Distinct Representatives in Special Set Families in Graphs

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Abstract

In 2017, Hedetniemi asked the question "for which graphs G does the indexed family $\{N_G(v)|v \in V(G)\}$ of open neighborhoods have a system of distinct representatives?" In [1], we answered that question. Now, we move on to other special set families in graphs and examine whether they do or do not have a system of distinct representatives.

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All graphs will be finite and simple. In [1], we called a graph G SDR-good if the indexed collection $\mathcal{N}(G) = \{N_G(v)|v \in V(G)\}$ has a system of distinct representatives (SDR). An SDR for $\mathcal{N}(G)$ is a one-to-one function $\Phi : V(G) \to V(G)$ such that $\Phi(v) \in N_G(v)$ for all $v \in V(G)$.

Theorem 1 *[1] A graph G is SDR-good if and only if G has a spanning subgraph the components of which are either single edges or cycles.*

Let $\mathcal{M}(G)$ be the set of all maximum matchings in G. We shall say that a graph is SDR-$\mathcal{M}(G)$-good if $\mathcal{M}(G)$ has a system of distinct representatives (SDR). An SDR for $\mathcal{M}(G)$ is a one-to-one function $\Phi : \mathcal{M}(G) \Rightarrow E(G)$ such that $\Phi(M) \in M$ for all $M \in \mathcal{M}(G)$. Note that the existence of such a function requires $|E(G)| \geq |\mathcal{M}(G)|$. So, $|E(G)| < |\mathcal{M}(G)|$ implies that G is not SDR-$\mathcal{M}(G)$-good.

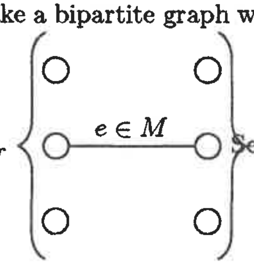
Considering some familiar families of graphs, we discover:

- When $G = K_3$, $\mathcal{M}(G)$ is the set of singleton subsets of $E(G)$, is SDR-$\mathcal{M}(G)$-good, and has a unique SDR.

- $G = K_4$, is not just SDR-$\mathcal{M}(G)$-good, but there are 2^3 ways to select distinct representatives for $\overline{\mathcal{M}(K_4)}$, because $\mathcal{M}(K_4)$ consists of 3 pairwise disjoint maximum matchings, each with 2 edges.
- K_5 has $|\mathcal{M}| > |E|$, so can't be SDR-$\mathcal{M}(G)$-good.
- K_6 has $|\mathcal{M}| = |E|$, and there are many SDRs of $\mathcal{M}(K_6)$. This can be demonstrated by creating a bipartite graph H with partite sets $E(K_6)$ and $\mathcal{M}(K_6)$. An edge e is adjacent to a matching M in this bipartite graph if and only if $e \in M$. The degree on each side is 3; therefore the bipartite graph is 1-factorizable. Any perfect matching in H gives a system of distinct representatives for K_6, and vice-versa. There are as many different SDRs of $\mathcal{M}(K_6)$ as there are perfect matchings in H.
- K_7 is not SDR-$\mathcal{M}(G)$-good, $|\mathcal{M}| > |E|$.

Proposition 2 *For K_n, $n \geq 7$, $|\mathcal{M}(K_n)| > |E(K_n)|$.*

Proof:
Determining the $|\mathcal{M}(K_n)|$. For positive integers r, let $f(r) = |\mathcal{M}(K_{2r})|$ and $g(r) = |\mathcal{M}(K_{2r-1})|$.
We have $f(1) = 1$ and we will get $f(r)$ in terms of $f(r-1)$ by the following trick. For $r > 1$, make a bipartite graph with bipartition $E(K_{2r})$, $\mathcal{M}(K_{2r})$:

Set of Edges in K_{2r}  Set of Maximum Matchings in K_{2r}

Clearly, the degree in this graph of each $e \in E$ is $f(r-1)$, and the degree of each $M \in \mathcal{M}$ is r. Therefore, counting the edges of the bipartite graph in two ways, we have $\frac{2r(2r-1)}{2} f(r-1) = r|\mathcal{M}| = rf(r)$ implies $f(r) = (2r-1)f(r-1)$. Therefore, $|\mathcal{M}(K_{2r})| = f(r) = (2r-1)(2r-3)\dots 1$.
Now, each maximum matching in K_{2r-1} misses one vertex, and maximum matchings missing different vertices are different. Therefore, $g(r) = (2r-1)f(r-1) = (2r-1)(2r-3)\dots 1$ for $r > 1$. Suppose that $n > 6$.
If $n = 2r$, $r > 3$, then $|\mathcal{M}(K_n)| = f(r) = (2r-1)(2r-3)\dots 1 > (2r-1)(2r-3) > (2r-1)r = |E(K_n)|$.
If $n = 2r-1$, $r > 3$, then $|\mathcal{M}(K_n)| = g(r) = (2r-1)(2r-3)\dots 1 > (2r-1)(2r-3) > (2r-1)(r-1) = |E(K_n)|$. □

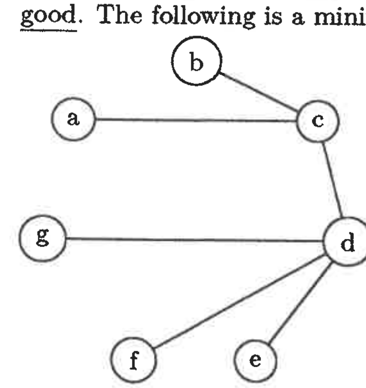
Considering the familiar family of complete bipartite graphs, we observe $G = K_{a,b}$ with $a \leq b$ can only be SDR-$\mathcal{M}(G)$-good if $|\mathcal{M}(G)| = \frac{b!}{(b-a)!} \leq ab = |E(G)|$ which only holds when $a = 1$, or $a = 2$ & $b \leq 3$, or $a = b = 3$.

Obviously $\mathcal{M}(K_{1,b})$ has a unique SDR for all $b \geq 1$. $\mathcal{M}(K_{2,2})$ has 4 different SDRs. Some inelegant fiddling around shows that $\mathcal{M}(K_{2,3})$ has exactly 2 different SDRs, and an analysis such as we gave for K_6 shows that $\mathcal{M}(K_{3,3})$ as many SDRs as there are maximum matchings in a certain biregular bipartite graph with 6 vertices of degree 3 on one side and 9 vertices of degree 2 on the other.

We know that $|E(G)| < |\mathcal{M}(G)|$ implies that G is not SDR-$\mathcal{M}(G)$-good, so we wonder if $|\mathcal{M}| \leq |E|$ is necessary and sufficient for G to be SDR-$\mathcal{M}(G)$-good.

Proposition 3 $|\mathcal{M}| \leq |E|$ *is necessary but not sufficient for G to be SDR-$\mathcal{M}(G)$-good.*

Proof:
We need only give an example in which $|M| \leq |E|$ but G is not SDR-$\mathcal{M}(G)$-good. The following is a minimum counterexample:



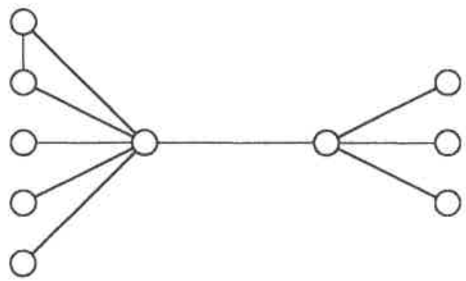
In the example, each maximum matching ($|M| = 2$) is made up of one pendant edge incident to each of vertices c and d, giving $|\mathcal{M}(G)| = 2 \times 3 = 6 = |E(G)|$. Since only 5 edges are eligible to represent maximum matchings, clearly $\mathcal{M}(G)$ has no SDR. □

In the counterexample in the proof of Proposition 3, edge cd appears in no maximum matching in the graph. We will call such an edge an "$\mathcal{M}$-bad" edge.

This example generalizes to a family of graphs; double broomstick graphs whose central path has an odd number of edges. In each of those graphs, the odd number of edges in the central path, k, means there are $(k+1)/2$

M-bad edges in each graph. The size of the graph can be increased by adding pendant stars with the bridging edge being M-bad.

These graphs are all trees. We can find connected graphs G which are not trees that are not SDR-$\mathcal{M}(G)$-good even though $|\mathcal{M}(G)| \leq |E(G)|$; for instance,



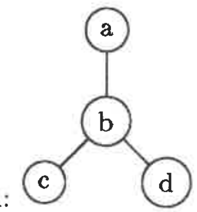
This is a unicyclic graph G such that $\alpha'(G) = 3$, $|\mathcal{M}(G)| = 9 < 10 = |E(G)|$, and G is not SDR-$\mathcal{M}(G)$-good. However, we have not found any 2-edge-connected SDR-$\mathcal{M}$-bad graph with $|\mathcal{M}(G)| \leq |E(G)|$.

Let $\mathcal{I}(G)$ be the set of maximum independent sets in G. We shall say that a graph is SDR-$\mathcal{I}(G)$-good if the set $\mathcal{I}(G)$ has a system of distinct representatives (SDR). An SDR for $\mathcal{I}(G)$ is a one-to-one function $\Phi : \mathcal{I}(G) \to V(G)$ such that $\Phi(I) \in I$ for all $I \in \mathcal{I}(G)$.

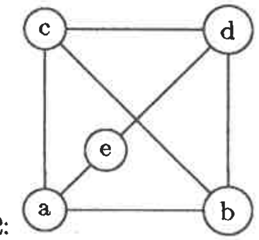
The line graph of a graph G, written $L(G)$, is the graph whose vertices are the edges of G with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G.

Theorem [Krausz, 1943]: For a simple graph G, there is a solution to $L(H) = G$ if and only if G decomposes into [maximal] complete subgraphs, with each vertex of G appearing in at most two of these complete subgraphs.
The size of a maximum matching in G is $\alpha'(G)$, and the size of a largest independent set in G is $\alpha(G)$.
Folk theorem: $\alpha'(G) = \alpha(L(G))$.

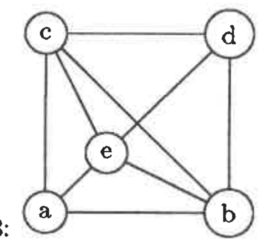
Recall Beineke (1968)[2]: A simple graph G is the line graph of some simple graph if and only if G does not contain any of the nine graphs below as an induced subgraph. Observe that 8 out of the 9 are SDR-$\mathcal{I}(G)$-good.



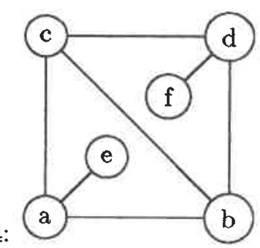
B1:
$\mathcal{I}(B1) = \{\{a, c, d\}\}$, SDR-$\mathcal{I}(G)$-good



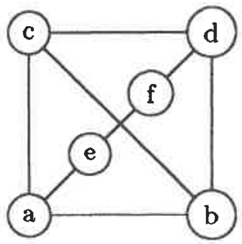
B2:
$\mathcal{I}(B2) = \{\{b, e\}, \{c, e\}, \{a, d\}\}$, SDR-$\mathcal{I}(G)$-good



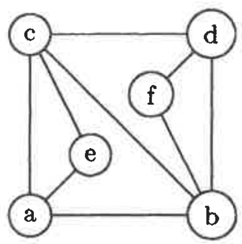
B3:
$\mathcal{I}(B3) = \{\{a, d\}\}$, SDR-$\mathcal{I}(G)$-good



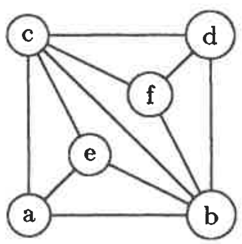
B4:
$\mathcal{I}(B4) = \{\{c, e, f\}, \{b, e, f\}\}$, SDR-$\mathcal{I}(G)$-good



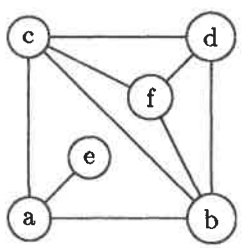
B5:
$\mathcal{I}(B5) = \{\{a,f\},\{a,d\},\{e,d\},\{b,e\},\{b,f\},\{c,e\},\{c,f\}\}$, NOT SDR-$\mathcal{I}(G)$-good



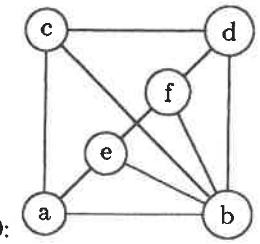
B6:
$\mathcal{I}(B6) = \{(e,f\},\{a,d\},\{a,f\},\{e,d\},\{e,b\},\{c,f\}\}$, SDR-$\mathcal{I}(G)$-good



B7:
$\mathcal{I}(B7) = \{\{d,e\},\{e,f\},\{a,d\},\{f,a\}\}$, SDR-$\mathcal{I}(G)$-good



B8:
$\mathcal{I}(B8) = \{\{e,b\},\{e,c\},\{e,d\},\{e,f\},\{a,f\},\{a,d\}\}$, SDR-$\mathcal{I}(G)$-good



B9:
$\mathcal{I}(B9) = \{\{e,c\},\{c,f\},\{e,d\},\{a,f\},\{a,d\}\}$, SDR-$\mathcal{I}(G)$-good

Proposition 4 *If G is SDR-$\mathcal{M}(G)$-good, then $L(G)$ is SDR-$\mathcal{I}(L(G))$-good*

The proposer of the original problem that inspired this paper, S. Hedetniemi, now proposes looking at a large class of similar problems, in which the roles of the indexed collections of open neighborhoods, or closed neighborhoods, are replaced by other iconic collections of graph elements. For instance, we could ask: for which finite simple graphs G does the list of maximal cliques in G have a system of distinct vertex representatives, or a system of distinct edge representatives? As with the two cases dealt with in this paper, the answers could be interesting, or not. There is opportunity to look at generalizations of neighborhoods, matchings, and independent sets, finding efficient algorithms and locating families of graphs for which the ratio of vertices or edges to some parameter covers a spectrum.

References

[1] Hedetniemi, Holliday, Johnson. Neighborhood representatives, *Congressus Numerantium* 231 (2018), 117-119.

[2] West, D. B. (1996). Introduction to Graph Theory. Prentice Hall.