GRAPH NIM

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ABSTRACT. The game of Nim is at least centuries old, possibly originating in China, but noted in the 16th century in European countries. It consists of several stacks of tokens, and two players alternate taking one or more tokens from one of the stacks, and the player who cannot make a move loses. The formal and intense study of Nim culminated in the celebrated Sprague-Grundy Theorem, which is now one of the centerpieces in the theory of impartial combinatorial games.We study a variation on Nim, played on a graph. Graph Nim, for which the theory of Sprague-Grundy does not provide a clear strategy, was originally developed at the University of Colorado Denver. Graph Nim was first played on graphs of three vertices. The winning strategy, and losing position, of three vertex Graph Nim has been discovered, but we will expand the game to four vertices and develop the winning strategies for four vertex Graph Nim. This work was published as a chapter in the Master's Thesis of Trevor Williams[8]

1. Introduction

Nim consists of several stacks of tokens called heaps, and two players alternate taking one or more tokens from a single heap. Now consider a generalization of this game in which the heaps are arranged in some order, and certain pairs of heaps are associated. Then, instead of selecting a single heap to remove tokens from a player can now select heaps that are associated and remove any number of tokens from any of those heaps. For example, Figure 1 shows an example game of Nim. In this example the heaps are associated in a triangular shape. Now a player can select any two heaps to remove tokens from. With more heaps we could develop even more complex arrangements.

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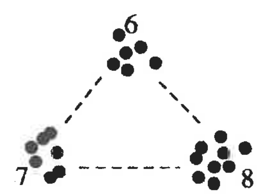


FIGURE 1. An Example of Generalizing Nim

This game can be visualized using a generalization of a graph which allows multiple edges between vertices. We may model this variation of Nim to with a graph by letting the heaps be edges, and associated heaps incident to some common vertex. Then on their turn each player selects a vertex and then removes any number of edges incident to that vertex. This game is called Graph Nim. An

FIGURE 2. An Example of Graph Nim

example of a position of this game is given in Figure 2.

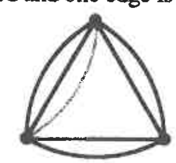
This generalization of Nim is similar to another generalization of Nim called Circular Nim[3]. In Circular Nim the heaps are arranged in a circle and, for a given k, a player may select k consecutive heaps to remove tokens from. Graph Nim differs from this in that the arrangement of the heaps need not be circular and the number of heaps a player may remove tokens from is given by the adjacencies of the graph. Graph Nim is not easily solved with the use of the Sprague-Grundy Theorem (which will be discussed later), but we use arguments that rely on the structural properties of the graphs. However, winning strategies and losing positions must be discovered on a graph-by-graph basis.

2. Graph Nim on a 3-vertex Graph

We will begin with Graph Nim on a 3-vertex graph, as shown in Figure 2. The winning strategy and losing position for this graph have been discovered at a Research Experience for Teachers at the University of Colorado-Denver. It is also commonly assigned as a homework problem in Dr. David Brown's Discrete Mathematics course, but this winning strategy and losing position have yet to be published, and will therefore be presented here.

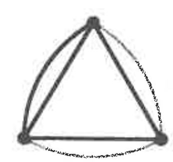
Before presenting the losing position of 3-vertex Graph Nim, we will first present a detailed example of how this game is played for clarity. Let the starting position of this example game be the graph shown in Figure 2.

The first player begins by selecting a vertex and now can delete any edges incident to that vertex. Suppose the top vertex is selected and one edge is removed.



Now the second player has this new graph to play on:

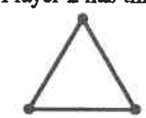
So the second player selects a vertex and can delete any edges incident to that vertex.



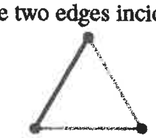
This gives Player 1 the following graph to play on:

So Player 1 selects the bottom left vertex and deletes an edge incident to it.

Then Player 2 has this graph to play on:



They select the bottom right vertex and delete two edges incident to it.



Player 1 plays on this graph and can simply remove the last edge and therefore wins.

FIGURE 3. An Example of How Graph Nim is Played

This example also shows the winning strategy for Graph Nim on a 3-vertex graph. When the game was reduced to a triangle Player 2 had no way to win. No matter what vertex Player 2 had selected they could only remove 1 or 2 edges. In either case all edges left in the game are incident to the same vertex and therefore Player 1 can win. This leads to the following theorem.

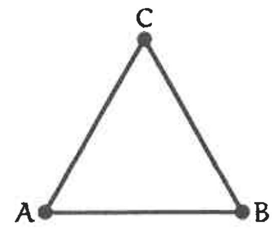


FIGURE 4. A Losing Position for Graph Nim on 3 Vertices

Theorem. *The losing position of Graph Nim played on 3-vertex graph is any 3-vertex graph such that there are an equal number of edges incident to every vertex.*

This theorem will be proved shortly, but first, some notation will be introduced. When playing Graph Nim on a 4-vertex graph it can be cumbersome to draw multiple edges between the vertices, as seen in Figure 5, therefore we will introduce a new way to represent the heaps.

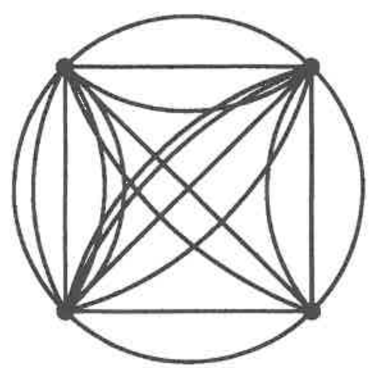


FIGURE 5. Graph Nim on a 4-vertex Graph Using the Multiple Edge Representation.

Definition. Define $w(x, y) \in \mathbb{N}$ to be the weight on the edge incident to the vertices x and y.

Now Graph Nim will be played as follows; on their respective turns each player selects a vertex and can reduce the weight of any number of edges incident to that vertex by non-negative integer amounts. On their turn a player must reduce the weight of at least 1 edge by at least 1, once an edge is weighted zero it can no longer be played. The winning player is the player that reduces the last edge to a weight of zero. A position of Graph Nim is given by the function $p(G)$ that

assigns weights to the edges of a graph, G. The losing position of a game will be denoted $\mathcal{L}p(G)$. Also, let K_n denote the complete graph on n vertices, that is, the n-vertex graph with each pair of vertices adjacent. Now the previous theorem will be restated using this new language. Where the vertex labeling is given by figure 4.

Theorem 1. $\mathcal{L}p(K_3) = \{w(A,B), w(A,C), w(B,C) \mid w(A,B) = w(A,C) = w(B,C)\}$.

Proof. Examine the simplest losing position possible in the game, which is when all the edges are weighted 1. If the all the edges are weighted 1 on Player 1's turn they must pick a vertex, and can reduce the weight of either one or two edges. If Player 1 reduces the weight on two edges to zero, then there is only one left for Player 2 to reduce to zero. If Player 1 reduces only one edge weight to zero then the resulting graph will have a vertex that is incident to 2 edges and two vertices incident to 1 edge, Player 2 simply has to pick the vertex incident to 2 edges and can then reduce the weight of the final two edges to zero. By the same argument, if all the edges have weight 2 then Player 2 can either force all the edges to have weight 1 or win the game. Since on any move a player can reduce edge weights by any integer value greater than or equal to 1, any game position in which all edges have equal weight will also be a losing position. □

3. Graph Nim on a 4-vertex Graph

The winning strategy and losing position for every 4-vertex graph will be presented in this section. These graphs will be referred to by their common graph theoretic names [7]. The following graphs have trivial solutions and will not be discussed: $\overline{K_4}$, $2K_2$, $K_{1,2} \cup K_1$, $K_{1,3}$. Now consider C_4, depicted in Figure 6. This graph is nontrivial, so we prove the following theorem about it's losing position.

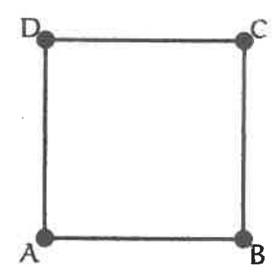


FIGURE 6. C_4

Theorem 2. $\mathcal{L}p(C_4) = \{w(A,B), w(B,C), w(C,D), w(D,A) \mid w(A,B) = w(C,D)$ and $w(B,C) = w(D,A)\}$, *where the labeling is as in Figure 6.*

Proof. Examine the simplest losing position possible in the game, which is when all the edges are weighted 1. If all the edges are weighted 1, on Player 1's turn they must pick a vertex, and can reduce the weight of either one or two edges. If Player 1 reduces the weight on two edges to zero, then there will be one vertex that is incident to the remaining 2 edges with weight greater than zero; Player 2 then simply has to select that vertex and reduce those 2 edges' weights to zero. If Player 1 reduces only one edge weight to zero then there will be 2 vertices incident to 2 edges, and the other 2 vertices will be incident to 1 edge. Without loss of generality let A and B be the vertices incident to 2 edges. If Player 2 selects either A or B and sets $w(A,B) = 0$, then the resulting game will only consist of 2 edges which are not incident to the same vertex. Therefore, Player 1 must reduce 1 of the 2 edge weights to zero and Player 2 will be able to reduce the other edge weight to zero. Since on any move a player can reduce edge weights by any integer value greater than or equal to 1, any game position in which opposite edges have equal weight will also be a losing position (note: in a graph on 4 vertices, if an edge xy is incident to two vertices, x and y, then the opposite edge is the edge that is not incident to either x or y). □

The winning strategy for Graph Nim on C_4 is to reduce the graph to the losing position. This can always be done: Player 1 simply has to select the vertex that is not incident to the 2 edges with the lowest weights and reduce the edges incident to that vertex to the same weight as the edge opposite it. If the initial position of the graph is the losing position, then Player 1 cannot win. This losing position was also found by M. Dufour and S. Heubach in the context of another variation of Nim called Circular Nim[3].

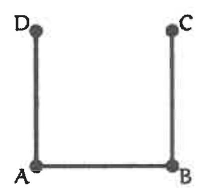


FIGURE 7. P_4

Now consider the strategy for Graph Nim on P_4, depicted in Figure 7. This is a special case of $p(C_4)$ where one of the edges already has an edge weight of zero, therefore, the winning strategy for Graph Nim on this graph is given by the winning strategy for Graph Nim on C_4. Without loss of generality let $w(A,D) > w(B,C)$, then Player 1 would simply select vertex A and reduce $w(A,B) = 0$ and $w(A,B) = w(B,C)$ the game is now in the losing position for Graph Nim on C_4. Then consider $K_3 \cup K_1$. This graph will have the same strategy as Graph Nim on K_3 because there is an isolated vertex. This strategy is discussed in the previous section.

Next consider $\overline{K_{1,2} \cup K_1}$, which is depicted in Figure 8. None of the winning strategies that have been presented thus far apply to this Graph Nim on this graph, and so we prove the following nontrivial theorem.

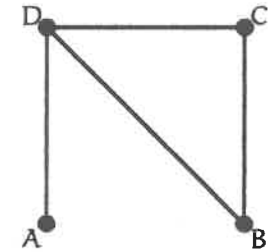


FIGURE 8. $\overline{K_{1,2} \cup K_1}$

Theorem 3. $\mathcal{L}p(\overline{K_{1,2} \cup K_1}) = \{w(A,D), w(D,C), w(D,B), w(C,B) \mid w(C,B) > w(A,D), w(D,C)$ and $w(D,B)\}$. *With respect to the labeling in figure 8.*

Proof. If $w(C,B) > w(A,D)$, $w(D,C)$ and $w(D,B)$ then Player 1 cannot reduce the game to $\mathcal{L}p(K_3)$ or $\mathcal{L}p(C_4)$ and therefore cannot win the game on the first move. □

The winning strategy for $\overline{K_{1,2} \cup K_1}$ is determined by $\mathcal{L}p(C_4)$. If the graph is labeled as in Figure 8 Player 1 will select vertex D, and reduce $w(D,C)$ and $w(D,B)$ to zero. Player 1 will also reduce $w(A,D)$ such that $w(A,D) = w(C,B)$. This will result in $\mathcal{L}p(C_4)$ for Player 2 to play on. This strategy has the condition that $w(A,D) \geq w(C,B)$. If this condition is not met, there is an alternative strategy based on $\mathcal{L}p(K_3)$. In this strategy Player 1 will again select vertex D, then reduce $w(A,D)$, $w(D,B)$, and $w(D,C)$ such that $w(A,D) = 0$ and $w(D,C) = w(D,B) = w(C,B)$. This will result in $\mathcal{L}p(K_3)$ for Player 2 to play on. This strategy has the conditions that $w(D,B) \geq w(C,B)$ and $w(D,C) \geq w(C,B)$.

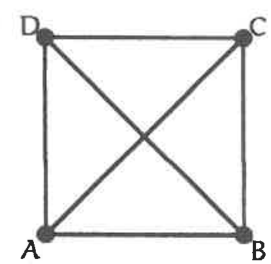


FIGURE 9. K_4

The winning strategy for Graph Nim on K_4, figure 9, is based on $\mathcal{L}p(\overline{K_{1,2} \cup K_1})$. Player 1 begins by finding the maximum edge weight in $p(K_4)$ the edge with this weight will be incident to 2 vertices, x and y; Player 1 should select one of the 2 vertices, a and b, that this edge is not incident to. Without loss of generality suppose Player 1 selects vertex a. Player 1 should then reduce $w(a,x)$, $w(a,y)$ and $w(a,b)$ such that $w(a,x) = w(a,y) = 0$ and $0 < w(a,b) < w(x,y)$. Unless $w(x,y) = w(b,x) = w(b,y)$ the game will be reduced to $\mathcal{L}p(\overline{K_{1,2} \cup K_1})$ for Player 2 to play on. This strategy will always work unless $w(x,y) = w(b,x) = w(b,y) = w(a,x) = w(a,y)$, but if $w(x,y) = w(b,x) = w(b,y) = w(a,x) = w(a,y)$ then Player 1 can select any vertex and remove all incident edges, the resulting position will be $\mathcal{L}p(K_3)$.

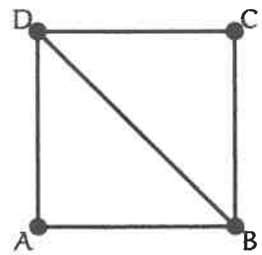


FIGURE 10. $\overline{K_2 \cup 2K_1}$

The winning strategy for Graph Nim on $\overline{K_2 \cup 2K_1}$, depicted in Figure 10, is the same as the winning strategy for Graph Nim on K_4, but with an extra condition. Again Player 1 begins by finding the maximum edge weight as in the strategy presented for Graph Nim on K_4 the edge with this weight will be incident to 2 vertices, x and y, and not incident to 2 vertices, a and b. Because $\overline{K_2 \cup 2K_1}$ only has 5 edges there is the possibility that $w(a,b) = 0$ in which case Player 1 cannot reduce the game to $\mathcal{L}p(\overline{K_{1,2} \cup K_1})$. If this is the case then Player 1 can only win if the game can be reduced to $\mathcal{L}p(C_4)$ or $\mathcal{L}p(K_3)$.

The losing positions and winning strategies for all 4-vertex graphs have now been developed. Some of these strategies and observations can be applied to graphs with more than 4 vertices.

4. Graph Nim on Other Graphs

As mentioned above, some losing positions and winning strategies can be applied to graphs with more than 4 vertices. For example, in the last section it was observed that Graph Nim on $2K_2$ was essentially Nim with 2 heaps. This can be generalized to any disjoint union of K_2's, because a player can only reduce the weight of a single edge weight in any given move. Also, Graph Nim on $K_{1,2} \cup K_1$, and $K_{1,3}$ could be won in a single move, because all the edges are incident to a

single vertex. This can also be generalized to any graph where all the edges are incident to a single vertex, which can be denoted $K_{1,n}$.

Since all the edges in a $K_{1,n}$ are incident to a single vertex we can think of the entire graph as a single heap in a game of Nim. This leads to another generalization of the winning strategies discussed earlier. Since $K_{1,n}$ can be thought of as a single heap then Graph Nim on any disjoint unions of $K_{1,n}$ is essentially Nim. Therefore, a winning strategy for Graph Nim played on $K_{1,n} \cup K_{1,m} \cup \cdots \cup K_{1,r}$ has been developed. Notice that a K_2 can also be denoted $K_{1,1}$, and therefore a disjoint union of K_2's is a special case of a disjoint union of complete bipartite graphs where one of the partite sets has size 1.

5. Sprague-Grundy Theorem and Graph Nim

Graph Nim is an impartial combinatorial game and as such The Sprague-Grundy Theorem applies. Recall from the Introduction:

Theorem (Sprague 1935, Grundy 1939[4, 6]). *Every impartial combinatorial game under the normal play convention is equivalent to a Nimber.*

The application of The Sprague-Grundy Theorem to Graph Nim will be presented here in detail. First, define an associative, commutative, binary operation called the *Nim-sum* or *Xor* and denoted $\oplus$. The Nim-sum is used to calculate what is called the *Nimber* of a game of Nim. The Nimber is simply the Nim-sum of all the heaps of the game, and is vital in determining the winning strategy of the game. Nim is in the losing position when the Nimber is 0. It has been proven that if the starting position does not have a Nimber of 0, then the first player can always make a move to a position that has a Nimber of 0, and can therefore always win[1].The Sprague-Grundy Theorem means that there is some function that assigns numbers, called Grundy Numbers, to the positions of an impartial combinatorial game and each Grundy number is equivalent to a Nimber. So if the appropriate function for a game can be determined, then the losing positions of the game, which are when the Grundy number is equal to zero, are known. The Sprague-Grundy Theorem states that there is some function that assigns Grundy Numbers to the positions of an impartial combinatorial game. The function is the mex, or minimum excluded value. Evaluating the mex of a position of a game is aided by the construction of a graph. This graph is built by creating a vertex for the initial position of the game. Next, a vertex for each position that can be obtained by a single move from the initial position of the game is created, and an arc from the first vertex to each of the new vertices is added. The process is continued for each new vertex until a vertex for each possible position of the game has been created. These graphs will be referred to as *Sprague-Grundy Graphs*.

The vertices of this graph will be labeled using the following algorithm. Start by labeling all vertices with no arcs leaving them "P", these are the terminal (final) positions of the game. Next label any vertex with an arc pointing to a "P" vertex as "N". Then label any vertex only pointing to "N" vertices as "P". These steps are repeated until every vertex is labeled. Now the mex will be used to assign Grundy Numbers to these positions. Begin by assigning the number 0 to all terminal positions. Now all vertices that only point to terminal positions will receive a mex value of 1. Then for every other vertex assign the value given by the mex of all the vertices they are adjacent to[2]. Then by The Sprague-Grundy Theorem, these Grundy Numbers are equivalent to a Nimber. Therefore, the positions that were assigned a value of zero are the losing positions of the game, note that all "P" vertices received a value of 0. The winning strategy is given by the graph, a player simply has to move the game to a position that has a Grundy Number of 0 on each turn. We know that this is always possible given that the game did not start at a position with Grundy Number 0, because each Grundy Number is equivalent to a Nimber.

Sprague-Grundy and 3-vertex Graph Nim. First examine The Sprague-Grundy Theorem applied to a graph with 3 vertices and edge weights restricted to be less than or equal to 1. Then the possible positions are simply the number of graphs with 3 vertices which is 4[5]. These positions are depicted in Figure 11. The Sprague-Grundy Graph for these possible positions is the graph depicted in Figure 12.

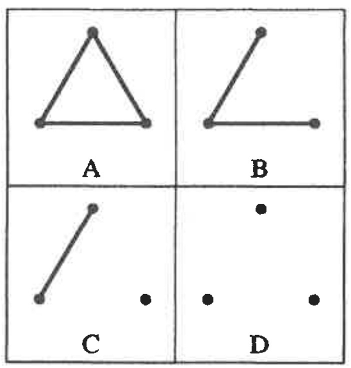


FIGURE 11. All Possible Positions of 3-vertex Graph Nim when Edge Weights are Less Than or Equal to 1

This confirms the losing position stated in Theorem 1, but the edge weights have been restricted to be less than or equal to 1. Now examine The Sprague-Grundy Graph with the edge weights less than or equal to 2. Then the number of possible positions has been expanded. Figure 13 shows the new positions that are possible if the edge weights are less than or equal to 2. Now adding these new positions into our Sprague-Grundy Graph, depicted in Figure 14, the graph has become a lot more complicated. Six vertices and 22 arcs were added to the graph, but through labeling the graph and using the mex function to calculate the Grundy

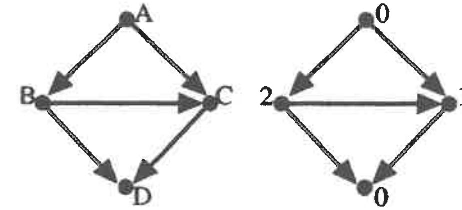


FIGURE 12. Sprague-Grundy Graph of 3-vertex Graph Nim when Edge Weights are Less Than or Equal to 1

Numbers, shown in Figure 14, there is only added one new vertex with Grundy Number zero, that is position E. This, also, confirms Theorem 1.

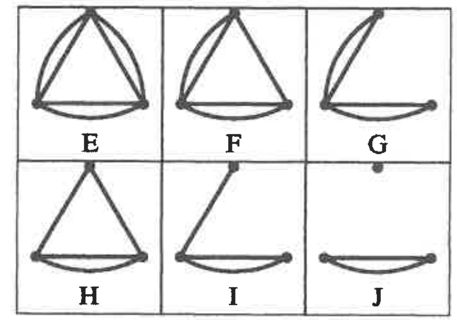


FIGURE 13. New Positions that are Possible if the Edge Weights are Less Than or Equal to 2 in 3-vertex Graph Nim.

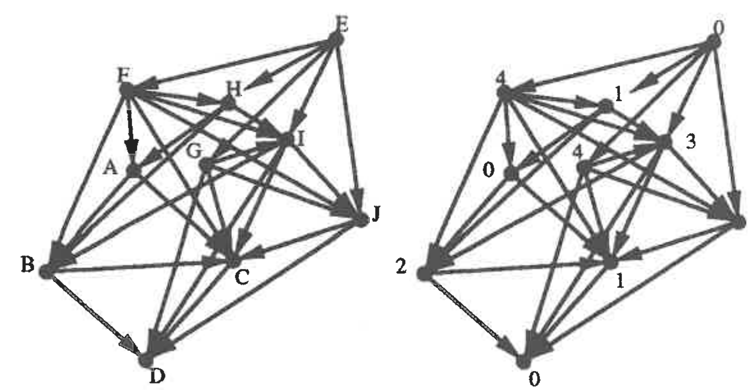


FIGURE 14. Sprague-Grundy Graph of 3-vertex Graph Nim when Edge Weights are Less Than or Equal to 2

Now doing the same thing for 4-vertex Graph Nim. Let the edge weights be restricted to be less than or equal to 1. Then the possible positions are simply the number of graphs with 4 vertices which is 11[5]. These positions are depicted in Figure 15. The Sprague-Grundy Graph for these positions is the graph in Figure 17.

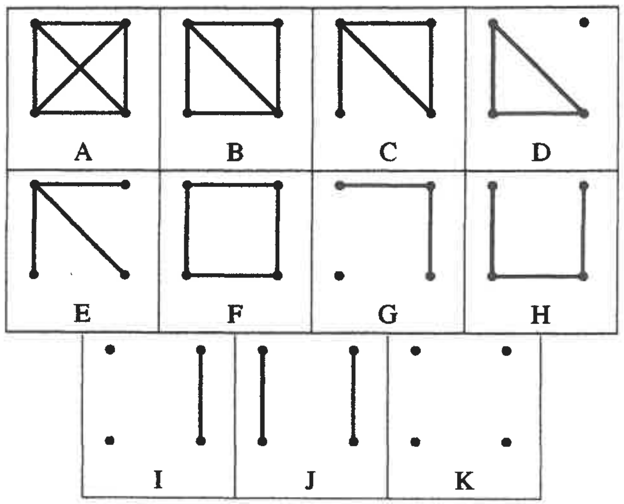


FIGURE 15. All possible positions of 4-vertex Graph Nim when edge weights are less than or equal to 1

Labeling the graph and assigning Grundy Numbers with the mex function, the graph in Figure 18 is obtained. The graph shows that positions D, F, and J are losing positions which confirm the findings in Theorem 1 and Theorem 2.

Conclusion. It is possible to use The Sprague-Grundy Theorem on Graph Nim. Examples have been shown for 3 and 4-vertex Graph Nim. The problem is that The Sprague-Grundy Graphs are huge and therefore are not practical. Arguments can be made about the adjacency in these graphs to deduce which vertices have to be a "P" vertex and have Grundy Number zero, but they would be similar to the arguments made without using The Sprague-Grundy Theorem.

6. Future Work

The winning strategies and losing positions for Graph Nim on 3 and 4-vertex graphs have been discovered, but there are many more graphs to work on. The obvious next step is to find winning strategies and losing positions for Graph Nim on 5-vertex graphs. Any 5-vertex graph with an isolated vertex is equivalent to a position of Graph Nim on 3 or 4 vertices. Also, any 5-vertex graph that is a disjoint union of complete bipartite graphs where one of the partite sets is size

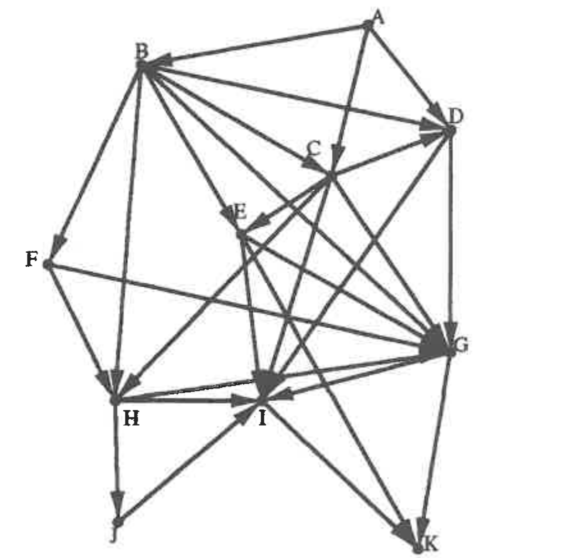


FIGURE 16. Sprague-Grundy Graph of 4-vertex Graph Nim when Edge Weights are Less Than or Equal to 1

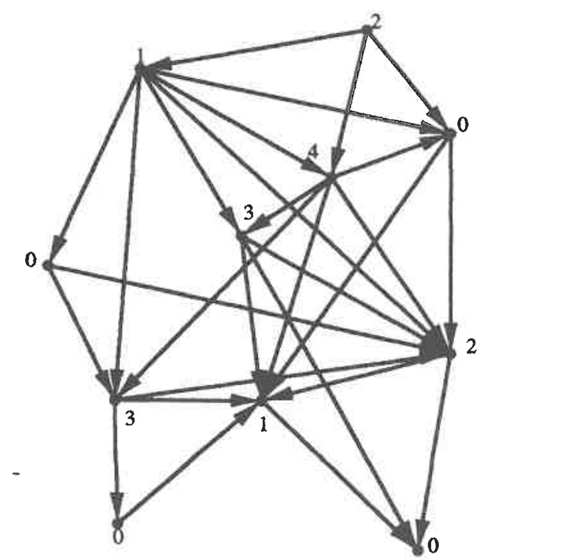


FIGURE 17. Labeled Sprague-Grundy Graph of 4-vertex Graph Nim when Edge Weights are Less Than or Equal to 1

1 has been solved. This takes care of a lot of the 5-vertex graphs, but doesn't solve them all. The Sprague-Grundy Theorem might be useful for discovering the "base" losing positions. The Sprague-Grundy Graph for 5-vertex Graph Nim when edge weights are less than or equal to 1 could be built, and used to find the "P" positions. This would be difficult, however, because there are 34 possible positions, even under the edge weight restriction. The Sprague-Grundy Theorem can also be applied to Circular Nim to confirm and extend the results of Dufour and Heubach.

There are also variations of the game that might be interesting to pursue. In standard Graph Nim the weighting of our edges was restricted to non-negative integers, but variations of the game could be played with other weight schemes, for example:

- Integer weighting and on a player's move they must choose a vertex and an operation (either addition or subtraction) then the player can add or subtract any integer amount from any edge incident to that vertex.
- Rational weighting and on a player's turn they must pick a vertex and must multiply the weights of any edges incident to the vertex by some fixed integer that they can split any way they see fit. Instead of reducing the weights to zero the goal is to get the weights to 1.

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