Game $\mathcal{R}$-Proper Chromatic Numbers

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Abstract

Consider the following two-person game on a graph G. The two players start with two color choices only, taking turns coloring any uncolored vertex with the restriction that any coloring must be a proper coloring. A third (or fourth, etc.) color can only be used when forced to maintain a proper coloring. One player, the minimizer, is trying to force the smallest number of colors possible. The other player, the maximizer, is trying to force the largest number of colors possible. This game proper chromatic number, denoted $\chi_{(E,g)}(G)$, is the minimum number colors used when both players play optimally. The advantage of the game proper chromatic number is that it is comparable to other published game chromatic variants, particularly the game chromatic number II and the game Grundy number.

This paper also considers extensions of the game proper chromatic number through generalized regions of the graph. Let $R = \{R_1, R_2, \ldots, R_t\}$ such that $\bigcup R_i = V(G)$. It is convenient to think of these R_i's as regions of interest in graph G. In particular, extensions to closed neighborhoods and open neighborhoods maintaining the restriction that all colorings must be "proper" in the sense that no R_i is monochromatic are considered for some natural classes of graphs. The minimum number of colors necessary provided each player plays optimally, following the rules established for the game proper chromatic number, is denoted $\chi_{(N[v],g)}(G)$ and $\chi_{(N(v),g)}(G)$ for the game closed neighborhood proper chromatic number and the game open neighborhood chromatic number, respectively.

Keywords: Chromatic Number, Proper Chromatic Number, Game Chromatic Number

Introduction

Let $G = (V, E)$ be a graph and $S \subseteq V(G)$ be an independent set when, for vertices $u, v \in S$, $uv \notin E(G)$. A vertex coloring is any partition of $V(G)$ into independent sets and $\chi(G)$ is the minimum number of sets needed to partition $V(G)$ into independent sets.

The game chromatic number was first introduced by Garnder in [6] but largely overlooked until rediscovered by Bodlaender in [2]. This idea was quickly developed in [5] and [3]. The game chromatic number is based on a modification of the coloring problem where two players take turns coloring the vertices of a graph. The first player wants to completely color the graph while the second player is trying to force a premature end to the game by leaving some vertex (or vertices) uncolorable from the list of available colors. The game chromatic number, $\chi_g(G)$, is the smallest number of colors necessary to guarantee the first player wins. See [1] for a good survey of game colorings on families of graphs.

In [4], Chen, Schlep, and Shreve introduced the game chromatic number II which adds another condition to the second player. In this version of the game, the second player can only use colors already played by the first player unless forced to use a new color to guarantee the graph maintains a proper coloring. The game chromatic number II, denoted $\chi_g^*(G)$, is the smallest number of colors necessary to guarantee the first player wins. Chen, Schelp, and Shreve showed the following relationships between their game chromatic number II, the game chromatic number, and the chromatic number of a graph.

Theorem 1 *[4] For any graph G, $\chi(G) \leq \chi_g^*(G) \leq \chi_g(G) \leq \Delta + 1$.*

In [7], Havet and Zhu introduced the greedy coloring game. In this game, the two players take turns choosing an uncolored vertex and color it in a greedy manner. One player, the minimizer, is trying to minimize the number of colors used to completely color the graph while the other player, the maximizer, is trying to maximize the number of colors used to completely color the graph. This leads to two greedy coloring games depending on whether minimizer or maximizer plays first. Assuming both players play optimally, the minimizer-first game Grundy number, denoted $\Gamma_g^{MIN}(G)$, is the number of colors used when minimizer has the first move; the maximizer-first game Grundy number, $\Gamma_g^{MAX}(G)$, is the number of colors used when maximizer has the first move. For natural classes of graphs these two games are equal. Note, however, that for a particular graph, the two games can be arbitrarily far apart.

Proposition 2 *[7] Suppose $\mathcal{H}$ is a class of graphs such that if $G \in \mathcal{H}$, then $2G \in \mathcal{H}$ and $G \uplus K_1 \in \mathcal{H}$. Then, $\Gamma_g^{MIN}(\mathcal{H}) = \Gamma_g^{MAX}(\mathcal{H})$.*

When classes of graphs meet the conditions of Proposition 2, then the minimizer-first game Grundy number is abbreviated to the game Grundy number, denoted $\Gamma_g(G)$. Clearly, $\chi(G) \leq \Gamma_g(G)$, but it is unknown if $\Gamma_g(G) \leq \chi_g(G)$ for all graphs.

Our new game proper chromatic number, in an effort to bridge these other coloring ideas, uses rules from both the game greedy coloring and game coloring. However, we will first detour to proper parameters and proper chromatic numbers.

Since colorings involve partitioning the vertex set into independent sets, we will consider an alternate definition of an independent set. A set $S \subseteq V(G)$ is an independent set if for each edge uv, $|S \cap \{u, v\}| < |\{u, v\}|$. That is, the intersection of S and each edge is a proper subset of that edge. Now, replace any edge with the closed neighborhood $N[v]$ of a vertex v. This new definition for a set S such that, for any vertex v, $|S \cap N[v]| < |N[v]|$ is the enclaveless parameter introduced by Slater in [11].

Now consider any arbitrary collection $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ of nonempty subsets of $V(G)$ such that $\bigcup_i R_i = V(G)$. An $\mathcal{R}$-proper set is defined to be a set S such that for any R_i, $|S \cap R_i| < |R_i|$ for $1 \leq i \leq t$. The maximum cardinality of an $\mathcal{R}$-proper set is denoted $\mathcal{M}(\mathcal{R})$ or $\mathcal{M}(\mathcal{R}; G)$. Specific instances of $\mathcal{R}$-proper sets include an independent set when $\mathcal{R}$ is the collection of edges, an enclaveless set when $\mathcal{R}$ is the collection of closed neighborhoods, and an open enclaveless set when $\mathcal{R}$ is the collection of open neighborhoods. For more information on $\mathcal{R}$-proper sets and other $\mathcal{R}$-parameters see [8] and [9].

First introduced in [8] and developed in [10], the $\mathcal{R}$-proper chromatic number, denoted $\chi_{\mathcal{R}}(G)$, is defined to be the minimum number of $\mathcal{M}(\mathcal{R})$-sets, that is, $\mathcal{R}$-proper sets, that partition $V(G)$. To avoid a chromatic number that is not defined for all graphs, $\chi_{\mathcal{R}}(G)$ is determined by requiring each R_i with $|R_i| \geq 2$ for $1 \leq i \leq t$ to be $\mathcal{R}$-proper. When $|R_i| = 1$, we ignore the fact that no color class c can be proper on the region R_i. This paper will consider the game version of several $\mathcal{R}$-proper colorings.

In our game, two players take turns choosing an uncolored vertex to color. One player, the minimizer, is trying to minimize the number of colors used to completely color the graph and a second player, the maximizer, is trying to maximize the number of colors used to completely color the graph. Assuming both play optimally, the minimizer-first game $\mathcal{R}$-proper chromatic number, denoted $\chi_{(\mathcal{R},g)}^{MIN}(G)$, is the number of colors used when minimizer plays first; the maximizer-first game $\mathcal{R}$-proper chromatic number, denoted $\chi_{(\mathcal{R},g)}^{MAX}(G)$, is the number of colors used when maximizer plays first. Here, $\mathcal{R}$ denotes the collection of nonempty subsets of $V(G)$.

For a particular graph, $\chi_{(\mathcal{R},g)}^{MIN}(G)$ and $\chi_{(\mathcal{R},g)}^{MAX}(G)$ can be arbitrarily far apart. For example, let $\mathcal{R}$ be the collection of edges so that $\chi_{\mathcal{R}}(G) = \chi_E(G) = \chi(G)$. Then, consider $K_{t,t}^*$ obtained from the complete bipartite graph $K_{t,t}$ by removing a perfect matching.

Proposition 3 *If $t \geq 3$, then $\chi^{MIN}_{(E,g)}(K^*_{t,t}) = t$ and $\chi^{MAX}_{(E,g)}(K^*_{t,t}) = 3$.*

Proof. Suppose minimizer plays first. Then maximizer responds by playing on the vertex joined to the one minimizer played by an edge of the removed matching. Clearly, one color is needed for every removed edge and so $\chi^{MIN}_{(E,g)}(K^*_{t,t}) = t$.

Suppose maximizer plays first. Denote the two partitions of $V(G)$ as V_1 and V_2. Since $t \geq 3$, maximizer is guaranteed to be able to color one vertex in V_1 color 1 and another color 2 which will force at least a third color in V_2. Thus, $\chi_{(E,g)}(K^*_{t,t}) \geq 3$.

Without loss of generality, the maximizer colors some vertex $v \in V_1$. Minimizer responds by playing the same color on another vertex in V_1 so that no vertex in V_2 can be placed into color class 1. To force a third color, maximizer will choose a third vertex in V_1 to color with color 2. If $t \geq 4$, minimizer will color a fourth vertex in V_1 with color 2 so that no vertices in V_2 can be in color class 2. Otherwise, $t = 3$ and minimizer will play color 2 on the vertex joined to the one maximizer played by an edge of the removed matching so that no other vertices in $K^*_{t,t}$ can be in color class 2. All uncolored vertices of V_2 will be in color class 3 and no other color classes can be forced in V_2. Any remaining vertices in V_1 can be in color class 1 or color class 2. Regardless, no other color class can be forced since none of the remaining vertices in V_1 are adjacent to a vertex in color class 1. Thus, $\chi_{(E,g)}(K^*_{t,t}) = 3$.

□

Despite the fact that for a particular graph G, $\chi^{MIN}_{(\mathcal{R},g)}(G)$ and $\chi^{MAX}_{(\mathcal{R},g)}(G)$ can be very different, for natural classes of graphs, these values are equivalent. For class $\mathcal{H}$, define $\chi^{MIN}_{(\mathcal{R},g)}(\mathcal{H}) = \max\{\chi^{MIN}_{(\mathcal{R},g)}(G) : G \in \mathcal{H}\}$ and $\chi^{MAX}_{(\mathcal{R},g)}(\mathcal{H}) = \max\{\chi^{MAX}_{(\mathcal{R},g)}(G) : G \in \mathcal{H}\}$. For readability, the proof of the following proposition is written under the assumption that the $\mathcal{R}$-proper partitions involve only the vertex set of G and modifications for other partitions are discussed afterward. For such $\mathcal{R}$-proper partitions only involving the vertex set of G, let G^+ denote a copy of G with an extra isolated vertex. These ideas follow from those in [12] and [7].

Proposition 4 *Let $\mathcal{H}$ be a class of graphs such that if $G \in \mathcal{H}$, then $2G$ and G^+ are also in $\mathcal{H}$. Let $\mathcal{R}$ be an arbitrary partition of $V(G)$. Then $\chi^{MIN}_{(\mathcal{R},g)}(\mathcal{H}) = \chi^{MAX}_{(\mathcal{R},g)}(\mathcal{H})$.*

Proof. Assume $\chi^{MAX}_{(\mathcal{R},g)}(\mathcal{H}) = k$. There exists $G \in \mathcal{H}$ such that $\chi^{MAX}_{(\mathcal{R},g)}(G) > k - 1$, that is, maximizer has a strategy, called MAXWIN-strategy, to ensure that the maximum color used in the $\mathcal{R}$-proper chromatic game is at least k.

Assume that G has an odd number of vertices. Let the two copies of G in $2G$ be G_1 and G_2. Assume minimizer colors a vertex of G_1 in their first move. Then, maximizer colors a vertex of G_2 in their first move according to MAXWIN-strategy. From then on, whenever minimizer colors a vertex of G_2, maximizer also colors a vertex of G_2, using MAXWIN-strategy. Because G_1 has an odd number of vertices whenever minimizer colors a vertex in G_1, there exists an uncolored vertex in G_1 that maximizer can color. Thus, $\chi^{MIN}_{(\mathcal{R},g)}(2G) > k - 1$. If G has an even number of vertices, the same argument shows maximizer has a strategy for the minimizer first coloring game on $(2G)^+$. This proves $\chi^{MIN}_{(\mathcal{R},g)}(\mathcal{H}) \geq k$.

Now assume that $\chi^{MIN}_{(\mathcal{R},g)}(\mathcal{H}) = k$. There exists a graph $G \in \mathcal{H}$ such that $\chi^{MIN}_{(\mathcal{R},g)}(G) > k - 1$. Then, for G^+, maximizer can color the isolated vertex for their first move and follow the strategy for the minimizer first game afterward. Thus, $\chi^{MAX}_{(\mathcal{R},g)}(\mathcal{H}) \geq k$.

Therefore, $\chi^{MIN}_{(\mathcal{R},g)}(\mathcal{H}) = \chi^{MAX}_{(\mathcal{R},g)}(\mathcal{H})$

□

Note that the proof of this proposition is easily modified for edge-set partitions by changing vertices to edges and the isolated vertex to an isolated K_2. Other partitions that involve a mix of vertices and edges can also be proven in the same way by coloring vertices and edges with G^+ representing G with an isolated vertex. As all other details are exactly the same, the formal write-up of these modifications is omitted.

Game Proper Chromatic Number

In this section, we consider the game proper chromatic number where $\mathcal{R}$ is the collection of edges and compare this game coloring to other published game colorings. Because forests follow the hypotheses of Proposition 4, we start by considering forests for the game proper chromatic number.

Theorem 5 *For any forest F, $\chi_{(E,g)}(F) \leq 3$.*

Proof. Assume there exists a tree such that $\chi_{(E,g)}(T) \geq 4$. Then, T must contain the induced subgraph S_T with a coloring isomorphic to the one depicted in Figure 1. Minimizer would avoid this coloring by choosing one of the center vertices earlier unless each endpoint in S_T that is colored 1 is also the center vertex of another double star. This results in an induced caterpillar, C_T, with a coloring isomorphic to the one depicted in Figure 2. This coloring, however, is not optimal for the minimizer who can always avoid it.

Consider the centers of the three double stars and label them A, B, and C. Minimizer can reduce the number of colors on C_T provided they can

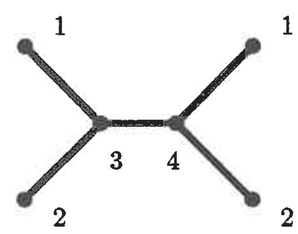

Figure 1: Induced subgraph S_T in Theorem 5.

force at least one vertex in color class 1 or color class 2 in each of the three centers A, B, and C. Assume minimizer is the first to play on C_T. Then, minimizer plays color 1 on either vertex in B; without loss of generality, assume it is the one adjacent to region A. At best, maximizer can block one vertex in region A, either by playing color 2 on the endpoint of the vertex adjacent to the one colored by minimizer in B (so the vertex adjacent to these must receive a third color) or by playing either color on the endpoint adjacent to the other center vertex in A. In either case, minimizer can color one of the center vertices in A with color 1 or color 2.

Assume maximizer is the first to play on C_T. Maximizer will not choose to play on A, B, or C unless no other plays are available. So, maximizer will play on one of the endpoints. If this is an endpoint of B, then minimizer responds by playing the parent in the opposite color and play proceeds as before. If this endpoint is one of the endpoints of A, then minimizer responds by coloring the closest center vertex in B with color 1. Maximizer can, at best, force only one vertex in A to be in color class 3 leaving the other available to be in color class 1 or color class 2, depending on maximizer's earlier plays.

Center C follows the same pattern as A so that minimizer can always keep the total number of colors on C_T to at most 3. Thus, $\chi_{(E,g)}(F) \le 3$. □

Havet and Zhu [7] showed that $\chi_g(G)$ can be arbitrarily large in comparison to $\Gamma_g(G)$, but it is unknown if $\chi_g(G)$ is always an upper bound for

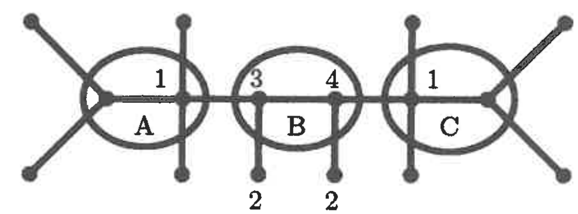

Figure 2: Caterpillar C_T for Theorem 5.

the game Grundy number. The game proper chromatic number, however, is always an upper bound for the game Grundy number.

Theorem 6 *For any graph G, $\Gamma_g(G) \le \chi_{(E,g)}(G)$.*

Proof. Assume $\chi_{(E,g)}(G) = k$. Let S be a strategy for coloring the vertices that achieves $\chi_{(E,g)}(G) = k$. Remove all vertices in color classes 1 and 2 so that the remaining vertices were colored using a greedy strategy and call the resulting subgraph H. Since the vertices in H were colored using the forced strategy of S, $\Gamma_g(H) \le k - 2$.

Claim: $\Gamma_g(G) \le \Gamma_g(H) + 2$.

Suppose not. Then, maximizer has a strategy that forces (in a greedy manner) at least one new color class on $G - H$. But, maximizer could have followed this strategy in S so some vertex $v \in V(G - H)$ would be in a new color class in S which contradicts the optimality of S.

Thus, $\Gamma_g(G) \le \Gamma_g(H) + 2 \le k - 2 + 2 = k = \chi_{(E,g)}(G)$.

□

The game proper chromatic number can also be compared to the game chromatic number II. One advantage of the game proper chromatic number is the comparability it has with other game chromatic variants.

Theorem 7 *For any graph G with at least one edge, $\chi_g^*(G) \le \chi_{(E,g)}(G)$.*

Proof. Note that, since G contains at least one edge, $\chi_{(E,g)}(G) \ge 2$. Assume $\chi_{(E,g)}(G) = k$ and S is the coloring strategy to achieve this value.

Case 1: S has at least two vertices of color 1. Minimizer colors one of these vertices with color 1 and maximizer colors the other with color 1. Minimizer then colors any of the color two vertices from S with color 2. Play proceeds following the same strategy as S. This resulting coloring,

R, uses the same number of colors as S. Furthermore, maximizer has no better options. If a more optimal strategy than R existed for maximizer then on maximizer's first play, they could choose the better vertex to be a color 1, which would result in S not being an optimal strategy. R, however, may not be optimal for minimizer. Regardless, $\chi_g^*(G) \le \chi_{(E,g)}(G)$.

Case 2: S has at least two vertices of color 2 and one vertex of color 1. This is equivalent to the previous case with colors 1 and 2 switched.

Case 3: S has only one vertex of color 1, label it u, and only one vertex of color 2, label it w. Because there are only two vertices colored 1 and 2, for every vertex $v \in V(G)$ other than u and w, both uv and vw are in the edge set of the graph since every vertex forced into a higher color class must be adjacent to both a color 1 and a color 2 vertex. Assume u and w are independent. Then, in the $\chi_g^*(G)$ coloring, minimizer plays color 1 on vertex u, maximizer is forced to play color 1 on vertex w, and all other vertices are able to be colored following the same strategy as S, reducing the total number of colors necessary so that $\chi_g^*(G) < \chi_{(E,g)}(G)$. If u and w are not independent, then $N[u] = V(G)$. So, minimizer plays color 1 on u and maximizer can play color 2 on any vertex which might as well be w. If maximizer had a better choice of a vertex for color 2, maximizer would have done so under strategy S. Thus, maximizer has achieved optimal play so that $\chi_g^*(G) \le \chi_{(E,g)}(G)$.

□

Combining our results with the results of Chen, Schelp, and Shreve stated in Theorem 1, we state the following conjecture relating these game chromatic variations.

Conjecture 8 *For any graph G with at least one edge,* $\chi(G) \le \chi_g^*(G) \le \Gamma_g(G) \le \chi_{(E,g)}(G) \le \chi_g(G) \le \Delta + 1$.

Game Neighborhood Proper Chromatic Numbers

In this section, we consider two game neighborhood proper chromatic numbers: the game closed neighborhood proper chromatic number, where $\mathcal{R}$ is the collection of closed neighborhoods, and the game open neighborhood proper chromatic number, where $\mathcal{R}$ is the collection of open neighborhoods. In the closed neighborhood proper chromatic game, a play is valid only if the play does not create a closed neighborhood that is entirely monochromatic. The game closed neighborhood proper chromatic number, $\chi_{(N[v],g)}(G)$ is the number of colors when both players play optimally. In the open neighborhood proper chromatic game, a play is valid only if the play does not create an open neighborhood that is entirely monochromatic. The game

open neighborhood proper chromatic number, $\chi_{(N(v),g)}(G)$ is the number of colors used when both players play optimally.

First, consider the game closed neighborhood proper chromatic number, which turns out to be trivial.

Theorem 9 *For any graph G with at least one edge,* $\chi_{(N[v],g)}(G) = 2$.

It will take two lemmas to prove this theorem.

Lemma 10 *Let graph G be a connected graph. Then* $\chi_{(N[v],g)}(G) \le 3$.

Proof. Suppose there exists a graph G with $\chi_{(N[v],g)}(G) = k$ where $k \ge 4$. Let x be a vertex in color class k. Since x is in color class k, x must have a neighbor v_r such that $N[v_r] - x$ is entirely contained in color class r for $1 \le r \le k-1$.

Now consider all neighbors of x in color class r such that $r \ge 3$. For v_r to be in color class r, v_r must have neighbors u_j such that $N[u_j] - v_r$ is entirely contained in color class j for $1 \le j \le r-1$. But this means that v_r has neighbors in color classes 1 through $r-1$ for all $r \ge 3$ and so $N[v_r] - x$ is not entirely contained in a single color class. This implies that x can always be in color class 3 so that $\chi_{(N[v],g)}(G) \le 3$.

□

Lemma 11 *Let graph G be a connected graph with vertices u and v such that $uv \notin E(G)$. Then,* $\chi_{(N[v],g)}(G+uv) \le \chi_{(N[v],g)}(G)$.

Proof. Note that maximizer has a strategy to achieve $\chi_{(N[v],g)}(G)$ and this same strategy must produce a valid coloring for $G + uv$. Suppose, however, that $\chi_{(N[v],g)}(G+uv) > \chi_{(N[v],g)}(G)$. Since G is connected, Lemma 10 implies that if such a graph exists then $\chi_{(N[v],g)}(G+uv) = 3$ and $\chi_{(N[v],g)}(G) = 2$. If maximizer has a strategy that achieves $\chi_{(N[v],g)}(G+uv) = 3$ where vertices u and v are not in color class 3, then that same strategy would imply $\chi_{(N[v],g)}(G) = 3$. So, only u or v can be in the third color class. Without loss of generality, assume it is u. Maximizer can force this third color only if $N[v] - u$ is entirely contained in a single color class. Call this color class 1. Furthermore, there must exist vertex $v_2 \in N_G[u]$ such that $N[v_2] - u$ is entirely contained in color class 2. Minimizer, however, can always avoid this situation in $G + uv$. If maximizer plays first on v or v_2, then minimizer can respond by coloring the other vertex (v_2 or v, respectively) the same color as maximizer played. Vertex u is now forced into color class 1 or 2.

Minimizer will not choose to play first on v or v_2 unless all vertices adjacent to v and all vertices adjacent to v_2, other than u, have been colored. Then, minimizer will play vertex u into either color class 1 or color class 2.

Note that v and v_2 are now colorable using colors 1 and 2. Thus, there is no connected graph where $\chi_{(N[v],g)}(G+uv) > \chi_{(N[v],g)}(G)$.

□

Note that this lemma could be written as $\chi_{(N[v],g)}(G) \leq \chi_{(N[v],g)}(G-e)$ for any e that is not a bridge. This means that the largest possible value for $\chi_{(N[v],g)}(G)$ on n vertices is achieved by a tree.

Proof of Theorem 9. Since each component of G can be considered separately, assume G is connected. Using Lemma 11, an upper bound for $\chi_{(N[v],g)}(G)$ can be found by considering a spanning tree, T, of G. Root T at any vertex v_r such that $\deg v_r \geq 2$. This is always possible unless $T = K_2$ in which case, it is clear that $\chi_{(N[v],g)}(K_2) = 2$. Minimizer will color vertices using the following algorithm. Maximizer colors vertex v, then:

1. Minimizer responds by coloring the parent of v the other color.
2. If the parent of v is already colored, minimizer responds by coloring a child of v the other color.
3. If the parent of v and all children are already colored, minimizer colors any uncolored vertex that is adjacent to a colored vertex the opposite color.
4. If no such vertices exist, then minimizer starts the game by coloring a parent of an endpoint.

Claim: $\chi_{(N[v],g)}(T) \leq 2$.

Suppose maximizer can force a third color on some vertex v. For this to occur, there exists vertex u, a neighbor of v, such that $N[u]-v$ is entirely contained in color class 1. There also exists vertex w, another neighbor of v, such that $N[w]-v$ is entirely contained in color class 2.

Case 1: Suppose that v is the parent of both u and w and only children of v have open neighborhoods potentially contained in a single color class. Since all of u's children are in color class 1, maximizer has to be the player to play color 1 on u. But, as soon as maximizer colors u with color 1, minimizer will color v with color class 2. By assumption, the parent of v, v_p, can not have $N[v_p]-v$ contained entirely within color class 2 and so coloring v with color 2 is possible.

Case 2: Without loss of generality, suppose that u is the parent of v and w is the child of v. As before, maximizer must be the player to color vertex w with color 2 because minimizer would change the color class by the algorithm. If u is uncolored when maximizer colors w, then minimizer would color v with color class 1. So, u must be colored before w. Furthermore, for u_p, the parent of u, and u to be in the same color class, maximizer must be the player to color u since minimizer would have colored u with color 2.

But, since the parent of u is colored and v is uncolored, minimizer would color vertex v or another child of u with color class 2. If another child of u is colored color 2, then $N[u]-v$ is not monochromatic and v can be in either color class. If v is colored color 2, then the fact that w is not already colored guarantees that v can be in color class 2 (and that w will be forced to be in color class 1).

Since T is a tree, u and w cannot both be parents of v, so $\chi_{(N[v],g)}(T) \leq 2$.

Since $\chi_{(N[v],g)}(G) \leq \chi_{N[v],g)}(T) \leq 2$ and $\chi_{(N[v],g)}(G) \geq 2$, provided G contains an edge, $\chi_{(N[v],g)}(G) = 2$.

□

The open neighborhood chromatic number, $\chi_{N(v)}(G)$, is not trivial and thus the game open neighborhood chromatic number is interesting for further study.

Theorem 12 (Sinko, [10]) *For any graph G,* $\chi_{N(v)}(S(G)) = \chi(G)$

Thus, there exist graphs such that $\chi_{(N(v),g)}(G) \geq k$ for any k. For example, the subdivided complete graph K_k has $\chi_{(N(v),g)}(S(K_k)) = k$ as seen in Figure 3. While the non-gamed version of the open neighborhood proper chromatic number is clearly a lower bound for the game open neighborhood proper chromatic number, it is also related to the sharp upper bound of the game version.

Theorem 13 *For any graph G,* $\chi_{N(v)}(G) \leq \chi_{(N(v),g)}(G) \leq \chi_{N(v)}(G)+1$.

Proof. Clearly, $\chi_{N(v)}(G) \leq \chi_{(N(v),g)}(G)$. Assume there exists a graph G such that $\chi_{(N(v),g)}(G) \geq \chi_{N(v)}(G)+2$. To achieve this, G must contain a subdivided double star as a subgraph (which may or may not be induced) with a coloring isomorphic to the coloring given in Figure 4. Minimizer can play on the joints of the double star unless the endpoints colored 1 are also joints of double stars, leading to the subgraph S_G shown in Figure 5. Consider the regions A, B and C on S_G. To reduce the total number of colors, minimizer needs to guarantee that in each one of these regions, at least one vertex of degree three is colored with either color 1 or color 2. Note that while B must be disjoint from regions A and C, regions A and C could overlap one another.

Maximizer will not choose to play in any of these regions with color 1 or 2 unless no other option is allowed, so we can assume minimizer is the first to play in these regions. Minimizer will play color 1 on either of the two degree three vertices in region B; without loss of generality assume it is the one closest to region A. At most, maximizer can force a third color on vertex u by playing color 2 on one of the endpoints "adjacent" to u (that is, at distance two), leaving the other degree three vertex in region A available

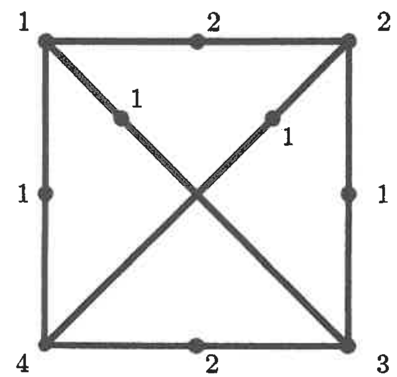

Figure 3: $\chi_{(N(v),g)}(S(K_k) = 4$

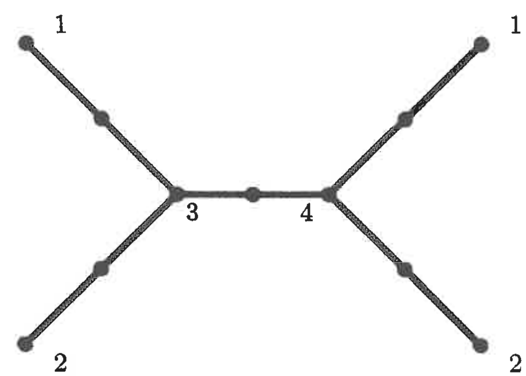

Figure 4: A subdivided double star for Theorem 13.

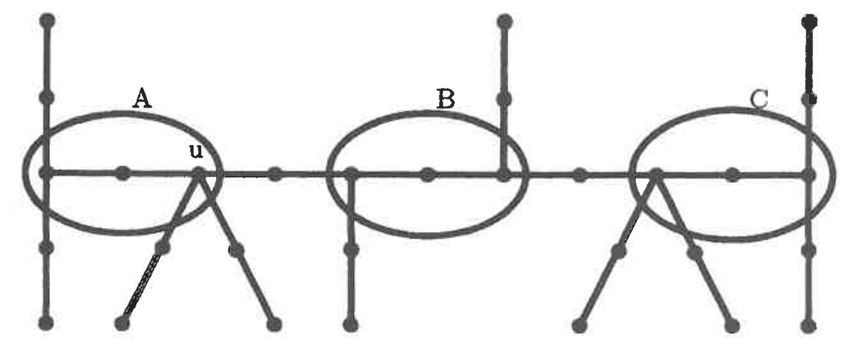

Figure 5: S_G for Theorem 13.

for minimizer to play color 1. Alternatively, maximizer can block the other degree three vertex in region A from being either color 1 or color 2 but not both.

If maximizer is the first to play on S_G, maximizer will play on one of the endpoints. This first play, at most, can force either u or the other degree three vertex in A to be in color class 3, but the other vertex must be available for color class 1 or color class 2, depending on maximizer's first play.

If regions A and C have no overlap, a similar strategy guarantees minimizer can color one of the degree three vertices in region C with color 1 or color 2. If A and C have some overlap, assume this overlap is a vertex in color class 3 as otherwise region C already has a vertex contained in either color class 1 or color class 2. Consider the other degree three vertex in region C. At most, maximizer can block this vertex from being in either color class 1 or color class 2 by choosing to play on an endpoint vertex. However, maximizer cannot block both color classes, so minimizer responds by playing the other color on this vertex resulting in all three regions A, B, and C containing either a color 1 or a color 2 vertex on one of the degree three joints. This will reduce the total number of colors in the optimal game and so $\chi_{(N(v),g)}(G) \leq \chi_{N(v)}(G) + 1$.

□

The following result from [10] combined with Theorem 13 will allow us to claim that for all forests $\chi_{(N(v),g)}(F) \leq 3$.

Theorem 14 (Sinko, [10]) *For any tree T with $n \geq 3$, $\chi_{N(v)}(T) = 2$.*

Corollary 15 *For any forest F, $\chi_{(N(v),g)}(F) \leq 3$*

Last, the game open neighborhood chromatic number is determined for all paths and cycles.

Theorem 16 *For P_7, $\chi^{MIN}_{(N(v),g)}(P_7) = 2$ and $\chi^{MAX}_{(N(v),g)}(P_7) = 3$. For P_8, $\chi^{MIN}_{(N(v),g)}(P_8) = 3$ and $\chi^{MAX}_{(N(v),g)}(P_8) = 2$. For any other n, $\chi_{(N(v),g)}(P_n) = \begin{cases} 2 & \text{if } n < 7 \\ 3 & \text{if } n > 8 \end{cases}$.*

Proof. Clearly, $2 \leq \chi_{(N(v),g)}(P_n) \leq 3$. To force a third color, maximizer needs to color a vertex distance four away from a previously colored vertex.

Assume $n \leq 6$ and minimizer plays first. Minimizer plays on either the center vertex, if n is odd, or either of the two center vertices if n is even. Since the length of the path is at most six, there are no vertices at distance four from this vertex. Each time maximizer plays, minimizer responds by checking to see if there is an uncolored vertex at distance four. If so, minimizer colors this vertex using the same color maximizer just used. Such a coloring is always possible. If no such coloring was possible, then the vertex minimizer colors must be at distance four from the vertex maximizer colored and there must be another already colored vertex distance two from the vertex minimizer is considering on the other side which means the length of the path must be at least seven. If maximizer plays first, minimizer follows the same strategy. Since the length of the path is no more than six, each vertex has at most one vertex at distance four on which minimizer can play to block maximizer from forcing a third color. Thus, for $n < 7$, $\chi_{(N(v),g)}(P_n) = 2$.

Assume $n = 7$ and minimizer plays first. Minimizer starts by playing on the center vertex which has no vertex at distance four for the maximizer to choose. So, maximizer chooses randomly among the remaining vertices, each of which has exactly one vertex at distance four. Call the pairs of vertices at distance four partners. Minimizer responds to maximizer by coloring the partner vertex the same as the color maximizer used. Because these partners are in the same order across the center, a coloring is valid on one side if it is valid on the other side, so minimizer can always play the same color as maximizer and thus $\chi^{MIN}_{(N(v),g)}(P_7) = 2$. If maximizer plays first, they will choose to play on the center vertex because otherwise minimizer follows the same strategy as before responding by playing the same color at distance four. Minimizer will respond by playing on one of the two vertices at distance two from the center. Note that to be open neighborhood proper colored, this vertex must be the other color from the center vertex. If minimizer chooses any other vertex to play, maximizer immediately forces a third color. The newly colored vertex by minimizer has one partner at distance four, and, the only valid open neighborhood proper color for this vertex is the same color used on its partner. This is what maximizer plays. Now, minimizer is choosing first among the partners so that maximizer can respond by playing the other color on the partner vertex to minimizer's play always forcing the need for a third color, so that $\chi^{MAX}_{(N(v),g)}(P_7) = 3$.

Assume $n = 8$ and minimizer plays first. No matter where minimizer plays, there exists a vertex at distance four. Maximizer responds by playing this vertex and changing colors to force a third color on the middle vertex between the partners so that $\chi^{MIN}_{(N(v),g)}(P_8) = 3$. Assume maximizer plays first. Each vertex has exactly one partner vertex that is distance four away. Minimizer always responds by playing the same color on this partner vertex. Due to the relationship between these partner vertices, so long as minimizer always responds by using the same color as maximizer, this coloring is a valid play provided maximizer's previous move was valid. Minimizer can follow this strategy to color the entire path, so $\chi^{MAX}_{(N(v),g)}(P_8) = 2$.

Assume $n \geq 9$ and minimizer plays first. Regardless of where minimizer plays, there exists a vertex at distance at least four from minimizer's play. Maximizer colors this vertex the second color which forces a third color to completely color the path. Now assume maximizer plays first. Maximizer colors either the center vertex, if n is odd, or either of the two center vertices, if n is even. This guarantees that there are two vertices of distance four from this first vertex. Minimizer can color one of them the same color as maximizer, but maximizer is guaranteed to be able to switch the color of the other distance four vertex on their second play, forcing a third color to completely color the graph. Thus, $\chi_{(N(v),g)}(P_n) = 3$ if $n > 8$.

□

Theorem 17 *For C_8, $\chi^{MIN}_{(N(v),g)}(C_8) = 3$ and $\chi^{MAX}_{(N(v),g)}(C_8) = 2$. For any* *other* n, $\chi_{(N(v),g)}(C_n) = \begin{cases} 2 & \text{if } n < 5 \\ 3 & \text{if } n \geq 5, n \neq 8 \end{cases}$

Proof. Clearly, $2 \leq \chi_{(N(v),g)}(C_n) \leq 3$.

Assume $n < 5$. To force a third color, maximizer must play a vertex along a path of length 4 from a colored vertex. Since $n < 5$, no such vertex exists and so $\chi_{(N(v),g)}(C_n) = 2$ for $n < 5$.

Assume $n \geq 5$ and $n \neq 8$. Assume minimizer plays first. Then, there exists a vertex along a path of length four from the vertex minimizer colors. This path may not be the shortest path. Maximizer changes the color for this vertex which forces a third color to completely color the cycle. Assume maximizer plays first. For all cycles of length at least five, except for length eight, there are two distinct vertices along paths of length four from the first vertex maximizer colors. At most, minimizer can color one of them the same color as maximizer played leaving at least one vertex for maximizer to change the color to force a third color. So, for $n \geq 5$ and $n \neq 8$, $\chi_{(N(v),g)}(C_n) = 3$.

Consider C_8 and assume minimizer plays first. There exists a single vertex along a path of length four from the vertex minimizer colors. Maximizer colors this vertex using the second color which forces a third color to completely color the graph. Thus, $\chi^{MIN}_{(N(v),g)}(C_8) = 3$. Assume maximizer plays first. For each vertex in C_8, there is exactly one partner vertex that is along a path of length four from the original. Each time maximizer plays, minimizer responds by coloring the partner vertex the same color as the maximizer used. Because of the symmetry in the graph, minimizer's color choice is valid because it is a valid color anytime maximizer's choice for the partner vertex was valid. Thus, minimizer is always able to keep the colors down to two and so $\chi^{MAX}_{(N(v),g)}(C_8) = 2$.

□

Extensions

To construct the $\mathcal{R}$-proper definitions, we considered an alternate definition of independent sets. Another definition for independent sets states that $S \subseteq V(G)$ is an independent set if, for every edge uv, $|S \cap \{u,v\}| \leq 1$. Replacing edges with an arbitrary collection $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ results in a set S such that for every region R_i, $|S \cap R_i| \leq 1$ for $1 \leq i \leq t$. These are defined to be the $\mathcal{R}$-limited parameters and more about these parameters can be found in [9]. Of course, we can define chromatic parameters where we look for partitions of $V(G)$ into these $\mathcal{R}$-limited sets to define $\mathcal{R}$-limited chromatic numbers. These chromatic numbers are discussed in [10]. However, these $\mathcal{R}$-limited chromatic numbers have not been gamed yet, and these generalized game $\mathcal{R}$-limited chromatic numbers may provide insight into the connections between these and other chromatic numbers.

Another area for study are the game $\mathcal{R}$ achromatic numbers. The $\mathcal{R}$ achromatic number, $\psi_{\mathcal{R}}(G)$, is defined to be the largest number such that each pair of color classes is contained in some R_i, that is, no pair of color classes can be combined into a single color class. Of course, a game also could be played on these graphs where the goal is to maximize the total number of color classes.

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