Changes in Genus Ranges of 4-Regular Graphs by Insertions of Certain Subgraphs

Hayden Hunter   Nataša Jonoska   Masahico Saito
Department of Mathematics and Statistics
University of South Florida

Abstract

A rigid vertex is a vertex with a prescribed cyclic order of its incident edges. An embedding of a rigid vertex graph preserves such a cyclic order in the surface at every vertex. A cellular embedding of a graph has the complementary regions homeomorphic to open disks. The genus range of a 4-regular rigid vertex graph Γ is the set of genera of closed surfaces that Γ can be cellularly embedded into. Inspired by models of DNA rearrangements, we study the change in the genus range of a graph Γ after the insertion of subgraph structures that correspond to intertwining two edges. We show that such insertions can increase the genus at most by 2, and decrease by at most by 1, regardless of the number of new vertices inserted.

1 Introduction

A rigid vertex is a vertex with a prescribed cyclic order of its incident edges [2]. An embedding of a rigid vertex graph in a surface preserves such a cyclic order in the surface at every vertex. If a general graph Γ seen as a 1-complex is embedded into a closed surface F, then the complement of Γ forms a family of regions. If each region is homeomorphic to an open disk, we say that Γ is cellularly embedded into this closed surface F [6]. The genus range of a 4-regular rigid vertex graph Γ is the set of genera of closed surfaces that Γ can be cellularly embedded into [2]. It was shown in [2] that the genus range of a 4-regular graph differs from the genus range of the same graph when the vertices are considered rigid.

To each closed surface embedding of a graph one can consider a neighborhood of the graph within the surface. This neighborhood forms a ribbon graph (see next section for details). Conversely, from any ribbon graph associated to a graph Γ a cellular embedding of Γ in a closed surface can be obtained. The genus of an embedding is therefore computed from the

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number of boundary components of a ribbon graph using the Euler characteristic formula. Each 4-valent rigid vertex in a ribbon graph may achieve one of two different connectivity of the boundary components at that vertex. Hence, for a graph of n rigid 4-valent vertices, there are up to 2^n constructions of ribbon graphs. The genus range, i.e., the set of all possible surface genera where a graph can be embedded, is obtained through the number of boundary components of all possible ribbon graphs.

A word w over an alphabet Σ is a *double occurrence word* (DOW) if each symbol in Σ appears in w precisely 0 or 2 times. Such words are also known as unsigned Gauss codes. Each DOW has a corresponding 4-regular rigid vertex graph Γ where $V(\Gamma)$ consists of the symbols within the DOW w and $E(\Gamma)$ are determined by the adjacent symbols in w [5, 2].

Let u be a word with distinct symbols, a *single occurrence word* (SOW), such that a DOW w and u share no symbol. We consider the situations when u is inserted in two different locations of w, or, when u and its reverse u^R are inserted in w thereby constructing another DOW. The former case is called a *repeat insertion*, while the latter case is called *return insertion* [5].

The DOWs and the particular repeat/return insertions are motivated from models of DNA recombination [1]. Scrambled genes in certain ciliate species can be represented with DOWs and the gene rearrangement process can be modeled through removal of subwords in these DOWs [1, 3]. In this case, the 4-valent rigid vertex graphs associated with the corresponding DOWs can be seen as the spatial arrangement of the DNA during the recombinant processes. It has been observed that most prevalent scrambling (over 90% of the scrambled genes) in the genome of some species can be described through repeat/return insertions in DOWs [4]. Therefore, changes of properties of graphs associated with DOWs through repeat/return insertions have been studied [5, 7].

In this paper, we study changes in the genus ranges of 4-valent rigid vertex graphs that correspond to repeat/return insertions in their corresponding DOWs. We observe that such insertions (regardless of the number of symbols) change the genus of an associated ribbon graph by ρ where $\rho \in \{-1, 0, 1, 2\}$. If the inserted word has even number of symbols, then $\rho \neq -1$. Since removal of repeat/return subwords in a DOWs can correspond to a DNA rearrangement step, we consider the relationship between the genus range of a DOW minimum number of steps (called the nesting index) needed to reduce a given DOW to the empty word by a sequence of deletions of repeat and return subwords. Although the maximum genus range is bounded by twice the nesting index, we conjecture that this bound is not achieved.

2 Preliminaries

2.1 Graphs and Words

A *graph* $\Gamma = (V, E, \eta)$ is a 3-tuple where $V = V(\Gamma)$ is the set of *vertices*, $E = E(\Gamma)$ is the set of *edges* and $\eta : E \to \{\{u, v\} \mid u, v \in V(\Gamma)\}$ such that for $e \in E$, $\eta(e) = \{u, v\}$ is the set of endpoints of E. In this paper all graphs are finite. A *directed graph* $\Gamma = (V, E, s, t)$ has a *source function* $s : E \to V$ and *target function* $t : E \to V$ where for $e \in E$, $s(e) = u$ indicates the source (start) of e and $t(e) = v$ is the target (end) of e. If there are no ambiguities we write $e = (u, v) = uv$. A *walk*, W, in an undirected graph Γ is a sequence of vertices and edges

$$W := (v_1, e_1, v_2, \dots, v_{n-1}, e_{n-1}, v_n)$$

where $\eta(e_i) = \{v_i, v_{i+1}\}$ for all $i \leq n - 1$. In the case of a directed graph, the walk satisfies $s(e_i) = t(e_{i-1}) = v_i$ for $2 \leq i \leq n$. A graph is *connected* if for all $u, v \in V$, there exists a walk from u to v. We say that a directed graph is *strongly connected* if for any pair of vertices $u, v \in V$, there exists a walk from u to v and a walk from v to u. For a graph Γ, an *Eulerian circuit* is a walk $W = (v_1, e_1, v_2, \dots, v_{n-1}, e_{n-1}, v_n)$ where each $e \in E(\Gamma)$ appears exactly once in W and $v_1 = v_n$.

A closed surface S is a compact surface without boundary. A closed orientable surface S is either the sphere S^2 or the connected sum, $S = \#_g T$, of g copies of the torus T where g is called the *genus* of S (the genus of S^2 is 0). For the remainder of the paper all surfaces are orientable.

A graph Γ can be embedded into a closed surface S as a 1-complex. The set $S - \Gamma$ forms a family of regions. We call the embedding of Γ into S a *cellular embedding* if each region in $S - \Gamma$ is homeomorphic to an open disk.

A (finite) set Σ is called an *alphabet* where elements of Σ are called *symbols*. If $a_0, \dots, a_n \in \Sigma$ then the concatenation $w = a_0 \cdots a_n$ forms a *word* over Σ and we say that the symbols a_i ($i = 1, \dots, n$) appear in w. We denote the set of symbols in a word w as $\Sigma_{[w]} = \{a \in \Sigma \mid a \text{ appears in } w\}$. We denote the set of all non-empty words over Σ as Σ^+. The empty word ε is a word that contains no symbols. Then $\Sigma^* = \Sigma^+ \cup \{\varepsilon\}$. If $w = u_1 v u_2$ so that $u_1, v, u_2 \in \Sigma^*$, we say that v is a subword of w which we denote as $v \sqsubseteq w$. If $w = a_0 \cdots a_n$ then the *reverse* of w is $w^R = a_n \cdots a_0$. Two words $u = a_1 \cdots a_n$ and $v = b_1 \cdots b_n$ are *equivalent*, or $u \sim v$, if there exists a bijection $f : \Sigma_{[u]} \to \Sigma_{[v]}$ such that $f(a_1) \cdots f(a_n) = b_1 \cdots b_n$.

2.2 Double Occurrence Words and Assembly Graphs

In this subsection we recall definitions and set notations for double occurrence words and assembly graphs [3].

Definition 1 (Double occurrence words). We say that $w \in \Sigma^*$ is a *double occurrence word* (DOW) if for every $a \in \Sigma$, a appears in w 0 or 2 times.

The set of DOWs is denoted as Σ_{DOW}. For example, the word 121323 is in Σ_{DOW}. For the remainder of the paper, we set $\Sigma = \mathbb{N}$.

For a rigid vertex v with $\deg(v) = 4$, we label its incident vertices e_1, e_2, e_3, e_4. We consider the equivalence relation among orders of these edges generated by

$$(e_{i+3}, e_{i+2}, e_{i+1}, e_i) \sim (e_i, e_{i+1}, e_{i+2}, e_{i+3}) \sim (e_{i+1}, e_{i+2}, e_{i+3}, e_i)$$

where the subscript addition is mod 4.

Definition 2 (Rigid vertex). A *rigid vertex* is a vertex with a prescribed cyclic order of its incident edges up to this equivalence relation.

A visual representation for a rigid vertex is shown in Figure 1. Consecutive edges within the cyclic orientation are called *neighbors*. As depicted in Figure 1, neighbors are separated by an angle of "90 degrees" while non-neighbors are separated by an angle of "180 degrees". In particular, in Figure 1, e_2 and e_4 are neighbors of e_1 and e_3 is a non-neighbor of e_1 [3]. An embedding of a rigid vertex graph in a surface is required to preserve a given cyclic order of adjacent edges at every vertex, so that a neighborhood of every vertex on the surface is as depicted in Figure 1.

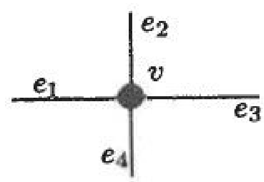


Figure 1: 4-valent rigid vertex

Definition 3 (Transverse Eulerian circuit). A *transverse Eulerian circuit* in a 4-valent rigid vertex graph is an Eulerian circuit where every two consecutive edges in the circuit are non-neighbors.

Example 1. The graph in Figure 2 has no transverse Eulerian circuit while the graph in Figure 3 does.

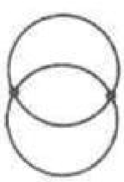


Figure 2: A 4-valent rigid vertex graph with no transverse Eulerian circuit

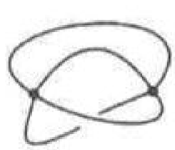


Figure 3: A 4-valent rigid vertex graph with a transverse Eulerian circuit

Definition 4 (Assembly graph). An *assembly graph* is a 4-regular rigid vertex graph with a transverse Eulerian circuit.

Each double occurrence word $w = a_0 \cdots a_n$ has a corresponding assembly graph $\Gamma = (\Sigma_{[w]}, E)$. The 4-valent rigid vertices of the assembly graph are $a_i \in \Sigma_{[w]}$ and the Eulerian circuit is determined by w such that the ith edge in the circuit has end-points $\eta(e_i) = \{a_i, a_{i+1}\}$ ($i+1$ is taken $\bmod n$). Because each vertex appears twice in the circuit, every vertex has four incident edges, and the consecutive edges in the circuit indicate the non-neighboring edges. The transverse Eulerian circuit also gives an orientation of Γ such that for $e \in E(\Gamma)$, $s(e) = a_i$ and $t(e) = a_{i+1}$ for some $i \in \{0, \dots, n\}$. More precisely, given a vertex $v \in \Sigma_{[w]}$ there exists $a_i, a_j \in \Sigma_{[w]}$ such that $i \neq j$ and $a_i = a_j = v$. Then the prescribed cyclic order of the edges incident to v is $(a_{i-1}a_i, a_{j-1}a_j, a_ia_{i+1}, a_ja_{j+1})$, so that Γ has a transverse Eulerian circuit [3]. Conversely, if a 4-regular rigid vertex graph has an Eulerian circuit, then listing the order in which the vertices appear in the circuit forms a double occurrence word.

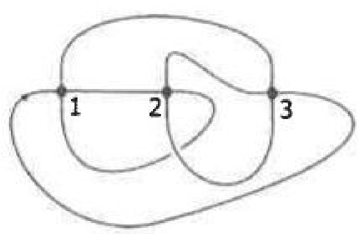


Figure 4: An assembly graph corresponding to 121323

Example 2. The corresponding assembly graph for the DOW 121323 is $\Gamma = (\{121323\}, E)$ where $E = \{12, 21, 13, 32, 23, 31\}$. A representation for the assembly graph of 121323 is shown in Figure 4.

2.3 Ribbon Graphs and Genus Ranges

Let Γ be an assembly graph. For each rigid vertex v in Γ, construct a square with v at its center. For each edge incident to v and w, we thicken the edge to construct a ribbon connecting the squares such that the edge is a centerline of the ribbon as shown in Figure 5. Ribbons are connected to squares in such a way that the resulting surface is orientable. We call this compact surface with boundary a *ribbon graph* of Γ. The set of all ribbon graphs of Γ is denoted $\mathcal{R}_\Gamma$. By attaching 2-cells (disks) along all boundary circles, we obtain a closed surface S in which the assembly graph Γ is cellularly embedded. Conversely, any cellular embedding of Γ in a closed surface S can be constructed in this way, and a regular (thin) neighborhood of Γ in S can be regarded as a ribbon graph of Γ.

Each vertex square has two possible ways to connect as demonstrated in Figure 7. The first is represented in Figure 6 where boundary connections are depicted as in Figure 7(A). We note that the ribbon graph is orientable

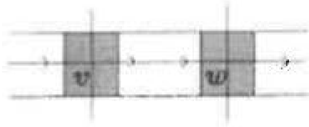


Figure 5: Connecting squares by ribbons [2]

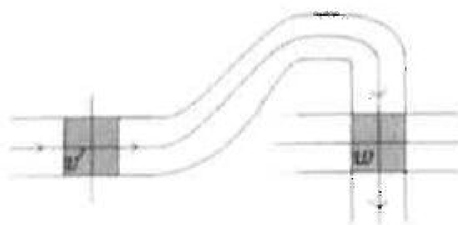


Figure 6: Visiting a square twice [2]

and the boundary components are oppositely oriented along the ribbons and without loss of generality we choose the orientations as depicted in Figure 7(A). The second way of attaching the ribbon is touching the band from bottom to top (cf. Figure 7(A) vs. Figure 7(B)). The boundary connections of Figure 7(B) is represented by Figure 7(C). For an assembly graph Γ of n vertices we have up to 2^n possible ribbon graphs and thus 2^n possible embeddings. Due to symmetry we can consider only 2^{n-1} cases. The ways that the embeddings of Γ change correspond to the way a ribbon connects to a vertex square in the corresponding ribbon graph, and the change of the boundary components of the ribbon graph R can be represented as in Figure 7. The boundary connection as represented in Figure 7(A) is called *type 1 vertex configuration* and the boundary connection in Figure 7(C) is *type 2 vertex configuration.*

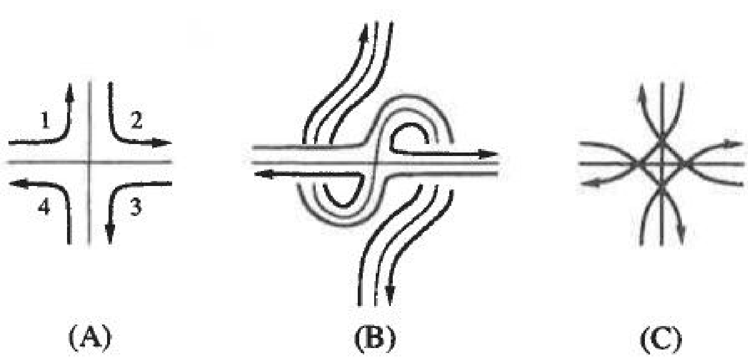


Figure 7: Boundary connection change at a vertex [2]. (A) *type 1* vertex configuration and (C) *type 2* vertex configuration

For an assembly graph Γ embedded in a closed surface S, with neighborhood ribbon graph $R \in \mathcal{R}_\Gamma$, the number of boundary components of R is denoted by $b(R)$. Using the Euler characteristic for the closed surface $\chi(S) = |V| - |E| + b(R)$, we calculate the genus $g(R) = g(S) = \frac{1}{2}(2 - \chi(S))$.

For an assembly graph with n vertices (and $2n$ edges) we have

$$\begin{aligned} g(S) &= \frac{1}{2}[2-(|V|-|E|+b(R))] \\ &= \frac{1}{2}[2-(n-2n+b(R))] = \frac{1}{2}[2+n-b(R)] \quad (1) \end{aligned}$$

The set of integers that represent the genera of the closed surfaces where an assembly graph Γ is embedded is the *genus range* of Γ, denoted $GR(\Gamma)$ [2].

For each boundary connection change at a vertex of Γ, the total number of boundary components in Γ can either increase by 2, decrease by 2, or remain the same as represented in Figure 8.

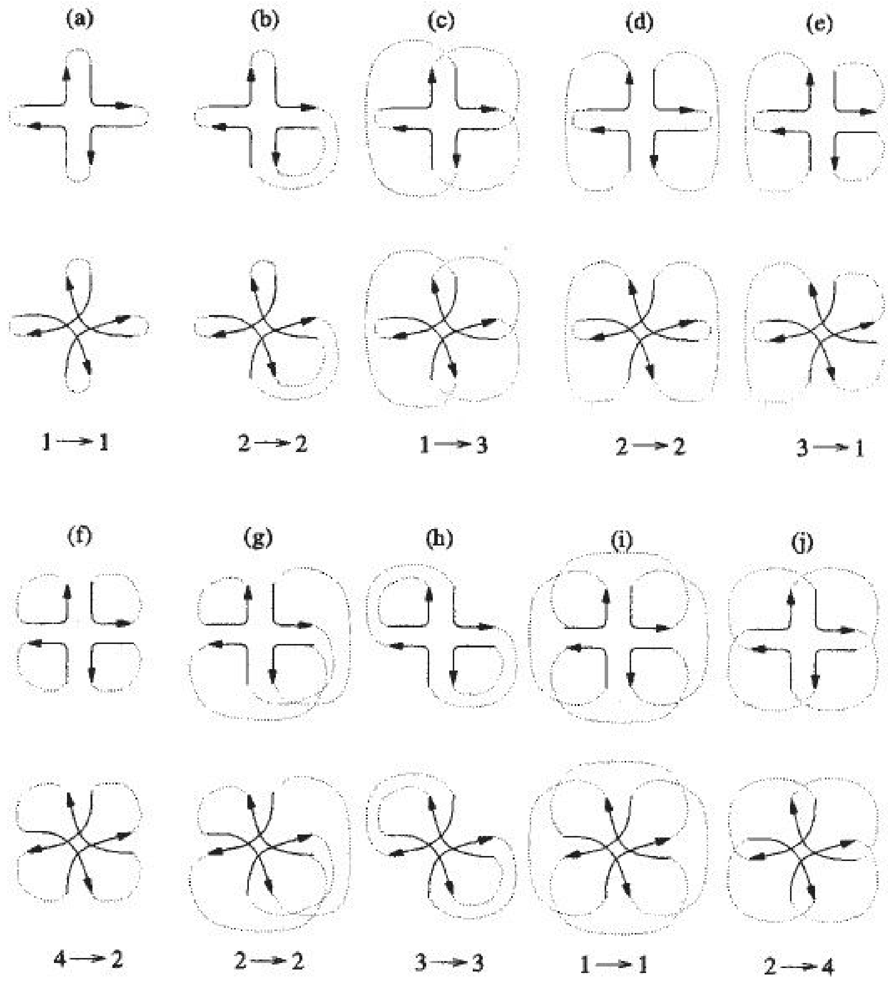


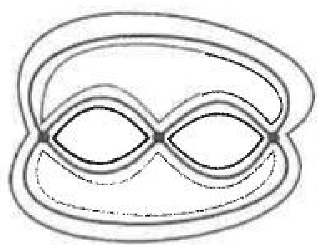
Figure 8: All possible boundary connection changes (figure from [2])

It is known that the genus range of an assembly graph consists of consecutive integers so that the genus range $GR(\Gamma)$ can be represented as a closed interval of integers $[a,b]$ where $a = \min GR(\Gamma)$ and $b = \max GR(\Gamma)$ [2].

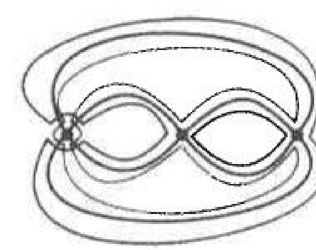
3 Genus Ranges of Assembly Graphs and Repeat/Return Insertions

In this section we consider assembly graphs and their genus ranges, in particular, how genus ranges change with insertion of repeat or return words. The assembly graph corresponding to the DOW w is denoted with Γ_w. We use the notation $\mathcal{R}_w$ instead of $\mathcal{R}_{\Gamma_w}$.

Example 3. Consider the assembly graph Γ_{123123} and the following two ribbon graphs $R, R' \in \mathcal{R}_{123123}$ as depicted in Figure 9a and Figure 9b. The genus of each ribbon graph may not necessarily be the same. As we can see in Figures 9a and 9b, $b(R) = 5$ while $b(R') = 3$. Thus we have $g(R) = 0 \neq 1 = g(R')$. Changing the types of vertex configurations at every vertex, one can show that $GR(\Gamma_{123123}) = [0, 1]$.



(a) R



(b) R'

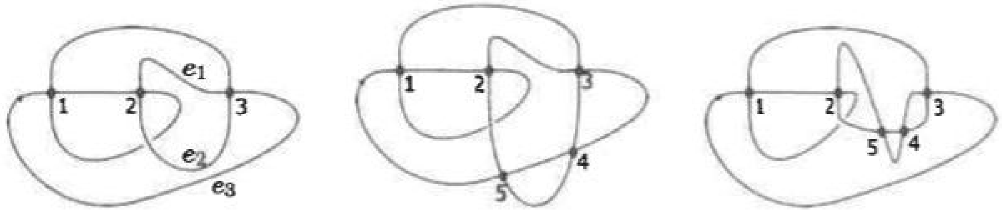
Figure 9: Γ_{123123} and two of its ribbon graphs R, R'

Properties of repeat and return insertions on DOWs have been studied to determine relationships between DOWs [5, 7]. We observe how repeat/return insertions of DOWs affect the genus ranges of the assembly graphs that correspond to these DOWs.

Definition 5 (Repeat and return insertions). Suppose $u, v, w \in \Sigma^*$ such that $uvw \in \Sigma_{DOW}$ For the word where $x \in \Sigma^+$ such that $uxvxw \in \Sigma_{DOW}$, we say that $uxvxw$ is a *repeat insertion* of uvw and for $uxvx^Rw$ we say it is a *return insertion* of uvw.

Example 4. Let $uvw = 121323 = uv'w'$ with $u = 1213$, $v = 23$, $v' = 2$, $w = \varepsilon$, and $w' = 3$. Let $x = 45$. Then $1213452345 = uxvxw$ is a repeat insertion of x and $1213452543 = uxv'x^Rw'$ is a return insertion of x in 121323. The corresponding graphs are depicted in Figures 10a, 10b, and 10c.

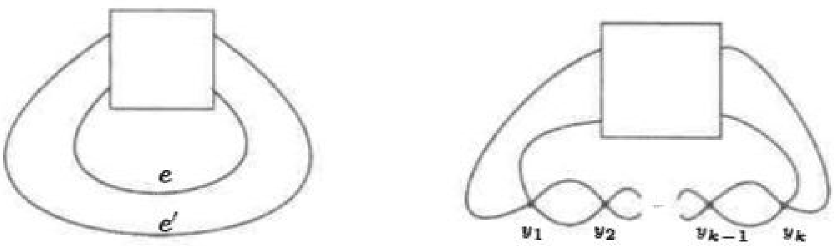
The repeat/return insertions definition is adapted from the definition in [5]. Now we look at how the properties of repeat and return insertions on DOWs affect their corresponding assembly graphs and genus ranges.



(a) Γ_{121323} (b) $\Gamma_{1213452345}$, 2-interlace of e_2 and e_3 (c) $\Gamma_{1213452543}$, 2-interlace of e_1 and e_2

Figure 10: Crossing edges and adding vertices 4 and 5 to construct 10b and 10c

Let $\Gamma = (V, E)$ be an assembly graph. A *k-interlace* between edges $e, e' \in E$ is the process of "intertwining" e and e' so that they cross k times, and adding vertices to these k crossings. A visual interpretation of a k-interlace of e and e' is represented in Figure 11.



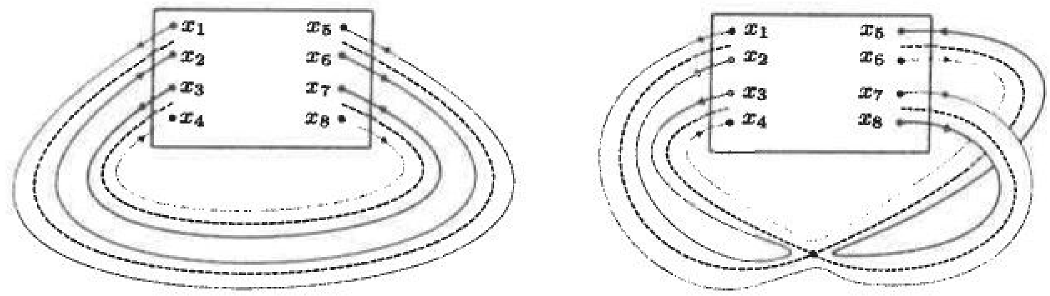
(a) Γ (b) Γ'

Figure 11: k-interlace of $e, e' \in E(\Gamma)$

A k-interlace between two edges represents the changes of the assembly graph by insertion of either a repeat and or a return insertions of the corresponding DOW. Let $uvw \in \Sigma_{DOW}$, let $y = y_1 \dots y_k$ be a SOW and $y' \in \{y, y^R\}$ be *SOW*s so that $uyvy'w \in \Sigma_{DOW}$. We can represent the assembly graph of the word uvw as in Figure 11a and the assembly graph of the word $uyvy'w$ as Figure 11b where $y_1, \dots, y_k$ are the newly added vertices to the k-interlace of e and e'. The graph Γ_{121323} is depicted in Figures 10a, and consider three of its edges e_1, e_2 and e_3. The 2-interlace of e_2 and e_3 corresponds to the repeat insertion of 45 as shown in 10b; and 2-interlace of e_1 and e_2 corresponds to the return insertion of 45 as shown in 10c.

We consider the changes of the genus of an arbitrary assembly graph Γ obtained by the k-interlace of two edges $e, e' \in E(\Gamma)$ for $k \geq 1$.

Definition 6. Let Γ' be obtained from Γ by a k-interlace of edges $e, e' \in E(\Gamma)$. For $R' \in \mathcal{R}_{\Gamma'}$, we say that it is an *extension* of $R \in \mathcal{R}_\Gamma$ if for every vertex of Γ, the vertex type configuration is the same at both R and R'.



(a) R (b) R'

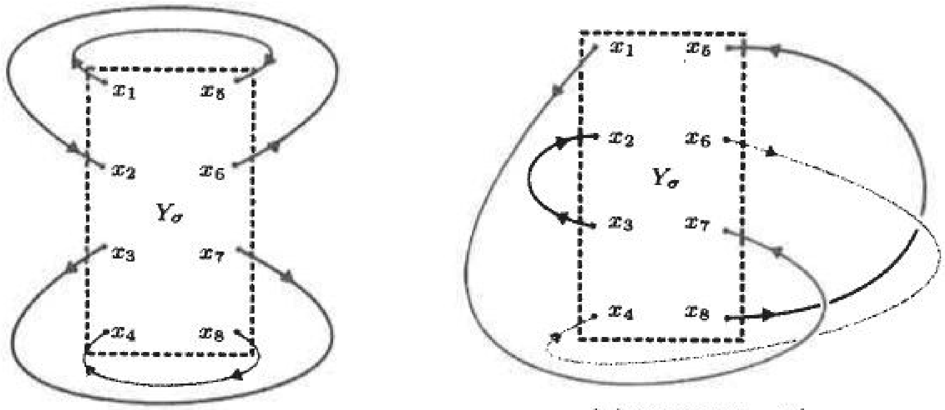
Figure 12: Extension of R' from R with type 1 configuruation at y

Consider $R \in \mathcal{R}_\Gamma$ and let $X = \{x_1, x_2, \ldots, x_8\}$ be eight points on the boundary components of R such that pairs of points in X belong to the same boundary component. Suppose the bijection $\tau : \{x_1, x_3, x_6, x_8\} \to \{x_2, x_4, x_5, x_7\}$ is such that $Y_\tau = \{(x_i, \tau(x_i)) | i = 1, 3, 6, 8\}$ indicates the ordered pairs of points belonging to the same components (indicating the orientation of the boundaries), two examples of Y_τ are presented in Figures 12a and 12b. Let $\sigma : \{x_2, x_4, x_5, x_7\} \to \{x_1, x_3, x_6, x_8\}$ be a bijection such that $Y_\sigma = \{(x_i, \sigma(x_i)) | i = 2, 4, 5, 7\}$ indicates the pairs of points that belong to the same boundary component in the remaining portion of the boundary curves. The directed graph $G_{(\tau,\sigma)} = (X, Y = Y_\tau \cup Y_\sigma)$ is called the *X-connection graph*, or X-CG of $R \in \mathcal{R}_\Gamma$. This graph is comprised of one, two, three or four disjoint cycles.

Let Γ' be obtained from Γ by a k-interlace of edges $e, e' \in E(\Gamma)$ and let $R' \in \mathcal{R}_{\Gamma'}$ be an extension of R. Then the X-CG of R' can be defined as $G_{(\tau',\sigma)}$ as the boundary connections for each vertex $v \in \Gamma$ are unchanged in R'. An example of these X-CG graphs are depicted in Figures 13a and 13b which are based on the ribbon graphs in Figures 12a and 12b. The edges defined with Y_σ remain the same in R' as they are in R.

Theorem 1. *Let $\Gamma = (V, E)$ be the assembly graph, $e, e' \in E$, and let Γ' be obtained from Γ by a 1-interlace of e and e'. Fix a ribbon graph $R \in \mathcal{R}_\Gamma$ and let $R' \in \mathcal{R}_{\Gamma'}$ be an extension of R. Then $g(R') = g(R) + \rho$ where $\rho \in \{-1, 0, 1, 2\}$.*

Proof. Let y be the newly added vertex in the 1-interlace Γ'. There are two possible types of configurations of boundary components at y in R'.
Case 1: Type 1 configuration at y in R'. Because R is orientable, the boundary components on a ribbon graph have opposite orientation, and without loss of generality, we choose $X = \{x_1, \ldots, x_8\}$ and assume orientation as depicted in Figure 13a. This defines Y_τ with $\tau(x_1) = x_5$, $\tau(x_6) = x_2$, $\tau(x_3) = x_7$ and $\tau(x_8) = x_4$.

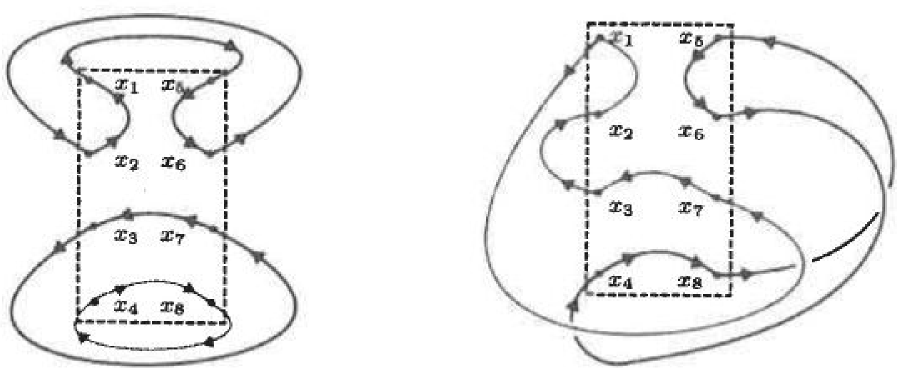


(a) X-CG for R (b) X-CG for R'

Figure 13: X-CGs for R and R'

We let $G_{(\tau,\sigma)}$ be the X-CG of R. If the configuration at the new vertex y is of type 1, the X-CG graphs of R' changes to $G_{(\tau',\sigma)}$ where $\tau'(x_1, x_3, x_6, x_8) = (x_7, x_2, x_4, x_5)$ as shown in Figures 13a and 13b. The rest of the connections of the boundary components are indicated with the same Y_σ for both R and R' and there are twenty-four possible cases for σ.

The change in the number of disjoint cycles that comprise $G_{(\tau,\sigma)}$ versus $G_{(\tau',\sigma)}$ gives the change in the number of boundary components between R and R'. Let $\mathcal{C}(G_{(\tau,\sigma)})$ be the total number of disjoint cycles of $G_{(\tau,\sigma)}$. Then we have $\mathcal{C}(G_{(\tau',\sigma)}) - \mathcal{C}(G_{(\tau,\sigma)}) = b(R') - b(R)$.



(a) R (b) R'

Figure 14: R and R' with fixed $Y_\sigma = \{\boldsymbol{x_2x_1, x_7x_3, x_5x_6, x_4x_8}\}$

The Table 1 contains all changes in the number of boundary components for the 24 choices of σ. The highlighted row for Y_σ is illustrated in Figure 14.

Y_σ	$C(G_{(\tau,\sigma)})$	$C(G_{(\tau',\sigma)})$	$b(R')-b(R)$
$\{x_2x_1, x_4x_3, x_5x_6, x_7x_8\}$	2	1	−1
$\{x_2x_1, x_4x_3, x_7x_6, x_5x_8\}$	1	2	1
$\{x_2x_1, x_5x_3, x_4x_6, x_7x_8\}$	1	2	1
$\{x_2x_1, x_5x_3, x_7x_6, x_4x_8\}$	2	1	−1
$\{x_2x_1, x_7x_3, x_4x_6, x_5x_8\}$	2	3	1
$\{x_2x_1, x_7x_3, x_5x_6, x_4x_8\}$	3	2	−1
$\{x_4x_1, x_2x_3, x_5x_6, x_7x_8\}$	1	2	1
$\{x_4x_1, x_2x_3, x_7x_6, x_5x_8\}$	2	3	1
$\{x_4x_1, x_5x_3, x_2x_6, x_7x_8\}$	2	1	−1
$\{x_4x_1, x_5x_3, x_7x_6, x_2x_8\}$	1	2	1
$\{x_4x_1, x_7x_3, x_2x_6, x_5x_8\}$	3	2	−1
$\{x_4x_1, x_7x_3, x_5x_6, x_2x_8\}$	2	1	−1
$\{x_5x_1, x_2x_3, x_4x_6, x_7x_8\}$	2	3	1
$\{x_5x_1, x_2x_3, x_7x_6, x_4x_8\}$	3	2	−1
$\{x_5x_1, x_4x_3, x_2x_6, x_7x_8\}$	3	2	−1
$\{x_5x_1, x_4x_3, x_7x_6, x_2x_8\}$	2	1	−1
$\{x_5x_1, x_7x_3, x_2x_6, x_4x_8\}$	4	1	−3
$\{x_5x_1, x_7x_3, x_4x_6, x_2x_8\}$	3	2	−1
$\{x_7x_1, x_2x_3, x_4x_6, x_5x_8\}$	1	4	3
$\{x_7x_1, x_2x_3, x_5x_6, x_4x_8\}$	2	3	1
$\{x_7x_1, x_4x_3, x_2x_6, x_5x_8\}$	2	3	1
$\{x_7x_1, x_4x_3, x_5x_6, x_2x_8\}$	1	2	1
$\{x_7x_1, x_5x_3, x_2x_6, x_4x_8\}$	3	2	−1
$\{x_7x_1, x_5x_3, x_4x_6, x_2x_8\}$	2	3	1

Table 1: Highlighted example in Figure 14

This computation was performed on *Mathematica* 11.0. The table indicates that the difference in boundary changes increases by 1 or 3 or decreases by 1 or 3, hence for Γ with n vertices we have

$$\begin{aligned} g(R') &= \frac{1}{2}(2 - b(R') + [n+1]) = \frac{1}{2}(2 - [b(R) + \delta] + [n+1]) \\ &= \frac{1}{2}(2 - b(R) + n) + \frac{1-\delta}{2} = g(R_0) + \frac{1-\delta}{2} = g(R) + \rho \qquad (2) \end{aligned}$$

where $\delta \in \{-1, -3, 1, 3\}$ implying $\rho \in \{-1, 0, 1, 2\}$.

Case 2: Type 2 configuration at y in R'. The ribbon graph R' and the X-CG of R' in this case are as depicted in Figures 15 and 16. We perform the same computations for the 24 cases of Y_σ in this case (see Appendix Table 2). These computations show that the difference in the number of boundary components in R and R' also increase by 1 or 3 or decrease by 1

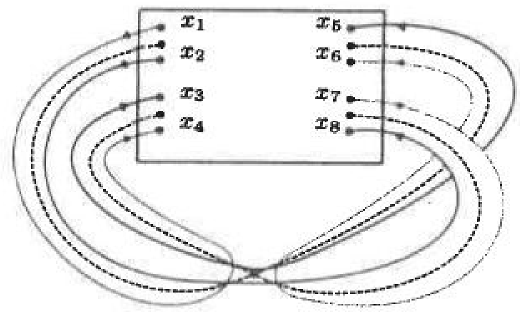


Figure 15: R'

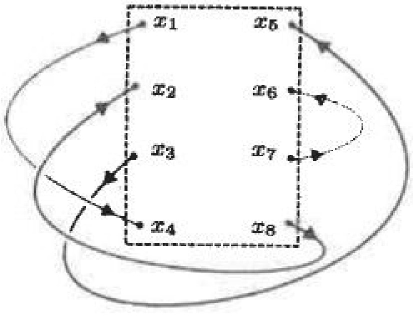


Figure 16: Exterior connection graph of R', $G_{(\tau',\sigma)}$

or 3, hence for a graph Γ with n vertices we obtain the same result as in Case 1. □

Let $\Gamma = \Gamma_w$ where $w = uv \in \Sigma_{DOW}$ and let $\Gamma' = \Gamma_{ubvb}$ be obtained by a 1-interlace of Γ. If $u, v \in \Sigma_{DOW}$ then we have a special case of Theorem 1. The assembly graph Γ_{ubvb} is called the *cross sum* of Γ_u and Γ_v which is discussed in [2]. The cross sum of Γ_u and Γ_v as shown in Figure 17. It has been shown in [2] that in this case that $GR(\Gamma_{ubvb}) = \{g_1 + g_2 \mid g_1 \in \Gamma_u,\ g_2 \in \Gamma_v\}$ holds.

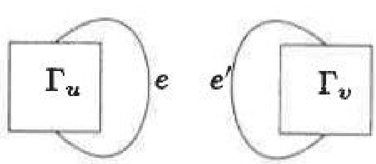


Figure 17: Γ_u,Γ_v

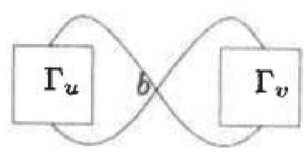


Figure 18: Cross sum of Γ_u,Γ_v

Theorem 2. *Let Γ be the assembly graph, $e, e' \in E(\Gamma)$, and Γ' be obtained by a 2-interlace of e and e'. Let $R \in \mathcal{R}_\Gamma$ and $R' \in \mathcal{R}_{\Gamma'}$ so that R' is an extension of R. Then $g(R') = g(R) + \rho$ where $\rho \in \{0, 1, 2\}$.*

Proof. Let y_1 and y_2 be the newly added vertices in Γ' obtained by the 2-interlace. Let R'_i be an extension of R for $i \in \{1, 2, 3\}$ where R'_1 has a type 1 configuration at both y_1 and y_2, R'_2 has a type 2 configuration at precisely one of the vertices y_1, y_2, and R'_3 has a type 2 configuration at both y_1 and y_2. The X-CGs for each of these ribbon graphs are shown in Figure 20 where the edges in Y_σ are the same in all three cases and are not depicted.

For the ribbon graphs R'_1 and R'_3 there exists precisely one boundary component that does not traverse the edges incident to vertices within Γ. Thus, in order to find $b(R'_j) - b(R)$ for $j \in \{1, 3\}$, we must add 1 to the total number of strongly connected components for R'_1 and R'_3's X-CGs

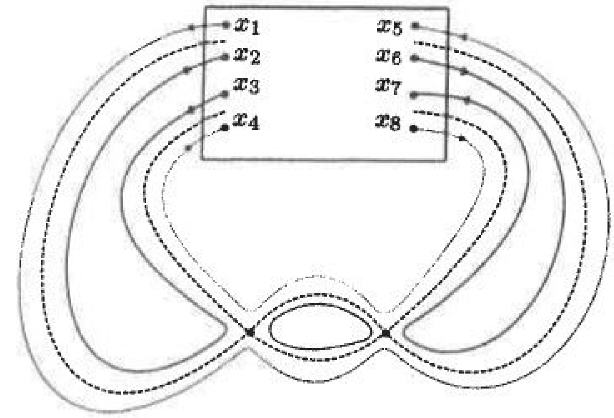
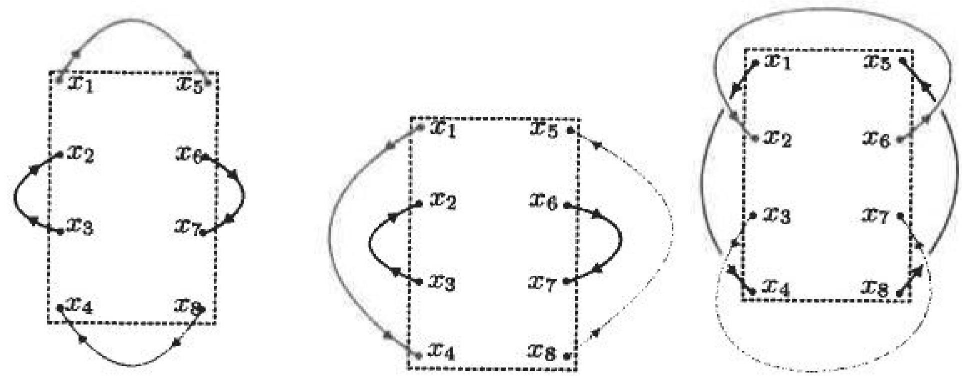


Figure 19: R'



(a) X-CG, $G_{(\tau'_1,\sigma)}$, of R'_1

(b) X-CG, $G_{(\tau'_2,\sigma)}$, of R'_2

(c) X-CG, $G_{(\tau'_3,\sigma)}$, of R'_3

Figure 20: X-CGs

so that $\mathcal{C}(G_{(\tau'_j,\sigma)}) + 1 - \mathcal{C}(G_{(\tau,\sigma)}) = b(R'_j) - b(R)$. The 24 differences in the total number of boundary components between R and its extension R' is computed by *Mathematica* 11.0, see Appendix Table 3. Thus the total number of boundary components changes by $\delta \in \{-2, 0, 2\}$.

$$
\begin{aligned}
g(R') &= \frac{1}{2}(2 - b(R') + [n+2]) = \frac{1}{2}(2 - [b(R) + \delta] + [n+2]) \\
&= \frac{1}{2}(2 - b(R) + n) + \frac{2-\delta}{2} = g(R) + \frac{1}{2}(2-\delta) = g(R) + \rho \qquad (3)
\end{aligned}
$$

where $\delta \in \{-1, 0, 1\}$ implying $\rho \in \{0, 1, 2\}$. □

Theorem 3. *Let Γ be the assembly graph, $e, e' \in E(\Gamma)$, and Γ' be obtained by a 3-interlace of e and e'. Let $R \in \mathcal{R}_\Gamma$ and $R' \in \mathcal{R}_{\Gamma'}$ so that R' is an extension of R. Then $g(R') = g(R) + \rho$ where $\rho \in \{-1, 0, 1, 2\}$.*

Proof. Let y_1, y_2 and y_3 be the newly added vertices to R'. Let R'_i for

$i \in \{1,2,3\}$ be extensions of R where R_1' has a type 1 configuration at vertices y_1,y_2 and y_3, R_2' has a type 2 configuration at precisely $0 < \ell < 3$ vertices in $\{y_1, y_2, y_3\}$, and R_3' has a type 2 configuration at vertices y_1, y_2, and y_3. Just as in the other two theorems, we observe the exterior connection graphs for each R_i' and count the difference in the number of boundary components. The exterior connection graphs for R_i' are shown in Figure 22.

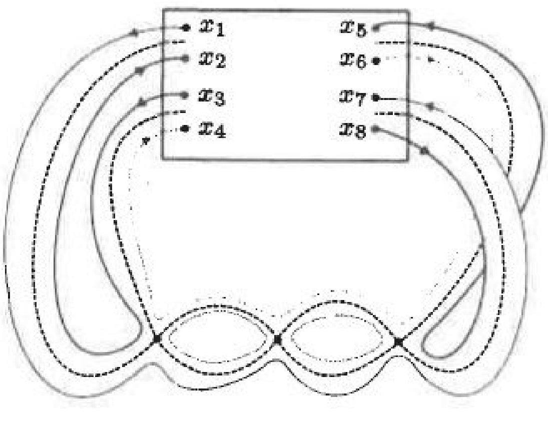
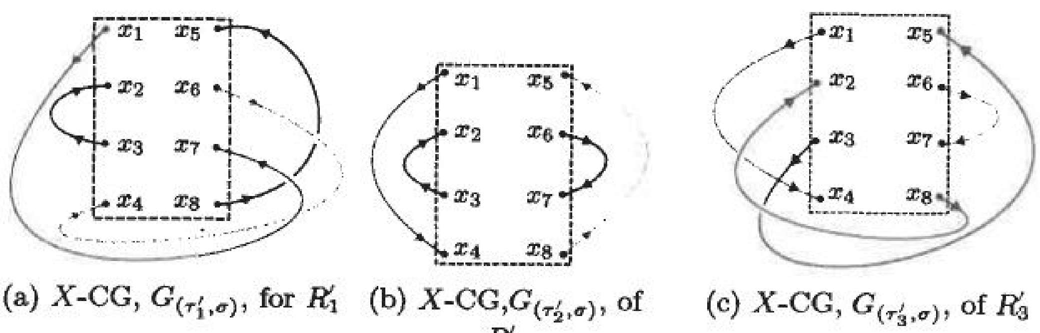


Figure 21: R'



(a) X-CG, $G_{(\tau_1',\sigma)}$, for R_1' (b) X-CG,$G_{(\tau_2',\sigma)}$, of R_2' (c) X-CG, $G_{(\tau_3',\sigma)}$, of R_3'

Figure 22: X-CGs for R'

The 24 differences in the total number of boundary components between R and its extension R' is computed by *Mathematica* 11.0, see Appendix Table 4. The difference in the number of boundary components between R and its extension R' is $\delta \in \{-1, 1, 3, 5\}$. Letting n be the number of vertices in Γ,

$$\begin{aligned} g(R') &= \frac{1}{2}(2 - b(R') + [n+3]) = \frac{1}{2}(2 - [b(R) + \delta] + [n+3]) \\ &= \frac{1}{2}(2 - b(R) + n) + \frac{3-\delta}{2} = g(R) + \rho \end{aligned} \quad (4)$$

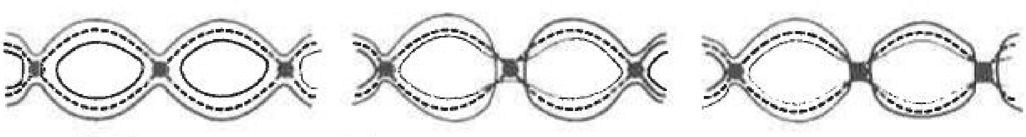
where $\delta \in \{-1, 1, 3, 5\}$. Hence we obtain $\rho \in \{-1, 0, 1, 2\}$. □

Theorem 4. *Let Γ be an assembly graph and let Γ_k be obtained by a k-interlace of two edges $e, e' \in E(\Gamma)$. Let $R \in \mathcal{R}_\Gamma$ and $R_k \in \mathcal{R}_{\Gamma_k}$ be an extension of R. Then for $k \geq 3$*

$$g(R_k) = \begin{cases} g(R_2) & \text{if } k \text{ is even} \\ g(R_3) & \text{if } k \text{ is odd} . \end{cases}$$

Proof. We consider the case k is even, as the case k is odd follows from a similar argument. Let R be fixed and consider the ribbon graphs for R_2 and R_k. Let $y_1, \ldots, y_k$ be the newly added vertices for R_k and z_1, z_2 be the newly added vertices for R_2. Consider the X-CGs for R_2.

If there are no type 2 configurations at $y_1, \ldots, y_k$ then the X-CG for R_k is the same as the X-CG for R_2 when there are no type 2 configurations at z_1 or z_2. If there are $1 \leq n \leq k-1$ type 2 configurations at $y_1, \ldots, y_k$, then the X-CG for R_k is the same as the X-CG for R_2 when there is precisely one type 2 configuration at either z_1 or z_2. If every vertex $\{y_1, \ldots, y_k\}$ is a type 2 configuration, then the X-CG for R_k is the same as the X-CG for R_2 for when there is a type 2 configuration at z_1 and z_2. Now we consider $b(R_k) - b(R_2)$.



(a) No type 2 configurations | (b) Type 2 configuration at a vertex y_i | (c) Type 2 configuration at vertices y_i and y_{i+1}

Figure 23: Section of k-interlace with/without type 2 configuration

From Figure 23a, we can see that each consecutive pair of vertices traverses a shared boundary component that does not exist in R_2. Thus, if there are no type 2 configurations at any of the vertices $y_1, \ldots, y_k$, then $b(R_k) = b(R_2) + k - 2$. Letting n be the number of vertices in Γ we have that

$$\begin{aligned} g(R_k) &= \frac{1}{2}(2 - b(R_k) + [n + k]) = \frac{1}{2}(2 - [b(R_2) + k - 2] + [n + k]) \\ &= \frac{1}{2}(2 - b(R_2) + n + 2) \quad = \quad g(R_2) \end{aligned}$$

Let R_k have a type 2 configuration at some vertex y_i. Choosing some y_j in the k-interlace, allow y_j to undergo a boundary connection change. Figures 23b and 23c shows that the outer portion of the boundary components traversing y_j will split, while the inner portion of the boundary components will connect together. Ultimately, the total number of boundary components do not change. Letting R_2 have precisely one type 2 configuration and R_k to have $1 \leq m \leq k-1$ type 2 connections or letting R_2 have two type 2 configurations and R_k have a type 2 configuration at every vertex $y_1, \dots, y_k$ leads to the same computation as shown in (5). Thus, for all possible cases, $g(R_k) = g(R_2)$. □

Corollary 1. *Let Γ be an assembly graph and let Γ_k be obtained by a k-interlace of two edges $e_1, e_2 \in E(\Gamma)$. Then for $k \geq 3$*

$$GR(\Gamma_k) = \begin{cases} GR(\Gamma_2) & \textit{If } k \textit{ is even} \\ GR(\Gamma_3) & \textit{If } k \textit{ is odd}. \end{cases}$$

Proof. The corollary follows immediately from Theorem 4. □

We note that in the above corollary we cannot set $k \geq 1$ for k odd. For an assembly graph Γ, if we let Γ' be obtained by a 1-interlace of $e, e' \in E(\Gamma)$ and Γ'' be a 3-interlace of the same two edges, it is not necessarly true that $GR(\Gamma') = GR(\Gamma'')$. Consider the DOW 123312 and the assembly graph Γ_{123312}. $\Gamma_{12343124}$ is obtained by a 1-interlace of the edges 33 and 21 and $\Gamma_{123456312456}$ is obtained by a 3-interlace of the edges 33 and 21. One can compute that $GR(\Gamma_{12343124}) = [1, 2]$ and $GR(\Gamma_{123456312456}) = [1, 3]$ so that $GR(\Gamma_{12343124}) \neq GR(\Gamma_{123456312456})$.

4 Reductions of DOWs and Genus Range

Using the developed machinery for insertions of DOWs we can determine an upper bound for the genus range of an assembly graph.

Definition 7. Let $w = u_1 v u_2 v' u_3 \in \Sigma_{DOW}$ where $v' \in \{v, v^R\}$. Then $w - v = u_1 u_2 u_3$ is called the *repeat/return removal* of v from w. We say that v is a maximal repeat/return word of w if for any $u \in \Sigma_{DOW}$ such that $v \sqsubseteq u \sqsubseteq w$ then $v = u$.

Example 5. For $w = 12341243$ we have $w - 12 = 3443$ and $w - 34 = 1212$.

Definition 8. A *reduction* of w is a sequence of words $(v_0, \dots, v_n)$ in which

(i) $v_0 = w$.
(ii) v_{k+1} is obtained from v_k by applying maximal repeat/return removal.
(iii) $v_n = \varepsilon$.

Definition 9 ([7]). The *nesting index* $\mathrm{NI}(w)$ of a DOW w is the smallest integer n for which $(v_0, \dots, v_n)$ is a reduction.

Example 6. Consider the DOW $w = 1213243545$. Then the sequence $(w, 12124545, 1212, \varepsilon)$ is a reduction of w.

Theorem 5. *Let Γ be an assembly graph and let $(\Gamma_0 = \Gamma, \Gamma_1, \ldots, \Gamma_m = \Gamma')$ be a sequence of assembly graphs where Γ_{i+1} is obtained from Γ_i by an interlace. Then*

$$\begin{aligned} \max GR(\Gamma') - \max GR(\Gamma) &\geq 2m, \\ \min GR(\Gamma) - \min GR(\Gamma') &\leq m. \end{aligned}$$

Proof. Let Γ be an assembly graph, Γ' be obtained from a k-interlace of $e, e' \in E(\Gamma)$, and $R' \in \mathcal{R}_{\Gamma'}$ be an extension of $R \in \mathcal{R}_\Gamma$. From Theorems 1, 2, 3, and 4, we have $g(R) - 1 \leq g(R') \leq g(R) + 2$.

Let $R_i \in \mathcal{R}_{\Gamma_i}$. Because R_i is an extension of R_{i-1} for $i \in \{1, \ldots, n\}$, it must be the case that $g(R_{i-1}) - 1 \leq g(R_i) \leq g(R_{i-1}) + 2$. Thus, $g(R_0) - m \leq g(R_m) \leq g(R_0) + 2m$. Hence we obtain the inequalities. □

Corollary 2. *Let $w \in \Sigma_{DOW}$ and Γ_w be its corresponding assembly graph. Then* $\max GR(\Gamma_w) \leq 2\,\mathrm{NI}(w)$.

Proof. Let $(u_0, \ldots, u_m)$ be a minimal reduction of w. From this reduction, form the sequence of assemly graphs $(\Gamma_{u_m}, \Gamma_{u_{m-1}}, \ldots, \Gamma_{u_0})$. We can see that $\Gamma_{u_{i-1}}$ is obtained from Γ_{u_i} by a $\frac{|u_{i-1} - u_i|}{2}$-interlace. Let $R_{u_i} \in \mathcal{R}_{\Gamma_{u_i}}$ so that for each $i \in \{1, \ldots, n\}$, $R_{u_{i-1}}$ is an extension of R_{u_i}. From Theorem 5, $\max GR(\Gamma_{u_0}) - \max GR(\Gamma_{u_m}) \leq 2m$. Because $GR(\Gamma_{u_m}) = GR(\Gamma_\varepsilon) = [0, 0]$, we have that $\max GR(\Gamma_{u_0}) = \max GR(\Gamma_w) \leq 2m = 2\,\mathrm{NI}(w)$. □

We note that for any k, there is w such that $\mathrm{NI}(w) - \max GR(\Gamma_w) = k$. For let

$$w = 12213443 \cdots (2k-1)(2k)(2k)(2k-1).$$

then $\mathrm{NI}(w) = k$ and $\max GR(\Gamma_w) = 0$.

We conjecture that the equality in Corollary 2 does not hold for any non-empty word. It is of interest to know, for given n and m, the minimum of $[2\,\mathrm{NI}(\Gamma_w) - \max GR(\Gamma_w)]$ over all DOW w with m symbols and $\mathrm{NI}(\Gamma_w) = n$.

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5 Appendix

Given an assembly graph Γ, The appendix contains all tables that show the difference in the number of boundary components of $R \in \mathcal{R}_\Gamma$ and $R' \in \mathcal{R}_{\Gamma'}$ where Γ' is a k-interlace of Γ and R' is an extension of R. Every table contains all 24 possible connections for an X-CG graph of a given ribbon graph.

Y_σ	$\mathcal{C}(G_{(\tau,\sigma)})$	$\mathcal{C}(G_{(\tau',\sigma)})$	$b(R') - b(R)$
$\{x_2x_1, x_4x_3, x_5x_6, x_7x_8\}$	2	1	-1
$\{x_2x_1, x_4x_3, x_7x_6, x_5x_8\}$	1	2	1
$\{x_2x_1, x_5x_3, x_4x_6, x_7x_8\}$	1	2	1
$\{x_2x_1, x_5x_3, x_7x_6, x_4x_8\}$	2	3	1
$\{x_2x_1, x_7x_3, x_4x_6, x_5x_8\}$	2	1	-1
$\{x_2x_1, x_7x_3, x_5x_6, x_4x_8\}$	3	2	-1
$\{x_4x_1, x_2x_3, x_5x_6, x_7x_8\}$	1	2	1
$\{x_4x_1, x_2x_3, x_7x_6, x_5x_8\}$	2	3	1
$\{x_4x_1, x_5x_3, x_2x_6, x_7x_8\}$	2	3	1
$\{x_4x_1, x_5x_3, x_7x_6, x_2x_8\}$	1	4	3
$\{x_4x_1, x_7x_3, x_2x_6, x_5x_8\}$	3	2	-1
$\{x_4x_1, x_7x_3, x_5x_6, x_2x_8\}$	2	3	1
$\{x_5x_1, x_2x_3, x_4x_6, x_7x_8\}$	2	1	-1
$\{x_5x_1, x_2x_3, x_7x_6, x_4x_8\}$	3	2	-1
$\{x_5x_1, x_4x_3, x_2x_6, x_7x_8\}$	3	2	-1
$\{x_5x_1, x_4x_3, x_7x_6, x_2x_8\}$	2	3	1
$\{x_5x_1, x_7x_3, x_2x_6, x_4x_8\}$	4	1	-3
$\{x_5x_1, x_7x_3, x_4x_6, x_2x_8\}$	3	2	-1
$\{x_7x_1, x_2x_3, x_4x_6, x_5x_8\}$	1	2	1
$\{x_7x_1, x_2x_3, x_5x_6, x_4x_8\}$	2	1	-1
$\{x_7x_1, x_4x_3, x_2x_6, x_5x_8\}$	2	1	-1
$\{x_7x_1, x_4x_3, x_5x_6, x_2x_8\}$	1	2	1
$\{x_7x_1, x_5x_3, x_2x_6, x_4x_8\}$	3	2	-1
$\{x_7x_1, x_5x_3, x_4x_6, x_2x_8\}$	2	3	1

Table 2: X-CG for R' with type 2 configuration

Y_σ	$\mathcal{C}(G_{(\tau,\sigma)})$	$\mathcal{C}(G_{(\tau_1',\sigma)})+1$	$b(R_1') - b(R)$
$\{x_2x_1, x_4x_3, x_5x_6, x_7x_8\}$	2	2	0
$\{x_2x_1, x_4x_3, x_7x_6, x_5x_8\}$	1	3	2
$\{x_2x_1, x_5x_3, x_4x_6, x_7x_8\}$	1	3	2
$\{x_2x_1, x_5x_3, x_7x_6, x_4x_8\}$	2	4	2
$\{x_2x_1, x_7x_3, x_4x_6, x_5x_8\}$	2	2	0
$\{x_2x_1, x_7x_3, x_5x_6, x_4x_8\}$	3	3	0
$\{x_4x_1, x_2x_3, x_5x_6, x_7x_8\}$	1	3	2
$\{x_4x_1, x_2x_3, x_7x_6, x_5x_8\}$	2	4	2
$\{x_4x_1, x_5x_3, x_2x_6, x_7x_8\}$	2	2	0
$\{x_4x_1, x_5x_3, x_7x_6, x_2x_8\}$	1	3	2
$\{x_4x_1, x_7x_3, x_2x_6, x_5x_8\}$	3	3	0
$\{x_4x_1, x_7x_3, x_5x_6, x_2x_8\}$	2	2	0
$\{x_5x_1, x_2x_3, x_4x_6, x_7x_8\}$	2	4	2
$\{x_5x_1, x_2x_3, x_7x_6, x_4x_8\}$	3	5	2
$\{x_5x_1, x_4x_3, x_2x_6, x_7x_8\}$	3	3	0
$\{x_5x_1, x_4x_3, x_7x_6, x_2x_8\}$	2	4	2
$\{x_5x_1, x_7x_3, x_2x_6, x_4x_8\}$	4	4	0
$\{x_5x_1, x_7x_3, x_4x_6, x_2x_8\}$	3	3	0
$\{x_7x_1, x_2x_3, x_4x_6, x_5x_8\}$	1	3	2
$\{x_7x_1, x_2x_3, x_5x_6, x_4x_8\}$	2	4	2
$\{x_7x_1, x_4x_3, x_2x_6, x_5x_8\}$	2	2	0
$\{x_7x_1, x_4x_3, x_5x_6, x_2x_8\}$	1	3	2
$\{x_7x_1, x_5x_3, x_2x_6, x_4x_8\}$	3	3	0
$\{x_7x_1, x_5x_3, x_4x_6, x_2x_8\}$	2	2	0

(a) Difference in number of boundary components between R_1' and R

Y_σ	$\mathcal{C}(G_{(\tau,\sigma)})$	$\mathcal{C}(G_{(\tau_2',\sigma)})$	$b(R_2') - b(R)$
$\{x_2x_1, x_4x_3, x_5x_6, x_7x_8\}$	2	2	0
$\{x_2x_1, x_4x_3, x_7x_6, x_5x_8\}$	1	3	2
$\{x_2x_1, x_5x_3, x_4x_6, x_7x_8\}$	1	1	0
$\{x_2x_1, x_5x_3, x_7x_6, x_4x_8\}$	2	2	0
$\{x_2x_1, x_7x_3, x_4x_6, x_5x_8\}$	2	2	0
$\{x_2x_1, x_7x_3, x_5x_6, x_4x_8\}$	3	1	-2
$\{x_4x_1, x_2x_3, x_5x_6, x_7x_8\}$	1	3	2
$\{x_4x_1, x_2x_3, x_7x_6, x_5x_8\}$	2	4	2
$\{x_4x_1, x_5x_3, x_2x_6, x_7x_8\}$	2	2	0
$\{x_4x_1, x_5x_3, x_7x_6, x_2x_8\}$	1	3	2
$\{x_4x_1, x_7x_3, x_2x_6, x_5x_8\}$	3	3	0
$\{x_4x_1, x_7x_3, x_5x_6, x_2x_8\}$	2	2	0
$\{x_5x_1, x_2x_3, x_4x_6, x_7x_8\}$	2	2	0
$\{x_5x_1, x_2x_3, x_7x_6, x_4x_8\}$	3	3	0
$\{x_5x_1, x_4x_3, x_2x_6, x_7x_8\}$	3	1	-2
$\{x_5x_1, x_4x_3, x_7x_6, x_2x_8\}$	2	2	0
$\{x_5x_1, x_7x_3, x_2x_6, x_4x_8\}$	4	2	-2
$\{x_5x_1, x_7x_3, x_4x_6, x_2x_8\}$	3	1	-2
$\{x_7x_1, x_2x_3, x_4x_6, x_5x_8\}$	1	3	2
$\{x_7x_1, x_2x_3, x_5x_6, x_4x_8\}$	2	2	0
$\{x_7x_1, x_4x_3, x_2x_6, x_5x_8\}$	2	2	0
$\{x_7x_1, x_4x_3, x_5x_6, x_2x_8\}$	1	1	0
$\{x_7x_1, x_5x_3, x_2x_6, x_4x_8\}$	3	1	-2
$\{x_7x_1, x_5x_3, x_4x_6, x_2x_8\}$	2	2	0

(b) Difference in number of boundary components between R_2' and R