Reconstruction Numbers for Unicyclic Graphs

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Abstract

The *Reconstruction Number* of a Graph G, denoted $RN(G)$, is the minimum number k such that there exists k vertex deleted subgraphs of G which determine G up to isomorphism. More precisely, $RN(G) = k$ if and only if there are vertex deleted subgraphs $G_1, G_2, \ldots, G_k$, such that if H is any graph with vertex deleted subgraphs $H_1, H_2, \ldots, H_k$, and $G_i \cong H_i$, $i = 1, 2, \ldots k$, then $G \cong H$. A *unicyclic graph* is a connected graph with exactly one cycle. In this paper we find reconstruction numbers for various types of unicyclic graphs. With one exception, all unicuclic graphs considered have $RN(G) = 3$.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. If G is a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, we will denote the vertex deleted subgraph $G - v_i$ by G_i, and define the *deck* of G, $D(G)$, to be the set $D(G) = \{G_1, G_2, \ldots, G_n\}$. The elements of $D(G)$ are sometimes referred to as *cards*. One of the foremost unresolved conjectures in graph theory is the *Reconstruction Conjecture* which claims that a finite simple undirected graph with at least three vertices is determined up to isomorphism by its deck of vertex deleted subgraphs. More precisely, we say that G is *reconstructible* if and only if for every graph H such that $D(H) = \{H_1, H_2, \ldots, H_n\}$ and $G_i \cong H_i$, $i = 1, 2, \ldots n$, we have $G \cong H$. The Reconstruction conjecture states that graphs with at least three vertices are reconstructible. Note that K_2 is not reconstructible since $D(K_2) = D(2K_1)$. The reader is referred to [2] for a survey on this topic.

A family of graphs is said to be reconstructible if every member of the family is reconstructible. Examples of reconstructible families include regular graphs, trees, disconnected graphs, separable graphs without end-vertices, and unicyclic graphs. Some notable families which have not been proven reconstructible include bi-degreed graphs, bipartite graphs, separable graphs and planar graphs. Given the difficulty of resolving the recon-

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struction conjecture, it is not surprising that there are numerous variations on the reconstruction theme, one of which is the reconstruction number of a graph, introduced by Harary and Plantholt in [6]. If G is reconstructible, we define the *reconstruction number* of G, denoted $RN(G)$, to be the minimum number k such that there exist vertex deleted subgraphs $G_1, G_2, \ldots, G_k$, such that if H is any graph with vertex deleted subgraphs $H_1, H_2, \ldots, H_k$, and $G_i \cong H_i$, $i = 1, 2, \ldots k$, then $G \cong H$. In the literature, this parameter is also referred to as the *ally reconstruction number* (as opposed to the *adversary reconstruction number*) [7], and other types of reconstruction numbers such as the *class reconstruction number* [6] and the *degree associated reconstruction number* [10] have also been studied. This paper deals only with the reconstruction number $RN(G)$ as described above.

Probably the most interesting result related to reconstruction numbers is due to Bollobás who proved that almost all graphs have $RN(G) = 3$ [1]. Several significant results on reconstruction numbers are due to Myrvold who showed that if T is a tree with at least 5 vertices, then $RN(T) = 3$ [9], and if D is a disconnected graph with at least two non-isomorphic components, then $RN(D) = 3$, [8]. We note that for any graph G the minimum reconstruction number possible is 3. This is because given any two vertex deleted subgraphs $G - u$ and $G - v$, if H is the graph obtained from G by adding or deleting the edge uv to obtain a graph with a different number of edges, then the decks of both G and H will contain these two cards.

The focus of this paper is the determination of reconstruction numbers for unicyclic graphs. A *unicyclic* graph is a connected graph with exactly one cycle. This is a work in progress and although a fair number of cases have been resolved, there is still a considerable amount of work that needs to be done. In all but one case considered thus far, we have found that $RN(G) = 3$ when G is unicyclic, the exceptional case being $RN(C_4) = 4$.

2 Definitions and Preliminary Results

Let G be a unicyclic graph and let C be the unique cycle subgraph it contains. Then the vertices of C are called *cycle vertices* and the edges of C are called *cycle edges*. If G has cycle vertices $c_1, c_2, \ldots, c_n$, then the trees that remain when the cycle edges of C are removed are called *branch clusters* of G. We will view each branch cluster of G as a rooted tree with root the unique cycle vertex it contains. If $\deg(c_i) = m + 2$ we will also refer the the branch cluster rooted at c_i as an m-cluster. A 0-cluster is called an *empty cluster*. The *rooted branches* of an m-cluster with root c are the maximal rooted subtrees in which c is a degree one vertex. Thus an m-cluster is the union of its m rooted branches. A pendant branch is a branch that is isomorphic to K_2. A *pin cushion* is a unicyclic graph where each vertex is either a cycle vertex or a degree one vertex.

Let T be a tree. If v is a vertex of T, then the *branches of* v are the maximal subtrees of T that contain v as an end vertex. The *weight of a branch* B is $|B|-1$. The *centroid of* T is the set of vertices whose branches all have weight less than or equal to $|T|/2$. It is well know that the centroid of T is either a single vertex or a pair of adjacent vertices.

The following result was presented at the 49th Southeastern International Conference on Combinatorics, Graph Theory and Computing, March 4-8, 2019. The proof has not yet been published, but we include the statement of the result for completeness. We plan to publish the proof of this result in the near future.

Claim 1. *If* G *is a pin cushion and* $G \ncong C_4$*, then* $RN(G) = 3$. $RN(C_4) = 4$.

It is a routine matter to prove that C_4 is reconstructible. To see that $RN(C_4) \neq 3$, note that $D(C_4)$ and $D(K_{1,3})$ have three cards in common. Aside from the outlier C_4, all unicyclic graphs we have determined reconstruction numbers for thus far have had reconstruction number equal to 3. If we are eventually able to show that C_4 is the only exception case, the proof will probably have a large number of cases.

We conclude this section with several elementary yet useful results.

Lemma 1. *If* $D(G)$ *contains two cards where one card is connected and the other card has no isolated vertices, then* G *is connected.*

Proof: If $D(G)$ contains a connected card G_i, then G is connected unless $\deg(v_i) = 0$, in which case every other card has an isolated vertex. ∎

Corollary. *If* $D(G)$ *contains two connected cards, then* G *is connected.*

Lemma 2. If $D(G)$ contains two cards and the maximum degree of a vertex that appears on either card is M, then the maximum degree of a vertex in G is at most $M+1$.

Proof: If G had a vertex v_i of degree $M+2$ or more, then every card except possibly G_i would have a vertex of degree at least $M+1$. ∎

3 Unicyclic Graphs With a Unique Nonempty Branch Cluster

In this section we find reconstruction numbers for various types of unicyclic graphs with a unique nonempty branch cluster. In the arguments that follow, we assume G is unicyclic with a C_n subgraph, and we select vertices v_1, v_2, v_3, with the goal of showing that the corresponding cards G_1, G_2, G_3, determine G up to isomorphism. To do this we assume H is a graph with vertices u_1, u_2, u_3, corresponding to cards H_1, H_2, H_3, and that $G_i \cong H_i$,

$i = 1, 2, 3$. Our goal is to show that $G \cong H$, and thus $RN(G) = 3$. To assert these assumptions regarding H, the vertices u_i, and cards H_i, we will simply state that H is a reconstruction of G based on the cards G_1, G_2, G_3.

Theorem 1. *Let G be a unicyclic graph with a unique nonempty M-cluster. Assume in addition that this M-cluster has two non-isomorphic rooted branches, and no pendant branches. Then $RN(G) = 3$*

Proof: We choose the vertices v_1, v_2 and v_3 as follows: choose v_3 to be the root of the M-cluster, v_1 a vertex in the M-cluster adjacent to the v_3, and v_2 any degree one vertex. Let H be a reconstruction of G from the cards G_1, G_2, G_3. Then H is connected since H_2 is connected and H_3 has no isolated vertices. Since H_1 has a C_n subgraph, call it C, so does H. Since the maximum degree in H_1 of a vertex on C is $M+1$, the maximum degree in H of a vertex on C is either $M + 1$ or $M + 2$. Now we can obtain H from H_3 by adding the vertex u_3 along with its incident edges. Since H_3 is an $(M+1)$-tree forest, and since H is connected and contains the cycle C, we must have $\deg_H(u_3) \geq M+2$. But u_3 must be a vertex on C, and thus $\deg_H(u_3) = M + 2$. Now that we know $\deg_H(u_3) = \deg_G(v_3) = M + 2$, we also know that G and H are both unicyclic, have the same number of edges, and hence $\deg_H(u_i) = \deg_G(v_i)$, $i = 1, 2, 3$.

Now if the $M + 2$ edges incident with u_3 are added to H_3 to obtain H, the subgraph C is formed by connecting the two end vertices of a P_{n-1} component on H_3 to u_3. For if C is formed by connecting u_3 to any other type of component, the resulting graph will have one M-cluster and a second nonempty cluster, but such a graph could not have a card isomorphic to H_1 which has a single $(M-1)$-cluster. We see then that G and H both have a unique nonempty branch cluster of the same order, and both of these branch clusters have the same multiset of branch weights. It only remains to show that the rooted branch cluster on G is isomorphic to the rooted branch cluster on H. Let N be the maximum branch weight and K the minimum branch weight in the M-cluster of G. We will choose v_1 and v_2 as described above, but with additional specifications that depend on whether $N = K$, $N = K+1$ or $N > K+1$.

The easiest case is $N > K+1$. In this case we choose v_1 from a branch of order K and v_2 from a branch of order N. Since the rooted branches of order K are easily identified on H_2, a graph isomorphic to H can be obtained by identifying the missing branch of order K on H_1 and replacing it. So we must have $G \cong H$.

Now suppose that $N = K + 1$. If possible, we choose v_1 and v_2 in branches such that v_2 is in a branch B of order N and $B - v_2$ is isomorphic (as a rooted tree) to the branch of order K that v_1 is in. If this is not possible, then simply choose v_2 in a branch of order N and v_1 in a branch of order K. Now if $B - v_2$ is isomorphic to the branch that v_1 is in, then

H_2 has two more rooted branches of type $B - v_2$ than H_1 does. We can replace the missing branch of order K on H_1 with a branch of this type and obtain a graph isomorphic to H. Again, $G \cong H$. Suppose we can't find v_1, v_2, and B where $B - v_2$ is isomorphic to the branch v_1 is in. The branches of order N can be identified on H_1. When we examine H_2, there is a unique branch that is the result of deleting a degree one vertex from one of the branches of order N. The other branches of order K are the actual branches of order K in H. We see that G and H have the same rooted branches and are isomorphic.

Finally, suppose all rooted branches have the same order. We choose v_1 and v_2 to be from rooted branches of isomorphism types A and B respectively where $A \ncong B$. Then H_2 has one more type A branch than H_1, hence the missing branch on H_2 can be replaced with one of type A to obtain a graph isomorphic to H. ∎

In the previous theorem we assumed the existence of two non-isomorphic branches. The next result deals with the case of a unique nonempty cluster where all branches are isomorphic and not pendant branches. The case of a unique nonempty cluster of pendant branches is handled in the next section.

Theorem 2. *Let G be a unicyclic graph with a unique nonempty M-cluster, $M \geq 2$. Assume in addition that all rooted branches in this M-cluster are isomorphic and are not pendant branches. Then $RN(G) = 3$*

Proof: We choose the vertices v_1, v_2 and v_3 as follows: choose v_3 to be the root of the M-cluster, v_2 a cycle vertex adjacent to the v_3, and v_1 a degree one vertex. Let H be a reconstruction of G based on these cards. Then H is connected since both H_1 and H_2 are, and H has a C_n subgraph, call it C, since G_1 does. The maximum degree in H_1 of a vertex on C is $M+2$, so the maximum degree in H of a vertex on C is at most $M+3$. Since H_3 is an $(M+1)$-tree forest and v_3 is a vertex of C, we must have $\deg_H(u_3) \geq M+2$. So $\deg_H(u_3) \in \{M+2, M+3\}$. Now there is exactly one component of H_3 that contains two or more neighbors of u_3 since otherwise every element of $D(H)$ other than H_3 would contain a cycle, and H_2 is a tree. So there is a unique component of H_3 that contains either two or three neighbors of u_3, and the remaining components each contain one neighbor. Denote the component that contains either two or three neighbors by A.

Suppose $A \cong P_{n-1}$. Then the end-vertices of A must be neighbors of u_3 in order to form the subgraph C. A third neighbor of u_3 in A is not possible since in that case it would not be possible to delete a vertex from H and obtain H_1 (a card with exactly one cycle of size n).

Suppose $A \ncong P_{n-1}$. Since H has a C_n subgraph, A must contain a P_{n-1} subgraph P whose end vertices are neighbors of u_3. Note that in this case all but one of the components of G_3 are isomorphic to A, the order of A is at least n, and as a result v_3 is the centroid of G_2. The vertex u_2 must be in A since otherwise H_2 would contain a cycle. By considering branch weights we see that u_3 is the centroid of H_2. But then H_2 has a centroidal branch of weight $n-1$ which is a contradiciton because G_1 has one centroidal branch of weight $n-2$ and all other centroidal branches of weight $|A| \geq n$.

We see then that $A \cong P_{n-1}$ and both G and H have the same number of edges and a unique nonempty branch cluster with all rooted branches in the cluster of equal weight. Now it is easy to identify the rooted branches of H from H_2. The vertex u_3 can be identified on H_2 as the unique vertex of degree greater than or equal to three that is either a vertex in the centroid of H_2, or that has a P_{n-1} branch that contains the centroid. In either case the rooted branches of the branch cluster of H are easily identified. Thus $G \cong H$. ∎

4 Unicyclic Graphs With a Unique Maximum Degree Pendant Cluster

In this section we consider one final, rather specific, type of unicyclic graph. In order to describe this type of graph, a new definition is needed. A family of rooted branch clusters is said to be *closed with respect to pruning* if anytime B is a rooted branch cluster in the family and v is a degree one vertex of B different from the root, then $B - v$ is also in the family. Since repeated prunings will yield an empty cluster, all such families have at least one 0-cluster.

Theorem 3. *Let G be unicyclic with a unique M-cluster of maximum root degree, that is, such that all other K-clusters of G have $K < M$. Assume in addition that the branches of the unique M-cluster are all pendant branches, and that the branch clusters of G are not closed with respect to pruning. Then $RN(G) = 3$.*

Proof: We first handle the case that G is a pin cushion with exactly one nonempty branch cluster, the M-cluster of pendant branches. We choose v_1 to be a degree one vertex, v_3 the root of the M-cluster, and v_2 a cycle vertex adjacent to v_3. Then, using arguments similar to those in previous cases, we see that H is connected, has a C_n subgraph, and that $\deg(u_3) = M + 2$. There is only way to obtain a unicyclic graph with a C_n subgraph by adding $M + 2$ edges from u_3 to H_3, and thus $G \cong H$.

We can assume then that G has two or more branch clusters. We choose v_3 to be the root of the M-cluster, v_2 a degree one vertex adjacent to v_3, and v_1 a degree one vertex from a branch cluster other than the M-cluster. Also, if it is possible to choose v_1 from a rooted branch B so that G does not have a rooted branch isomorphic to $B - v_1$, then do so. Let H be a reconstruction of G based on these cards. Then H is connected since both H_1 and H_2 are. Since H_2 has a C_n subgraph, call it C, so does H. Since the maximum degree in H_2 of a vertex on C is $M+1$, the maximum degree in H of a vertex on C is either $M+1$ or $M+2$. As in previous arguments, when adding u_3 to H_3 to obtain H, the degree of u_3 must be at least $M+2$, and since u_3 is a vertex of C, $\deg_H(u_3) = M+2$. Since $\deg_H(u_3) = \deg_G(v_3) = M+2$, we know that G and H are both unicyclic, have the same number of edges, and hence $\deg_H(u_i) = \deg_G(v_i)$, $i = 1, 2, 3$. On the card H_2, the neighbor of u_2 must be the root of an $(M-1)$-cluster of pendant branches, because otherwise the vertex deleted subgraph H_3 which has M isolated vertices would not be possible. We see then that G and H must have the same multiset of rooted branch clusters. If there is only one $(M-1)$-cluster of pendant branches on H_2, then $G \cong H$. Otherwise, recalling that the branch clusters are not closed with respect to pruning, there must have been a choice of rooted branch B such that $B - v_2$ is not a rooted branch of G. Since the rooted branches of G and H are the same, we can replace the branch $B - v_2$ with B and obtain H. So $G \cong H$. ∎

At this point, it is natural to consider the unicyclic graphs as described in the previous theorem, but with branch clusters that *are* closed with respect to pruning. We claim that these graphs have also been shown to have $RN(G) = 3$, but the proof is longer and quite a bit more complicated.

5 Conclusion

The goal of this paper has been to introduce the reader to the concept of reconstructions numbers for graphs, and in particular, to determine reconstruction numbers for several families of unicyclic graphs. As mentioned earlier, this is a work in progress. The cases presented here represent roughly half of the families for which we have found reconstruction numbers, and the proof that pin cushions have reconstruction number equal to 3 will appear in a future paper. Although we have not discovered any unicyclic graphs, other than C_4, with $RN(G) > 3$, a conjecture that $RN(G) = 3$ seems premature at this time. Given the considerable effort required to show $RN(G) = 3$ for various families of unicyclic graphs, perhaps a more tractable first goal might be to show that $RN(G) \leq 4$ for all unicyclic graphs.

Finally, we note that although it may not seem that computation of reconstruction numbers will have any serious impact on the resolution of the general reconstruction conjecture, this is not necessarily true. By focusing on only a few cards, one is forced to look more closely at the information contained on individual vertex deleted subgraphs. This closer examination may yield methods and techniques that can be applied to other families of graphs which are not known to be reconstructible. At the very least, it has been our experience that this closer inspection has frequently yielded a better understanding of the structure of the graphs being considered.

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