplane. A *k-cut* is any cut consisting of k edges. A *covering* for the $d-$cube is a set of cuts that cover all the edges of the $d-$cube. A *k-covering* is a covering containing k cuts. Two cuts (coverings) are *isomorphic* if there is a symmetry of the cube that maps one cut (covering) to the other one.

For the notational convenience, we don't distinguish between a cut and any of its corresponding cutting hyperplanes. In fact, for a given cut H, the two open halfspaces related to its hyperplane will be denoted by H^+ and H^-. This notational abuse will let us to color the end vertices of any edge in a cut by + and - according to their positions in H^+ and H^- respectively.
In a 3-dimensional cube a cut can only include 3, 4, 5, or 6 edges. As a matter of fact Figure 1 shows all the possible cuts for the 3-cube. The notation $4'$ refers to the cut consisting of 4 parallel edges, say parallel 4-cut, and the other one is called an angular 4-cut. An angular 2-cut over any square is defined similarly, i.e., a 2-cut in a 2-cube is angular when two edges share a vertex.

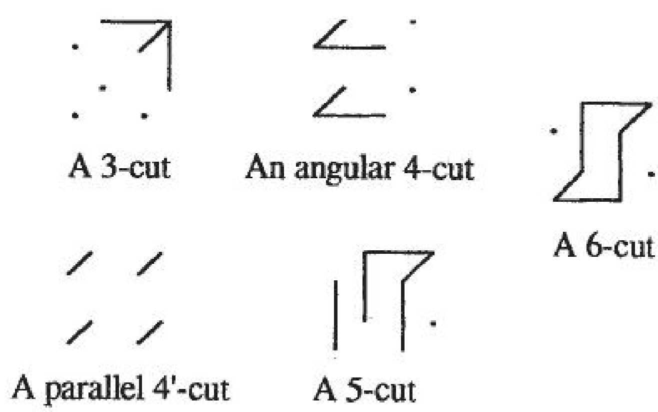

Fig. 1. All the non-isomorphic cuts of the 3-cube

It is well-known that three cuts are necessary and sufficient to cover all the edges of the 3-cube. Let $\{G, G', G''\}$ be a 3-covering for a given 3-cube, then the multiset $\{|G|, |G'|, |G''|\}$ is called an edge-number set for the corresponding covering. In the study of hypercube cut number problem, the set of 3-coverings and the corresponding list of edge-number sets have been a basic tool in the 4 and 5-dimensional cases and will continue to be fundamental for the higher dimensional cubes. The list of all edge number sets is $\{3,6,3\}$,$\{3,4,5\}$, $\{4,4,4\}$, $\{4,5,5\}$, $\{4,6,6\}$, $\{5,5,5\}$, $\{5,5,6\}$, $\{6,6,5\}$, and $\{6,6,6\}$, see [3,4,6]. In the list above, it is notable that the first two edge number sets $\{3,6,3\}$,$\{3,4,5\}$ are the only ones with a 3-cut, moreover, $\{4,4,4\}$, and $\{4,5,5\}$ are the only ones that do not have a 3-cut but contain an angular 4-cut. In the statement and proof of Theorem B, the only edge number set to be the main hypothesis is $\{4',4,4\}$ whose unique geometric characterization is drawn in Figure 2. The similar result for the other format of $\{4,4,4\}$, that is, $\{4',4',4'\}$ is much easier than Theorem B and its proof is omitted here.

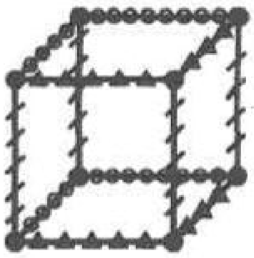

Fig. 2. The 3-covering for the edge number set $\{4', 4, 4\}$

3 The 5-cube Approach: A Summary

To obtain a noncomputational proof for $S(5) = 5$, we begin with a possible 4-covering $\{G, G', G'', H\}$ for the 5-cube and then we search for a contradicting edge that will be missed by all the cuts. We also assume that always there exists a 3-cube F_1 that is missed by the cut H, where the existence of a missing 3-cube is justified by a result from [3]. The following two theorems A and B cover three edge-number sets out of 9 that will complete a short proof for $S(5) = 5$, if they are extended similarly to all the edge number sets.

Theorem A-(The 3-cut Thm): Let $\{G, G', G'', H\}$ be a possible 4-covering for the 5-cube and let F_1 be a missing 3-cube for H. Then the edge number sets $\{3, 6, 3\}$ and $\{3, 4, 5\}$ can not be generated over F_1 by the remaining 3-covering $\{G, G', G''\}$.

Notice that $\{3, 6, 3\}$, and $\{3, 4, 5\}$ are the only edge number sets out of all the 3-coverings for a 3-cube with a 3-cut. Thus the theorem above is stating that for a possible 4-covering $\{G, G', G'', H\}$ of the 5-cube, if there is a missing 3-cube by G, then none of the other three cuts can be a 3-cut over this missing 3-cube. Theorem A and the original idea of the short proof can be found in [4]. Here we present a new simple proof for Theorem B that can be applied to prove similar results for the remaining edge number sets.

4 Cubical Terminologies & 3 Basic Lemmas

A *cubical rectangle* over the $d-$cube is any plane rectangle formed by two parallel edges, provided any usual square to be considered as a regular rectangle. By a *2-vertex coloring*, we mean a partial 2-vertex coloring on the $d-$cube that assigns a unique color of + or - to certain vertices of the hypercube.
In a 2-vertex coloring, an *alternating* cubical rectangle is the one whose four vertices are colored but does not contains any edge with vertices of the same color.
In other words, alternating cubical rectangles contain one diagonal with two +'s and the other one with two -'s. A *feasible coloring* is a 2-vertex coloring with no *alternating* cubical rectangle. For instance, consider any given hypercube cut K joint with the associated open half spaces K^+ and K^-. A natural assignment of colors to the vertices in K^+ by + and to the ones in K^- by -, clearly creates a feasible coloring, according to the convexity of K^+ and K^-.

An ordered pair is called a *coloring vector* if its components are + or -, that is, there are 4 possible coloring vectors: (+, +), (+, -), (-, +) (-, -). In a feasible 2-vertex coloring of the $d-$cube, by the lack of alternating cubical rectangles, the parallel edges with end vertices of different colors must have exactly the same coloring vectors, provided the edges have a convenient orietation.

In Figure 3, T and U form one pair of parallel facets of the 5-cube C^5, the same is true for F and $\overline{F}$. Those edges connecting T to U are called *horizontal edges* and they have orientation from left to right. The edges connecting F to $\overline{F}$ are called *vertical* ones and they are directed from up to down. In fact, the row coloring vectors (+, +), (+, -), (-, +), and (-, -), will assign colors only to horizontal edges and the column coloring vectors like $\binom{+}{+}$, $\binom{+}{-}$, $\binom{-}{+}$, and $\binom{-}{-}$ assign colors to vertical edges only. More precisely, the assignment of (+, -) to a horizontal edge E means that the left vertex of E in T will be colored + and the other vertex in U will be -. This coloring orientation of vertical and horizontal edges can be extended to all the edges in any direction, only the parallel edges must have the same orientation.

The following three lemmas are immediate consequences of feasibility of cut colorings:

Lemma 1 In any feasible coloring of a *cut* in the 5-cube, the coloring vectors assigned to all horizontal edges in the *cut* must be identical, either (+, -) or (-, +) only. The same holds for all vertical vectors in the cut, they must be one of the $\binom{+}{-}$ or $\binom{-}{+}$ only.

Lemma 2 Let $cub(x, y, z, w)$ and $cub(a, b, c, e)$ be two parallel squares of the 5-cube, where y and z are neighbors of x, and the parallelism defines a 1-1 correspondence φ between their vertices such that $a = \varphi(x)$, $b = \varphi(y)$, $c = \varphi(z), e = \varphi(w)$. Suppose for any cut K, $x \in K^+, y \in K^-, z \in K^-$ and $e \in K^+$. Then $w \in K^-$, $a \in K^+, b \in K^+, c \in K^+$.

Lemma 2 that can naturally be extended to higher dimensional cubes, just shows that the assigned colors of + for x, e and - for y, z will propagate the positive color to the vertices a, b, c, and also the negative color to w. In the next lemma by a diagonal of the 5-cube we mean the segment connecting two opposite vertices. Evidently, all the diagonals contain O, the center of the 5-cube, at the midpoint.

Lemma 3 (The Center Rule) For a given cut K if two opposite vertices are colored +, then the center $O \in K^+$, and any diagonal contains at least one end point vertex that is colored +.

The last lemma can also be explained by slightly generalizing the concept of cubical rectangles. In fact, a cubical rectangle can also be defined by any plane rectangle formed by four coplanar vertices of the d-cube.

For any subset of vertices S in the 5-cube suppose $cub(S)$, i.e., the cube hull of S, denotes the smallest cube containing S. To prove Lemma A and Theorem B, we will be following notations from Figure 3. In Figure 3, the 5-cube C^5 is the cube hull of four parallel 3-cubes F_1, F_2, $\overline{F_1}$ and $\overline{F_2}$. The following 3-cubes will also be highlighted later on. We define $U' = cub(a_2, b_2, c_2, e_2, \overline{a_2}, \overline{b_2}, \overline{c_2}, \overline{e_2})$, i.e., the higher facet of U. Also, $F' = cub(a_1, b_1, c_1, e_1, a_2, b_2, c_2, e_2)$ is the higher facet of F. The lower facet of the 4-cube T will be denoted by $T' = cub(d_1, f_1, g_1, h_1, \overline{d_1}, \overline{f_1}, \overline{g_1}, \overline{h_1})$.

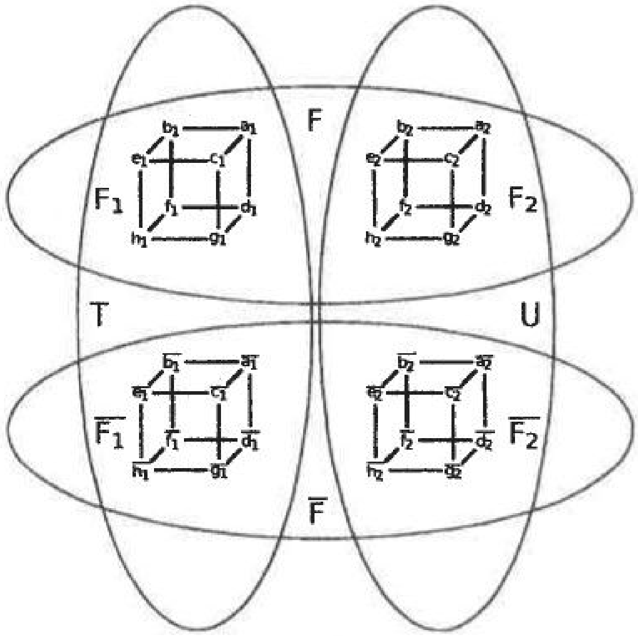

Fig. 3. A 5-cube is presented by 4 parallel 3-cubes F_1, F_2, $\overline{F_1}$ and $\overline{F_2}$

Theorem B Let $\{G, G', G'', H\}$ be a possible 4-covering for the 5-cube and let F_1 be a missing 3-cube for H. Then the edge number set $\{4', 4, 4\}$ can not be generated over F_1 by the remaining 3-covering $\{G, G', G''\}$.

To prove Theorem B, we need a lemma whose proof is based on the following fact over the cut G. For G and the 4'-cut over F_1, it is easy to see that the coloring vectors (+, -) together with $\binom{+}{-}$ lead to a missing facet for G in the 5-cube, as shown in the Figure 3. The latter is impossible since $S(4) = 4$ and the missing 4-cube can not be covered by the remaining three cuts, see Figure 4. The exact same situation occurs when $\binom{-}{+}$ comes together with (-, +). Hence in the next entire proof the coloring vectors of (-, +) joint with $\binom{+}{-}$ will be fixed for the cut G.

Lemma A Let $\{G, G', G'', H\}$ be a possible 4-covering for the 5-cube and let F_1 be a missing 3-cube for H, say $F_1 \subset H^+$. If the edge number set $\{4', 4, 4\}$ is generated over F_1 by the remaining 3-covering $\{G, G', G''\}$, then $O \in H^-$.

Proof The 3-cube F_1 is missed by H, and so must be covered by the three remaining cuts G, G', and G''. We consider the edge number set $\{4', 4, 4\}$ over F_1, by fixing a 4'-cut for G, a 4-cut for G' and a 4-cut for G''.

More precisely, as illustrated in Figure 4, we assume that:

$\{a_1, b_1, c_1, e_1\} \subset G^+$, $\{d_1, f_1, g_1, h_1\} \subset G^-$, $\{a_1, b_1, e_1, d_1, f_1, h_1\} \subset G'^+$,

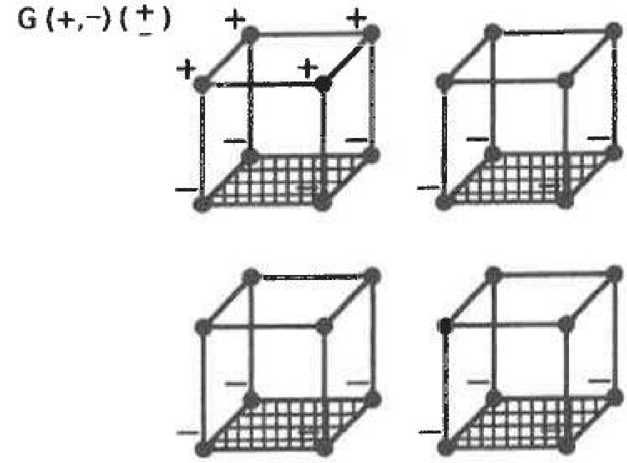

Fig. 4. A missing facet for the cut G, when (+, -) and $\binom{+}{-}$ is chosen

$\{c_1, g_1\} \subset G'^{-}$, $\{a_1, c_1, e_1, d_1, g_1, h_1\} \subset G''^{+}$, and $\{b_1, f_1\} \subset G''^{-}$.

Having fixed the horizontal and vertical coloring vectors $(-, +)$ and $\binom{+}{-}$ for G, then the coloring of G propagates to the neighboring cubes, applying horizontal and vertical coloring rule of Lemma 1. The latter leaves the 3-cubes F', and T' missed by G. We may also assume that the center of 5-cube $O \in G^{+}$ by symmetry, and so there is another 3-cube U' that will be missed by the cut G, by the center rule of Lemma 3. The following Figure 5 shows the fixed 3-covering $\{G, G', G''\}$ over F_1, the coloring of the missing face F_1 for H, and all the three new missing 3-cubes by G, after propagation of colors.

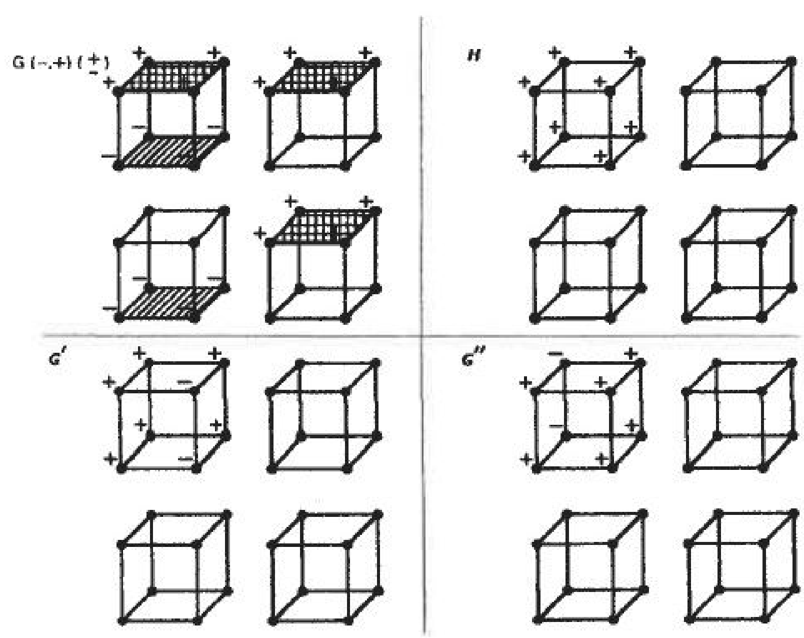

Fig. 5. Fixed 3-covering $\{G, G', G''\}$ over F_1, coloring of F_1 for H, and 3 new missing 3-cubes, F', T', U' by G, after propagation

To prove that $O \in H^-$, we recall that the 3-cubes F' and T' are missed by G. Therefore in H, each of the two squares $cub(a_2, b_2, c_2, e_2)$ and $cub(\overline{d_1}, \overline{f_1}, \overline{g_1}, \overline{h_1})$. must have at least 2 negative colors by the 3-cut Theorem and the fact that $S(3) = 3$. If one of the squares had at least three negatives and the other at least 2, then at least two negative vertices would be opposites; that is, $O \in H^-$. The only case left is that each square has exactly two neighboring negative vertices and no two of them are opposite. In this case, the positive vertices (and also the negative ones) must form corresponding parallel edges from the two squares $cub(a_2, b_2, c_2, e_2)$ and $cub(\overline{d_1}, \overline{f_1}, \overline{g_1}, \overline{h_1})$. We claim that this case can not occur, and so $O \in H^-$. Let $c_2, e_2, \overline{g_1}, \overline{h_1} \in H^+$ and $a_2, b_2, \overline{d_1}, \overline{f_1} \in H^-$ without loss of generality, see Figure 6.

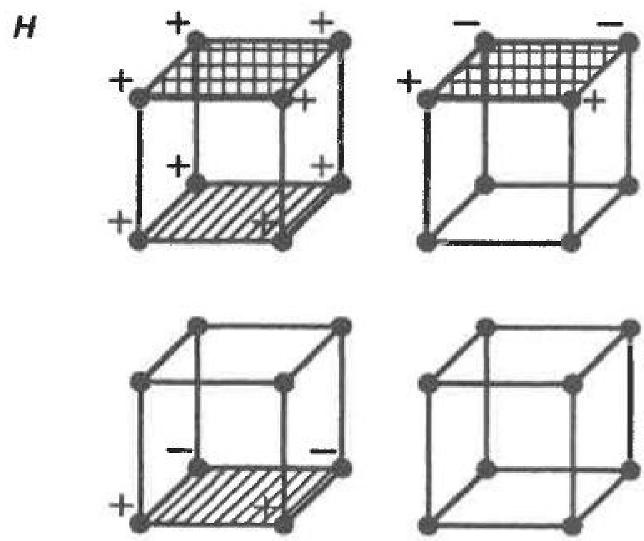

Fig. 6. The opposite squares in the 5-cube with exactly two "+" and exactly two "-" for the cut H

The latter creates angular 4-cuts for H in the 3-cubes F' and T'. So, the only possible edge number sets over these cubes are $\{4', 4, 4\}$ or $\{4, 5, 5\}$.

On the other hand, the edge number set $\{4', 4, 4\}$ is impossible, otherwise one of the G' or G'' would have a $4'$-cut over the 3-cube F' (or T'). The latter forbids G' or G'' to have an angular 2-cut over the square $cub(a_1, b_1, c_1, e_1)$ (or $cub(d_1, f_1, g_1, h_1)$), a contradiction. Thus the only acceptable edge number set over the 3-cube F' (or T') is $\{4, 5, 5\}$.

An edge number set $\{4, 5, 5\}$ implies that both G' and G'' generate 5-cuts over F' (or T'). The horizontal and vertical coloring vectors cannot be (-,+) or $\binom{-}{+}$ for any of the two cuts G' or G'', otherwise propagation of colors, applying Lemma 1, would produce at least 6 positive colors that contradicts to have a 5-cut. Thus we must use horizontal and vertical coloring $(+, -)$ and $\binom{+}{-}$ for both G' and G'', that implies the newly propagating colors are negative ones.

The edges $c_1 - c_2$ and $g_1 - \bar{g_1}$ are avoided by G, G', and H. Therefore they must be in G''. Thus $c_2, \bar{g_1} \in G''^-$. The coloring vectors in G'', applying Lemma 2, forcing the nodes $e_2, a_2 \in G''^-$ and consequently $\bar{c_2}, \bar{a_2} \in G''^-, \bar{e_2}, \bar{b_2} \in G''^-$, considering Lemma 1 and the given coloring vector $\binom{+}{-}$. Therefore U', the higher facet of U, would be missing by G'', that is impossible, since it is already missed by G. Thus the claim holds and $O \in H^-$ □

Proof of Theorem B We fix the covering $\{G, G', G''\}$ with the edge number set $\{4', 4, 4\}$ over F_1, the missing 3-cube for H, as before. Again, the coloring vectors of (-, +) and $\binom{+}{-}$ will be assumed for the cut G and, as a result, the horizontal and vertical colorings propagate colors to F_2 and $\bar{F_1}$ for G. Notice that $O \in H^-$ by the first part, therefore, $\bar{F_2} \subset H^-$ by the central rule of Lemma 3. We shall again assume that $O \in G^+$ by the symmetry. Hence, $\{\bar{a_2}, \bar{b_2}, \bar{c_2}, \bar{e_2}\} \subset G^+$, also by Lemma 3. The missing 3-cubes by G are highlighted and shown in Figure 5.

For the cut H, 3 or 4 of + colors over $cub(a_2, b_2, c_2, e_2)$ would either imply a 3-cut over F', or F' will be missed by H. Similarly, coloring three or more negatives over $cub(a_2, b_2, c_2, e_2) \subset U'$ is impossible. Thus, exactly two adjacent nodes of $\{e_2, b_2, a_2, c_2\}$ are positive (and 2 negative).

Therefore the highlighted 3-cube F' can only accept the edge number set $\{4', 4, 4\}$ or $\{4, 5, 5\}$. The colorings of G' and G'' again reject a $4'-$ cut, therefore the only choice remains to be $\{4, 5, 5\}$, i.e. G' and G'' must both provide a 5-cut over F'. Moreover, the horizontal coloring of G' and G'' must be $(+, -)$, otherwise they would not create 5-cuts. Therefore, we assume that $(+, -)$ to be the coloring for both of them.

Now, we claim that $cub(a_2, b_2, c_2, e_2)$ can not be all negative by G' or G''. Otherwise, suppose $\{a_2, b_2, c_2, e_2\} \subset G''^-$. First, let $O \in G''^+$, then $cub(\overline{d_1}, \overline{f_1}, \overline{g_1}, \overline{h_1}) \subset G''^+$, by Lemma 3. Consequently, there would be a 3-cut in T', a contradiction by the 3-cut Theorem. Second, let $O \in G''^-$. Then $\overline{a_2}, \overline{b_2}, \overline{e_2}) \subset G''^-$, by Lemma 3. Thus, there would be a 3-cut in U' that contradicts the 3-cut Theorem.

On the other hand, if $b_2 \in G'^-$, then $e_2, a_2, c_2 \in G'^-$ by Lemma 2, that is impossible by the argument above. Therefore, $b_2 \in G'^+$. By exactly the same similar argument, $c_2 \in G''^+$. Finally, $b_1 - b_2$ and $c_1 - c_2$ are avoided by G, G', G''. Therefore, $b_2, c_2 \in H^-$, which contradicts the original condition that exactly two adjacent nodes in $\{a_2, b_2, c_2, e_2\}$ must be positive. □

As the final note, we should recall that $\{3,6,3\}$, $\{3,4,5\}$ and $\{4,4,4\}$ are the only edge number sets with no overlaps and that they exactly cover 12 edges. The later fact has been essential part in the proof of $S(4) = 4$.

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