Genus, Skewness, Thickness and Coloring Theorems

Val Pinciu*

Abstract

A conjecture by Albertson states that if $\chi(G) \geq n$ then $cr(G) \geq cr(K_n)$, where $\chi(G)$ is the chromatic number of G and $cr(G)$ is the crossing number of G. This conjecture is true for $n \leq 16$, but it is still open for $n \geq 17$. In this paper we consider the statements corresponding to this conjecture where the crossing number of G is replaced with the genus $\gamma(G)$ (the minimum genus of the orientable surface on which G is embeddable), the skewness $\mu(G)$ (the minimum number of edges whose removal makes G planar), and the thickness $\theta(G)$ (the minimum number of planar subgraphs of G whose union is G.)

1 Introduction

Throughout this paper, $G = (V, E)$ denotes a connected simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. G is *planar* if it can be embedded in the plane, i.e., it can be drawn on the plane such that no edges cross each other. There are many relaxations of planarity, i.e., graph invariants that measure how close a graph is to a planar graph. Examples of relaxations of planarity include: the *crossing number* $cr(G)$ (the minimum number of edge crossings in any drawing of G in the plane), the *skewness* $\mu(G)$ (the minimum number of edges whose removal makes the graph planar), the *genus* $\gamma(G)$ (the minimum genus of the orientable surface on which G is embeddable), and the *thickness* $\theta(G)$ (the minimum number of planar subgraphs of G whose union is G).

A conjecture by Albertson [1] states that if $\chi(G) \geq n$ then $cr(G) \geq cr(K_n)$, where $\chi(G)$ is the chromatic number of G. The statement is trivial when $n \leq 4$, and it is equivalent to the Four Color Theorem ([3], [17]) when $n = 5$. Oporowski and Zhao [12] proved that every graph with

*Department of Mathematics, Southern Connecticut State University, New Haven, CT 06515 pinciuv1@southernct.edu

CONGRESSUS NUMERANTIUM 233 (2019), pp.119-124

crossing number at most two is 5-colorable, therefore the conjecture is true when $n = 6$. Albertson, Cranston, and Fox [1] verified the conjecture for $7 \leq n \leq 12$, and Barat and Toth [4] verified the conjecture for $13 \leq n \leq 16$. This conjecture is still open for $n \geq 17$.

In this paper we consider the statements corresponding to Albertson's Conjecture where the crossing number of G is replaced with the relaxations of planarity $\mu(G)$, $\gamma(G)$, and $\theta(G)$. In Sections 2 and 3 we show that the corresponding statements are true for all positive integers n when the crossing number of G is replaced with the genus or the skewness of G respectively. In Section 4 we show that the corresponding statement is true for infinitely many values of n, but not for all n, when the crossing number of G is replaced with the thickness of G.

2 Genus and a Coloring Theorem

The genus $\gamma(G)$ of a simple graph $G = (V, E)$ is the minimum genus of the orientable surface on which G is embeddable. While determining the genus of an arbitrary nonplanar graph is NP-complete [18], exact formulae are known for several classes of graphs, including the complete graphs on n vertices K_n:

Theorem 1 ([15]) $\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$.

In this section we show that the statement obtained by replacing $cr(G)$ with $\gamma(G)$ in Alberston's Conjecture is true:

Theorem 2 *Let G be any simple graph and let n be any positive integer. If $\chi(G) \geq n$ then $\gamma(G) \geq \gamma(K_n)$, where $\chi(G)$ is the chromatic number of G.*

The proof is elementary and uses Heawood's upper bound for the chromatic number of a graph embeddable on an orientable surface of a given genus [5] and the Ringel-Youngs Theorem [15].

3 Skewness and a Coloring Theorem

The *skewness* $\mu(G)$ of a simple graph $G = (V, E)$ is the minimum number of edges whose removal makes the graph planar. It is obvious that $\mu(G) \leq cr(G)$. However the skewness and the crossing number of a graph can differ widely, since $\mu(G)$ can be of size $O(|V|^2)$, while $cr(G)$ can be as large as $O(|V|^4)$. A general lower bound for the skewness of a graph is given by the following:

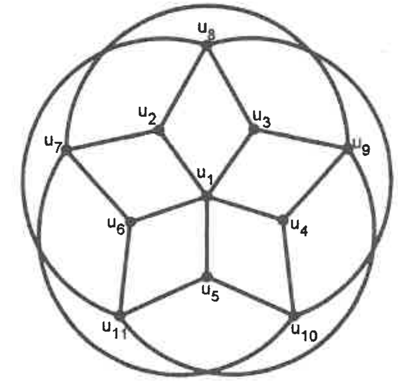

Figure 1: The Grötzsch graph has skewness 3 and crossing number 5.

Theorem 3 ([13]) *If $G = (V, E)$ is a connected simple graph with girth $g \geq 3$ then:*

$$\mu(G) \geq |E| - \frac{g(|V| - 2)}{g - 2}.$$

Equality holds if and only if G has a spanning planar subgraph that is face-regular of degree g.

It is easy to use Theorem 3 to show that Petersen's Graph has skewness and crossing number 2, while Heawood's Graph has skewness and crossing number 3. It is less obvious to show that Grötzsch's Graph (Figure 1) has skewness 3 and crossing number 5 (for more information see [13]). However, an exact formula for the skewness of an arbitrary nonplanar graph is unknown:

Theorem 4 ([10]) *Determining the skewness of an arbitrary nonplanar graph is* NP-complete.

Exact formulae are known for several classes of graphs, including the complete graphs on n vertices K_n, and the complete bipartite graphs on $m + n$ vertices $K_{m,n}$:

Theorem 5 ([9]) *If K_n and $K_{m,n}$ are the complete graph on n vertices and the complete bipartite graph on $m + n$ vertices respectively, then:*

1. $\mu(K_n) = \frac{(n-3)(n-4)}{2}$
2. $\mu(K_{m,n}) = mn - 2(m + n) + 4$

In this section we show that the statement that is obtained from Albertson's Conjecture by replacing $cr(G)$ with $\mu(G)$ is true for all positive integers n. Here is our main result:

Theorem 6 *Let G be any simple graph and let n be any positive integer. If $\chi(G) \geq n$ then $\mu(G) \geq \mu(K_n)$, where $\chi(G)$ is the chromatic number of G.*

This statement is equivalent with the fact that among all graphs requiring n colors, the complete graph on n vertices K_n is the one with the smallest skewness. The statement is trivial for positive integers $n \leq 4$, it is equivalent to the Four Color Theorem when $n = 5$, and it is equivalent to a generalization of the Five Color Theorem by Kainen [8] when $n = 6$. When $n \geq 7$, we provide a proof by induction on $|V(G)|$ using the following lemma:

Lemma 7 *Let G be any simple graph and let $n \geq 7$ be a positive integer. If $\mu(G) < \mu(K_n)$, then G has a vertex v such that $deg(v) < n - 1$.*

Proof. We can assume that $|V(G)| \geq n$. By combining Theorem 3 for $g = 3$ with Theorem 5 we obtain:

$$|E(G)| - 3(|V(G)| - 2) \leq \mu(G) < \mu(K_n) = (n-3)(n-4)/2$$

Through some algebraic manipulations we can derive that:

$$\sum_{v \in V(G)} [deg(v) - (n-1)] < (n-7)(n - |V(G)|) \leq 0$$

Therefore $deg(v) < n - 1$ for some $v \in V(G)$. □

4 Thickness and a Coloring Theorem

The thickness $\theta(G)$ of a simple graph $G = (V, E)$ is the minimum number of planar subgraphs of G whose union is G. Determining the thickness of an arbitrary nonplanar graph is NP-hard [11]. However exact formulae are known for several classes of graphs, including the complete graphs on n vertices K_n:

Theorem 8 ([2]) *The thickness of the complete graph K_n satisfies:*

$$\theta(K_n) = \left\lfloor \frac{n+7}{6} \right\rfloor,$$

except when $n = 9$ or 10 for which the thickness is 3.

In this section we consider the statement that corresponds to Albertson's Conjecture when $cr(G)$ is replaced with $\theta(G)$. Is it true that if $\chi(G) \geq n$ then $\theta(G) \geq \theta(K_n)$? The statement is trivially true for positive integers $n \leq 4$. Since K_5 is biplanar, this statement is equivalent to the Four Color Theorem when $n = 5$. Since K_6, K_7 and K_8 are biplanar, this statement is also true when $6 \leq n \leq 8$. However, the statement is false when $n = 9$. The Sulanke's Graph $K_{11} - C_5$ is a counterexample, since it is 9-critical and biplanar. This result was proved by Sulanke and reported by Gardner in [6]. Determining the truth value of this statement when $n = 10$, 11, or 12 is equivalent to Ringel's famous Earth-Moon problem: what is the largest chromatic number of any thickness 2 graph? When $n \geq 13$, our main result in this section states that the statement is true for infinitely values of n:

Theorem 9 *Let G be any simple graph and let $n \geq 13$ be a positive integer such that $n \equiv 1, 2, 3,$ or 4 (mod 6). If $\chi(G) \geq n$ then $\theta(G) \geq \theta(K_n)$, where $\chi(G)$ is the chromatic number of G.*

The proof follows from Theorem 8 and a result from [7] which states that the largest chromatic number for a thickness-t graph is in the set $\{6t-2, 6t-1, 6t\}$ when $t \geq 3$. It is still open whether the statement from Theorem 9 is true when $n \equiv 0$ or 5 (mod 6).

References

[1] M.O. Albertson, D.W. Cranston and J. Fox, Crossings, Colorings and Cliques, *Electron. J. Combin.*, **16** (2009) #R45.

[2] V. B. Alekseev and V. S. Gonchakov, Thickness of Arbitrary Complete Graphs, *Mat. Sbornik* **101** (1976), 212–230.

[3] K. Appel and W. Haken, Every Planar Map is Four Colorable, Part I. Discharging, *Illinois J. Math.*, **21** (1977), 429-490.

[4] J. Barat and G. Toth, Towards the Albertson Conjecture, *Electron. J. Combin.*, **17** (2010) #R73.

[5] P. J. Heawood, Map Colour Theorem, *Quarterly Journal of Mathematics* **24** (1890), 332–338.

[6] M. Gardner, Mathematical games, *Sci. Amer.*, **242** 1980, 14-19.

[7] T. Jensen and B. Toft, *Graph coloring problems*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., New York, 1995.

[8] P. Kainen, A Generalization of the 5-Color Theorem, *Proc. Amer. Math. Soc.* **45** (1974), 450–453.

[9] A. Kotzig, On a Certain Decomposition of a Graph, *Mat.-Fyz. Časopis* **5** (1955), 144–151.

[10] P. C. Liu and R. C. Gelmacher, On the Deletions of Nonplanar Edges of a Graph, *Congressus Numerantium* **24** (1979), 727–738.

[11] A. Mansfeld, Determining the Thickness of a Graph is NP-Hard, *Math. Proc. Cambridge Philos. Soc.* **93** (1983), 9–23.

[12] B. Oporowski, D. Zhao, Coloring graphs with crossings, *Discrete Math.*, **309** (2009),2948–2951.

[13] V. Pinciu and W. Soss, A Note on the Skewness of a Graph, *Congressus Numerantium* **227** (2016), 209–214.

[14] V. Pinciu, Skewness, Minimum Degree and Chromatic Number, *Congressus Numerantium* **228** (2017), 97–101.

[15] G. Ringel and J. M. Youngs, Solution of the Heawood Map-Coloring Problem, *Proc. Nat. Acad. Sci. USA* **60** (1968), 438–445.

[16] N. Robertson, The Smallest Graph of Girth 5 and Valency 4, *Bull. Amer. Math. Soc.* **70** (1964) 824–825.

[17] N. Robertson, D.P. Sanders, P.D. Seymour, and R. Thomas, The Four-Color Theorem, *J. Combin. Theory Ser. B* **70** (1997), 2–44.

[18] C. Thomassen, The graph genus problem is NP-complete, *Journal of Algorithms* **10 (4)** (1989), 568–576.