STRONGLY REGULAR GRAPHS FROM LARGE ARCS IN AFFINE PLANES

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Abstract

Strongly regular graphs with parameters (q^3+2q^2, q^2+q, q, q), $(q^3+q^2+q+1, q^2+q, q-1, q+1)$, and $(q^3, q^2+q-2, q-2, q+2)$ are constructed from k-arcs in affine planes of order q with $k = q+2, q+1, q$. In addition, strongly regular graphs with parameters $(nq^3-q^3+nq^2, nq^2-q^2+nq-q, 2qn-3q, qn-q)$ are constructed from maximal arcs of degree n in affine planes of order q. Each of these examples generalizes previously known examples when the affine planes were assumed to be Desarguesian.

Keywords: strongly regular graphs; arcs; affine planes

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1. Introduction

There are three main objectives in the study of strongly regular graphs. They are (1) to show no strongly regular graphs exist for a given set of (unknown) parameters, (2) to construct new strongly regular graphs with previously known parameters, and (3) to construct new strongly regular graphs with previously unknown parameters. The focus of this paper is on the second objective. That is, we construct strongly regular graphs whose parameters were previously known. This will be accomplished by considering arcs in non-Desarguesian planes. In order to do this, we first introduce Desarguesian planes.

A **projective plane** is a point-line incidence structure such that every pair of points is incident with a unique line, every pair of lines is incident with a unique point, and there exists four points, no three collinear. If the projective plane is finite, there exists an integer $q \geq 2$, called the *order* of the plane, such that every line is incident with $q+1$ points, every point is incident with $q+1$ lines, and there are q^2+q+1 points and q^2+q+1 lines.

An **affine plane** is also a point-line incidence structure such that any two distinct points lie on a unique line, each line has at least two points, given any line and any point not on that line, there is a unique line which contains the point and does not meet the given line, and there exist three non-collinear points. Similarly, if the number of points in an affine plane is finite, then there exists an integer $q \geq 2$, called the order of the plane, such that each line contains q points, each point is contained on $q+1$ lines, there are q^2 points and q^2+q lines.

An affine plane can be obtained from a projective plane by removing a line and all of the points on that line. This procedure can be reversed: two lines of an affine plane are *parallel* if they are equal or have no point in common. In an affine plane, parallelism is an equivalence relation, and the equivalence classes are called the *parallel classes*. By adding to an affine plane each parallel class as a point (called a *point at infinity*) and the set of all parallel classes as a line (called the *line at infinity*) and extending the incidence relation in such a way that the line at infinity is incident with all the points at infinity and no other points, and a point at infinity is incidence with a line not at infinity if and only if the corresponding parallel class contains the line, a projective plane is obtained and is called the *projective completion* of the affine plane.

An *isomorphism* between two incidence structures is a bijection between the points and lines of the first and the points and lines of the second, taking points to points and lines to lines such that a point-line pair is incident (in

the first incidence structure) if and only if their images are incident (in the second incidence structure). Isomorphism is an equivalence relation, and we call two incidence structures *isomorphic* if there exists an isomorphism between them. The affine plane obtained by deleting the line at infinity from the projective completion of the affine plane $\mathcal{A}$ is isomorphic to $\mathcal{A}$, but the affine plane obtained by deleting another line from the projective completion of the affine plane $\mathcal{A}$ need not be isomorphic to $\mathcal{A}$.

Given a division ring D, the incidence structure with the 1-dimensional subspaces of D^3 as points, the 2-dimensional subspaces of D^3 as lines and containment as incidence is a projective plane $\mathrm{PG}(2, D)$. A projective plane is **Desarguesian** if whenever two triangles are in perspective from a point, they are also in perspective from a line; that is, whenever ABC, $A'B'C'$ are triangles with AA', BB', CC' concurrent, they also have $AB \cap A'B'$, $AC \cap A'C', BC \cap B'C'$ collinear.

Theorem 1. *A projective plane is Desarguesian if and only if it is isomorphic to* $\mathrm{PG}(2, D)$*, for some division ring* D.

An affine plane is *Desarguesian* if its projective completion is Desarguesian. In the light of the theorems of E.H. Moore that finite fields have prime power order and for each prime power q, there is a unique field $GF(q)$ of order q, up to isomorphism and of J.H.M. Wedderburn that finite division rings are fields, it follows that finite Desarguesian projective planes have prime power order and for each prime power q, there is a unique Desarguesian projective plane of order q, which we denote by $\mathrm{PG}(2, q)$. For a recent, more geometric proof of this theorem of Wedderburn, see [2].

For the rest of this paper, we'll be concerned with finite projective and affine planes (without assuming that they are Desarguesian). All known finite projective and affine planes have prime power order, but there any many known finite non-Desarguesian projective and affine planes, the smallest of which have order 9.

Within planes, we can define various substructures. A k**-arc** of a projective plane is a set of k points, no three collinear. In 1947, Bose proved the following result related to k-arcs [3].

Theorem 2. *A* k*-arc of a finite projective plane of order* q *has* $k \leq q+2$. *If* q *is odd, then* $k \leq q+1$.

When k equals the upperbounds we obtain hyperovals and ovals, respectively. A **hyperoval** is a $(q+2)$-arc of a finite projective plane of order q. An **oval** is a $(q+1)$-arc of a finite projective plane of order q. In 1955, Segre proved that an oval of $\mathrm{PG}(2, q)$, q odd, is a *conic* [20]. There are many known large arcs, both in Desarguesian planes [4, 5, 7, 8, 12, 15, 17, 19, 21] and non-Desarguesian planes [6, 10, 11, 14, 18].

More generally, a (k, d)-**arc**, where $k, d > 1$, in a finite projective plane is a set of k points such that each line intersects the set in at most d points and at least one line intersects the set in exactly d points. The number of points k of a $(k; d)$-arc in $\mathrm{PG}(2, q)$ is at most $qd + d - q$. If $k = qd + d - q$, then the arc is called a **maximal arc**. Hyperovals are maximal arcs.

Utilizing arcs we can construct additional objects known as generalized quadrangles. A **generalized quadrangle of order** (s, t) is a point-line incidence structure such that every pair of points is incident with at most one line, given a point P and a line l not incident with P, there is a unique point incident with l and collinear with P, and every line is incident with $s+1$ points and every point is incident with $t+1$ lines. It follows that there are $(s+1)(st+1)$ points and $(t+1)(st+1)$ lines.

Theorem 3 ([1, 13]). *Each hyperoval of* $\mathrm{PG}(2, q)$ *gives rise to a generalized quadrangle of order* $(q-1, q+1)$.

Theorem 4 ([9]). *Each oval of* $\mathrm{PG}(2, q)$ *gives rise to a generalized quadrangle of order* (q, q).

Theorem 5 ([17]). *Each* q*-arc of* $\mathrm{PG}(2, q)$ *gives rise to a generalized quadrangle of order* $(q+1, q-1)$.

Furthermore, the concurrency graph of a generalized quadrangle of order (s, t) is a strongly regular graph with parameters $((t+1)(st+1)$, $(s+1)t, t-1, s+1)$. (The collinearity graph of a generalized quadrangle is also a strongly regular graph.) A regular graph Γ with v vertices and of degree k is said to be **strongly regular** if there exists integers λ and μ such that every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have μ common neighbors. Strongly regular graph parameters are denoted as (v, k, λ, μ).

Therefore, Theorems 3 - 5 yield families of strongly regular graphs. But, these graphs required the plane to be Desarguesian. In this paper, we construct strongly regular graphs, with the same parameters as the concurrency graphs of the generalized quadrangles from large arcs in Desarguesian planes, utilizing large arcs of non-Desarguesian planes.

2. Main Results

The parameters of the concurrency graph of a generalized quadrangle of order $(q+1, q-1)$ are $(q^3, q^2+q-2, q-2, q+2)$. The following result

pertains to a construction of a strongly regular graph from a q-arc in an affine plane. It is included for completeness and cohesiveness. However, its proof will be omitted as it can be found in [16].

Theorem 6. *Let* $\mathcal{A}$ *be an affine plane of order* q *(with point set* $\mathcal{P}$*) containing a* q*-arc* K*. Define a graph* $\Gamma(\mathcal{A}, K)$ *as follows: the vertex set is* $\mathcal{P} \times K$ *and* (P, A) *is adjacent to* (Q, B) *if and only if* $A \neq B$ *and either* $P = Q$ *or* $PQ \parallel AB$ *or* $A = B$ *and* $P \neq Q$ *and* PQ *is parallel to some tangent line of* K *at* A*. Then* $\Gamma(\mathcal{A}, K)$ *is a strongly regular graph with parameters* $(q^3, q^2+q-2, q-2, q+2)$.

We can also construct strongly regular graphs from maximal $\{k; n\}$-arcs in non-Desarguesian planes, as is witnessed in the following theorem. (Thank you to Stefaan De Winter and Jason Williford for the independent encouragement to consider this case.)

Theorem 7. *Let* $\mathcal{A}$ *be an affine plane of order* q *(with point set* $\mathcal{P}$*) containing a maximal* $\{k; n\}$*-arc* M*,* $2 \leq n \leq q-1$*. Define a graph* $\Gamma'(\mathcal{A}, M)$ *as follows: the vertex set is* $\mathcal{P} \times M$ *and* (P, A) *is adjacent to* (Q, B) *if and only if* $A \neq B$ *and* $P = Q$ *or* $PQ \parallel AB$*. Then* $\Gamma'(\mathcal{A}, M)$ *is a strongly regular graph with parameters* $(nq^3 - q^3 + nq^2, nq^2 - q^2 + nq - q, 2qn - 3q, qn - q)$.

Proof. The number of vertices is clearly $nq^3 - q^3 + nq^2$, as $|\mathcal{P}| = q^2$ and $|M| = nq + n - q$. The fact that Γ' is regular follows from the observation that the neighbors of (P, A) with second coordinate B must have $B \neq A$, for which there are $nq + n - q - 1$ such choices, and first coordiante Q must lie on the line parallel to AB on P, for which there are q choices. Therefore the valency is $q(nq + n - q - 1) = nq^2 - q^2 + nq - q$.
Next we show that the number of common neighbors given two adjacent vertices is $2qn - 3q$.
Case 1: Suppose that $(P, A) \sim (P, B)$ for $A \neq B$. There are $nq + n - q - 2$ common neighbors of the form (P, C) with $C \in M \setminus \{A, B\}$. Suppose that (X, C) is a common neighbor for $X \neq P$. If $(X, C) \sim (P, A)$ then $PX \parallel AC$ and if $(X, C) \sim (P, B)$ then $PX \parallel BC$. Since $A \neq B \neq C$, $BC = AC$. Thus, there are $(q-1)(n-2)$ common neighbors of the form (X, C). Hence, if $A \neq B$, (P, A) and (P, B) have $2qn - 3q$ common neighbors.
Case 2: Suppose that $(P, A) \sim (Q, B)$ for $A \neq B$ and $PQ || AB$. Suppose (X, C) is a common neighbor. Then $PX \parallel CA$ and $QX \parallel CB$ with $A \neq B \neq C$. If A, B, C are collinear, then $CA = CB$ and there are $q(n-2)$ such neighbors. If A, B, C are not collinear, then X is uniquely determined by C; in which case, there are $qn - q$ such choices for C. Thus, the number of common neighbors is $2qn - 3q$.

Thus, given any two adjacent vertices, there exist $2qn-3q$ vertices adjacent to both.

We now show that the number of common neighbors given two non-adjacent vertices is $qn-q$.

Case i: Suppose that $(P,A) \not\sim (Q,A)$ with $P \neq Q$. Let l_A be the line parallel to PQ on A and let B be one of the $n-1$ points in M on l_A. Then there are q choices for X such that X is on PQ which is parallel to l_A. Thus there are $q(n-1) = qn-q$ common neighbors.

Case ii: Let $(P,A) \not\sim (Q,B)$ with $A \neq B$. Then $P \neq Q$ and $PQ \nparallel AB$. Let $C \in M \setminus \{A,B\}$ and let l be the line on P parallel to AC and m be the line on Q parallel to BC. If A,B,C are not collinear, then $l \cap m = X$. So X is uniquely determined by C, of which there are $qn-q$ to choose from. If A,B,C are collinear then $l \parallel AC$ and $m \parallel BC$, which implies that $l||AB$ and $m \parallel AB$. But $l \neq m$, as $(P,A) \not\sim (Q,B)$. Thus, there does not exist a common neighbor (X,C) with C being collinear to A and B.

Thus, the number of common neighbors given two non-adjacent vertices is $qn-q$.

Therefore, $\Gamma'(\mathcal{A},M)$ is a strongly regular graph with parameters $(nq^3-q^3+nq^2, nq^2-q^2+nq-q, 2qn-3q, qn-q)$.

□

When $\mathcal{A}$ is Desarguesian, $\Gamma'(\mathcal{A},M)$ is the point graph of the partial geometry $T_2^*(M)$ of Thas [22].

When $n=2$, q is even and M is a hyperoval, and the parameters of the concurrency graph of a generalized quadrangle of order $(q-1,q+1)$ are (q^3+2q^2, q^2+q, q, q).

Corollary 1. *Let $\mathcal{A}$ be an affine plane of order q (with point set $\mathcal{P}$) containing a hyperoval H. Define a graph $\Gamma'(\mathcal{A},H)$ as follows: the vertex set is $\mathcal{P} \times H$ and (P,A) is adjacent to (Q,B) if and only if $A \neq B$ and either $P=Q$ or $PQ \parallel AB$. Then $\Gamma'(\mathcal{A},H)$ is a strongly regular graph with parameters (q^3+2q^2, q^2+q, q, q).*

This family of strongly regular graphs yields a symmetric (q^3+2q^2, q^2+q, q)-design whose polarity has no absolute points. Additionally, the sets $\{(P,A) : P \in \mathcal{P}, A \in H\}$ are cocliques of $\Gamma'(\mathcal{A},H)$, which partitions $\Gamma'(\mathcal{A},H)$ into cocliques. So if $\Gamma'(\mathcal{A},H)$ is geometric, the corresponding $GQ(q+1,q-1)$ admits a partition into ovoids. Moreover, for each ovoid in the partition, every pair of distinct points of the ovoid is regular. It remains to be shown whether $\Gamma'(\mathcal{A},H)$ is geometric or not.

Finally, the parameters of the concurrency graph of a generalized quadrangle of order (q,q) are $(q^3+q^2+q+1, q^2+q, q-1, q+1)$.

Theorem 8. *Let $\mathcal{A}$ be an affine plane of order q (with point set $\mathcal{P}$) containing an oval O. Define a graph $\Gamma''(\mathcal{A},O)$ as follows: the vertex set is $(\mathcal{P} \times O) \cup O$ and (P,A) is adjacent to (Q,B) if and only if either $A \neq B$ and either $P=Q$ or $PQ \parallel AB$ or $A=B$ and $P \neq Q$ and PQ is parallel to the tangent line of O at A; A is adjacent to B for all $A \neq B$ in O; and (P,A) is adjacent to B if and only if $A=B$. Then $\Gamma''(\mathcal{A},O)$ is a strongly regular graph with parameters $(q^3+q^2+q+1, q^2+q, q-1, q+1)$.*

Proof. The number of vertices is clearly q^3+q^2+q+1, as $|\mathcal{A}| = q^2$ and $|O| = q+1$. A vertex (P,A) having neighbors with the second coordinate B must have that $B \neq A$ and consist of pairs (X,B) with X on the line parallel to AB on P, for which there are q^2 of these, or have $B=A$ and consist of pairs (X,A) with X a point, other than P, on the line through P parallel to the tangent line on A in O, for which there are $q-1$ of these. The other neighbor of (P,A) is A. Hence, the valency of (P,A) is q^2+q.

Next we will show that the number of common neighbors given two adjacent vertices is $q-1$.

Case 1: Suppose that $(Q,C) \sim (P,A)$ and that (X,B) is a common neighbor such that $A \neq B \neq C$. If $A \neq C$, then a vertex (X,B) is a common neighbor of (P,A) and (Q,C) if and only if $A \neq B \neq C$ and $P=X$ or $PX \parallel AB$ and $Q=X$ or $QX \parallel CB$. So X lies on the line through P parallel to AB and on the line through Q parallel to CB. Since A, C, and B are on O, they cannot be collinear. Thus, AB and CB are not parallel, and, for each B, X is uniquely determined. Hence, in either case, there are exactly $q-1$ common neighbors of this form.

We will show there are not any common neighbors of another form. Since (Q,C) and (P,A) are adjacent, $P=Q$ or $PQ \parallel AC$. Since the tangent to O at A and the tangent to O at C are distinct, if $P=Q$, there are no further common neighbors. If $P \neq Q$, then $PQ \parallel AC$, so PQ cannot be parallel to the tangents of O and either A or C. Therefore, there are no further common neighbors.

Case 2: Suppose that $(P,A) \sim (Q,A)$ with $P \neq Q$. The vertex (X,A) with PX parallel to the tangent to O at A parallel to QX is a common neighbor. Such vertices only exist if PQ is parallel to the tangent to O at A. There are $q-2$ of them. Now consider (X,B) for $A \neq B$. If (X,B) is a common neighbor, then $PX \parallel AB \parallel QX$. There are q such neighbors; those (X,B) with X on the line PQ and B the other point of O on the line parallel to PQ. But this means that (P,A) and (Q,A) are not adjacent, because PQ is not parallel to the tangent to O at A. Therefore, there are no neighbors of the form (X, A) whenever (P, A) and (Q, A) are adjacent. Lastly, A is a common neighbor of (P, A) and (Q, A). Thus, the number of common neighbors is $q - 1$.

Note that (P, A) and B are not adjacent whenever $A \neq B$. So we will consider this case later on.

Case 3: Suppose that vertices A and B are adjacent, then their common neighbors are $C \in O \setminus \{A, B\}$. Thus there are $q - 1$ common neighbors.

Therefore, given two adjacent vertices, the number of common neighbors is $q - 1$. Now we will consider two non-adjacent vertices.

Case i: Suppose that $(Q, C) \not\sim (P, A)$. As shown in Case 1, there are $q - 1$ common neighbors, whether or not (Q, C) and (P, A) are adjacent. Furthermore, if (Q, C) and (P, A) are not adjacent, then $Q \neq P$ and there are two additional common neighbors, namely, (X, A), where X is the intersection of the line through P parallel to the tangent line to O at A with the line through Q parallel to AC, and (Y, C), where Y is the intersection of the line through Q parallel to the tangent line to O at C with the line through P parallel to AC. Therefore, the number of common neighbors given to non-adjacent vertices of the form (Q, C) and (P, A) is $q + 1$.

Case ii: As mentioned before, given the vertices (P, A) and (Q, A), they are not adjacent whenever they have a common neighbor of the form (X, B) for $B \neq A$. There are q points of this form. Additionally, they are both adjacent to the point A. Thus, there are $q + 1$ common neighbors.

Case iii: If $A \neq B$, then $(P, A) \not\sim B$. Their common neighbors are the vertices (X, B) with X on the unique line through P parallel to AB, for which there are q of these, together with the vertex A.

Thus, given two non-adjacent vertices, there are $q + 1$ vertices adjacent to both.

Therefore, $\Gamma''(\mathcal{A}, O)$ is a strongly regular graph with parameters $(q^3 + q^2 + q + 1, q^2 + q, q - 1, q + 1)$. □

Each of these strongly regular graphs gives rise to a symmetric $(q^3+q^2+q+1, q^2+q+1, q+1)$-design with a polarity with all points absolute. It does remain to be answered whether or not these graphs are geometric; that is, whether or not they are collinearity graphs of generalised quadrangles. As it stands, we only know that these graphs are pseudo-geometric; that is, have the same parameters as collinearity graphs of generalised quadrangles.

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