ABOUT COLORINGS OF (3,3)-UNIFORM COMPLETE CIRCULAR MIXED HYPERGRAPHS

N.A. NEWMAN, K. ROBLEE, V. VOLOSHIN
DEPARTMENT OF MATHEMATICS AND GEOMATICS
TROY UNIVERSITY
TROY, AL 36082
NEWMAN@TROY.EDU, KROBLEE@TROY.EDU, VVOLOSHIN@TROY.EDU

ABSTRACT. A mixed hypergraph is a triple $\mathcal{H}=(X,\mathcal{C},\mathcal{D})$, where X is the vertex set and each of $\mathcal{C}$ and $\mathcal{D}$ is a family of subsets of X, the $\mathcal{C}$-edges and $\mathcal{D}$-edges, respectively. A proper k-coloring of $\mathcal{H}$ is a mapping such that each $\mathcal{C}$-edge has two vertices with a common color and each $\mathcal{D}$-edge has two vertices with distinct colors. A mixed hypergraph $\mathcal{H}$ is called circular if there exists a host cycle on the vertex set X such that every edge ($\mathcal{C}$- or $\mathcal{D}$-) induces a connected subgraph of this cycle. We propose an algorithm to color the (3,3)-uniform, complete, circular, mixed hypergraphs for every value on its feasible set. In doing so, we show $\chi(\mathcal{H}) = 2$ and $\bar{\chi}(\mathcal{H}) = n/2$ when n is even and $\bar{\chi}(\mathcal{H}) = \frac{n-1}{2}$ when n is odd.

1. INTRODUCTION

In the classical theory of coloring graphs and hypergraphs [1, 5], we ask for colorings of the vertices so that each edge requires at least two vertices of different colors, and ask for the minimum number of colors required. It is natural to ask the dual question to color the vertices so that each edge requires at least two vertices of the same color, and ask for the maximum number of colors needed. It is also natural to ask the combination of the above two questions [3, 4, 5].

In the present paper we deal with such a combination of constraints on colorings and use the terminology of [4].

A mixed hypergraph is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the vertex set and each of $\mathcal{C}$ and $\mathcal{D}$ is a family of subsets of X, the $\mathcal{C}$-edges and $\mathcal{D}$-edges, respectively. Each element of $\mathcal{C} \cup \mathcal{D}$ is of size at least 2. In a mixed hypergraph, if a subset of vertices is a $\mathcal{C}$-edge and a $\mathcal{D}$-edge at the same time,

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then it is a *bi-edge*. A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a ***bihypergraph*** if $\mathcal{C} = \mathcal{D}$. A proper k-coloring of a mixed hypergraph is a mapping from the vertex set to a set of k colors so that each $\mathcal{C}$-edge has two vertices with a $\mathcal{C}$ommon color and each $\mathcal{D}$-edge has two vertices with $\mathcal{D}$ifferent colors. A mixed hypergraph is k-colorable (uncolorable) if it has a proper coloring with at most k colors (admits no proper colorings). A strict k-coloring is a proper coloring using all k colors. The minimum number of colors in a proper coloring of $\mathcal{H}$ is the *lower chromatic number* $\chi(\mathcal{H})$; the maximum number of colors in a strict coloring is the *upper chromatic number* $\bar{\chi}(\mathcal{H})$. We use $c(x)$ for the color of vertex x. The set of values k such that $\mathcal{H}$ has a strict k-coloring is called the ***feasible set*** of $\mathcal{H}$, denoted by $F(\mathcal{H})$. A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ has a ***gap*** at k if $F(\mathcal{H})$ contains elements larger and smaller than k, but omits k. $F(\mathcal{H})$ is called ***continuous*** if it has no gaps.

Definition 1.1. A mixed hypergraph $\mathcal{H}$ is called **circular** if there exists a host cycle on the vertex set X such that every $\mathcal{C}$-edge and every $\mathcal{D}$-edge induces a connected subgraph of the host cycle.

In other words, for circular mixed hypergraph there exists a circular ordering of the vertex set X, say, $X = \{x_0, x_1, ..., x_{n-1}, x_0\}$ such that every edge ($\mathcal{C}$- or $\mathcal{D}$-) induces an interval in this ordering. In [6] the lower chromatic number was investigated for the colorability and unique colorability of classical circular mixed hypergraphs, while the upper chromatic number was investigated in [7]. The generalizations of circular mixed hypergraphs have been investigated in [2]. In particular, it was shown that the feasible set of any mixed strong hypercactus is gap-free, and there are infinitely many mixed weak hypercacti such that the feasible set of any of them contains a gap. In this paper, we focus on a particular family of hypergraphs, the (3,3)-uniform complete circular hypergraphs.

Definition 1.2. Let $\mathcal{H}$=$(X, \mathcal{C}, \mathcal{D})$ be a circular mixed hypergraph. If every k consecutive vertices of X form a $\mathcal{C}$-edge, then we denote $\mathcal{C} = \mathcal{C}_k$. Similarly, if every l consecutive vertices of X form a $\mathcal{D}$-edge, then we denote $\mathcal{D} = \mathcal{D}_l$. The circular mixed hypergraph $\mathcal{H}$=$(X, \mathcal{C}_k, \mathcal{D}_l)$ is called (k, l)-uniform and is denoted by $\mathcal{KC}(n; k, l)$, where $n := |X|$. A (3, 2)-uniform circular mixed hypergraph $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_2) = \mathcal{KC}(n; 3, 2)$ is a complete circular mixed hypergraph where each consecutive triple must have two colors in common and every consecutive pair must be colored differently.

2. Coloring Algorithm

The algorithm below will properly color the vertices of $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3)$. Beginning with the initial vertices, we have 3 possible coloring patterns

exhibited by Parts 1, 2, and 3. We then color the next vertex while looking back at the two previous vertices' colors to ensure a proper coloring. We continue in this fashion and ensure x_{n-2} and x_{n-1} remain properly colored with x_0 and x_1. Dependant on the decision at each step, we end with a proper coloring of $\mathcal{H}$ with 2 to $\bar{\chi}(\mathcal{H})$. The coloring $(0, 0, 1, 1, 2, 2, ..., n/2 - 1, n/2 - 1)$ attains the maximum value of $\bar{\chi}(\mathcal{H})$ when n is even, and $(0, 0, 1, 1, 2, 2, ..., 0, 0, \frac{n-1}{2} - 1)$ when n is odd as seen in section 3.

INPUT: $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3) = \mathcal{KC}(n; 3, 3), n = |X|, X = \{x_0, x_1, ..., x_{n-1}\}$

OUTPUT: A proper coloring $c = (c(x_0), c(x_1), ..., c(x_{n-1}))$ of $\mathcal{KC}(n; 3, 3)$

Stop when $c(x_{n-1})$ = VALUE is TRUE.

Output: $c = (c(x_0), c(x_1), ..., c(x_{n-1}))$

Part 1 (a,a,b): Assigning our colors for x_0, x_1, x_2 as $(a, a, b, -, ..., -)$ we have the choice of a or b for $c(x_3)$ since $c(x_1) = a$ and $c(x_2) = b$. Note $c(x_{n-1}) \neq a$ and it must be the case that $c(x_{n-1}) = c(x_{n-2})$ or $c(x_{n-2}) = a$.

i. If $c(x_3) = a$, we have $(a, a, b, a, -, ..., -)$ and $c(x_4) = a$ or b and we have a bi-edge colored (b, a, b) or (b, a, a) leading us to part 2 or 3.

ii. If $c(x_3) = b$, we have $(a, a, b, b, -, ..., -)$ and $c(x_4) = a$ or c and we have a bi-edge colored (b, b, a) or (b, b, c). If the former, we repeat the beginning of part 1. If the latter, we introduce a new color and repeat the beginning of part 1 with the pattern (b, b, c).

Part 2 (a, b, a): Assigning our colors for x_0, x_1, x_2 as $(a, b, a, -, ..., -)$ we have the choice of a or b for $c(x_3)$ since $c(x_1) = b$ and $c(x_2) = a$. Note $c(x_{n-1}) = a$ or b. If $c(x_{n-1}) = a$, $c(x_{n-2}) \neq a$ and if $c(x_{n-1}) = b$, $c(x_{n-2}) = a$ or b.

i. If $c(x_3) = b$, we have $(a, b, a, b, -, ..., -)$, $c(x_4) = a$ or b and we have a bi-edge colored (a, b, a) or (a, b, b), respectively. If the former, we repeat the beginning of part 2. If the latter, we go to part 3.

ii. If $c(x_3) = a$, we have $(a, b, a, a, -, ..., -)$, $c(x_4) = b$ or c and we have a bi-edge colored (a, a, b) or (a, a, c), respectively. If the former, we repeat the beginning of part 1. If the latter, we introduce a new color and go to part 1.

Part 3 (a, b, b) Assigning our colors for x_0, x_1, x_2 as $(a, b, b, -, ..., -)$ we have the choice of a or c for $c(x_3)$ since $c(x_1) = b$ and $c(x_2) = b$. Note $c(x_{n-1}) = a$ or b. If $c(x_{n-1}) = a$, $c(x_{n-2}) \neq a$ and if $c(x_{n-1}) = b$, $c(x_{n-2}) = a$ or b.

i. If $c(x_3) = a$, we have $(a, b, b, a, -, ..., -)$, $c(x_4) = b$ or a and we have a bi-edge colored (b, a, b) or (b, a, a), respectively. If the former, we repeat the beginning of part2i. If the latter, we repeat part 3.

ii. If $c(x_3) = c$, we have $(a, b, b, c, -, ..., -)$, $c(x_4) = b$ or c and we have a bi-edge colored (b, c, b) or (b, c, c), respectively. If the former, we repeat the beginning of part 2. If the latter, we repeat part 3.

3. Coloring $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3)) = \mathcal{KC}(n; 3, 3)$

Lemma 3.1. *If $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3)) = \mathcal{KC}(3; 3, 3)$, then $\chi(H) = \bar{\chi}(H) = 2$.*

Proof. Clearly, $(0, 1, 1), (1, 0, 1), (1, 1, 0)$ are the only (non- isomorphic) colorings. □

Lemma 3.2. *If $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3)) = \mathcal{KC}(5; 3, 3)$, then $\chi(H) = \bar{\chi}(H) = 2$.*

Proof. From lemma 3.1, x_0, x_1, x_2 can be colored $(0, 1, 1), (1, 0, 1), (1, 1, 0)$. If $(0, 1, 1)$, $c(x_4) \neq 2$. If $c(x_3) = 2, c(x_4) \neq 0$ or 1.

Likewise, If $(1, 0, 1)$, $c(x_4) \neq 2$ and $c(x_3) \neq 2$. Likewise, If $(1, 1, 0)$, $c(x_3) \neq 2$. If $c(x_4) = 2, c(x_3) \neq 0$ or 1. Therefore, $\chi(H) = \bar{\chi}(H) = 2$. □

Lemma 3.3. *If $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3)) = \mathcal{KC}(n; 3, 3)$, then $\bar{\chi}(H) \geq n/2$, when n is even and $n > 2$, and $\bar{\chi}(H) \geq \frac{n-1}{2}$ when n is odd and $n > 5$.*

Proof. Even case: Let $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3) = \mathcal{KC}(n; 3, 3)$ with vertices x_0, x_1, ..., x_{n-1} with n even. Let $c(x_i) = \lfloor i/2 \rfloor$. Then for any three consecutive vertices (any bi-edge) we have the coloring pattern (a, a, b), (a, b, b), or (b, c, c) producing a proper coloring with $n/2$ colors. Parts 1ii, 2ii, 3ii provide possible colorings that may meet the value of $n/2$. The coloring mentioned is of the form 1ii, 3ii, 1ii, 3ii,... . The form 3ii, 1ii, 3ii, 1ii, ... is simply the previous iteration shifted "down" one vertex. That is, the Coloring Algorithm begins on x_{n-1}, x_0, x_1 rather than x_0, x_1, x_2. Any other iterations through the coloring algorithm will obviously provide fewer colors. It is easy to see the use of part 2 constitutes the form mentioned with two previous colors used on vertices reducing the amount of new colors able to be introduced.

Odd case: Let $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3) = \mathcal{KC}(n; 3, 3)$ with vertices $x_0, x_1, ..., x_{n-1}$ with n odd. We repeat the procedure from the even case producing the improper coloring $(0, 0, 1, 1, ..., \frac{n-1}{2} - 1, \frac{n-1}{2} - 1, \frac{n-1}{2})$ since x_{n-2}, x_{n-1}, x_0 will be colored $\frac{n-1}{2} - 1, \frac{n-1}{2}, 0$, respectively. $c(x_{n-1}) \neq 0$ since $c(x_0) = c(x_1) = 0$. Let $c(x_{n-1}) = y$. Then $c(x_{n-2}) = y$ or 0. If $y, x_{n-4}, x_{n-3}, x_{n-2}$ will produce an improper coloring. Thus the coloring must be $(0, 0, 1, 1, ..., \lfloor \frac{n-4}{2} \rfloor, \lfloor \frac{n-4}{2} \rfloor, 0, 0, \frac{n-1}{2} - 1)$ using $\frac{n-1}{2}$ colors. □

Lemma 3.4. *If $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3)) = \mathcal{KC}(n; 3, 3)$, then $\bar{\chi}(H) < n/2 + 1$, when n is even and $n > 2$, and $\bar{\chi}(H) < \frac{n-1}{2} + 1$ when n is odd and $n > 5$.*

Proof. Let $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3) = \mathcal{KC}(n; 3, 3)$ with vertices $x_0, x_1, ..., x_{n-1}$. For the sake of contradiction, let us assume $\mathcal{H}$ can be colored more than $n/2$ colors if n is even and more than $\frac{n-1}{2}$ if n is odd.

Given an arbitrary interval of five vertices $(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2})$, by our Coloring Algorithm these vertices must be colored 2 or 3 colors and will contribute to the size of a color class, $|C_\alpha|$, based on some partition of 5. Colors assigned to these vertices must be duplicated or "given" to another color class. This results in $\sum_{\alpha=0}^{n/2} |C_\alpha| \geq 2(\frac{n}{2} + 1) = n + 2$ when n is even, and $\sum_{\alpha=0}^{(n-1)/2} |C_\alpha| \geq 2(\frac{n-1}{2} + 1) = n + 1$ when n is odd, where $|C_\alpha|$ is the size of the α color class. In either case, we end with more color assignments than we have vertices. By the pigeonhole principle this contradicts the use of the aforementioned number of colors. □

Theorem 3.5. *If $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3)) = \mathcal{KC}(n; 3, 3)$, then $\bar{\chi}(H) = n/2$ when n is even, and $\bar{\chi}(H) = \frac{n-1}{2}$ when n is odd.*

Proof. Taking the previous two lemmas, we have the result. □

Lemma 3.6. *If $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3)) = \mathcal{KC}(n; 3, 3)$, then $\chi(\mathcal{H}) = 2$.*

Proof. We color the vertices $c(i) = 0$ if i is even and $c(i) = 1$ if i is odd. If n is even, any 3-interval will be colored either $\{0, 1, 0\}$ or $\{1, 0, 1\}$ which satisfies the restrictions. If n is odd, we have the same as before except $c(x_{n-1}) = 0, c(0) = 0$, and $c(1) = 1$. i.e. $(0, 0, 1)$ which still satisfies the restrictions. Therefore, $\chi(H) = 2$. □

Theorem 3.7. *If $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_3)) = \mathcal{KC}(n; 3, 3)$, then $F(H) = \{2, 3, 4, ..., \frac{n}{2}\}$ when n is even and $F(H) = \{2, 3, 4, ..., \frac{n-1}{2}\}$ when n is odd.*

Proof. Theorem 3.3 and lemma 3.4 satisfy the upper and lower chromatic numbers, respectively.

If n is even and there are more than 3 color classes, color the vertices as in Theorem 3.3. Recolor vertices x_{n-1} and x_{n-2} with 1. We now have a proper coloring using $\frac{n}{2} - 1$ colors. If there are still more than 3 color classes, recolor vertices x_{n-3} and x_{n-4} with 0. We now have a proper coloring using $\frac{n}{2} - 2$ colors. We continue alternating in this fashion until we have 3 color classes. We thus get $F(H) = \{2, 3, 4, ..., \frac{n}{2}\}$.

If n is odd and there are more than 3 color classes, color the vertices as in Theorem 3.3. Recolor vertices x_{n-1}, x_{n-3}, and x_{n-4} with 1. We now have a proper coloring using $\frac{n-1}{2} - 1$ colors. If there are still more than 3 color classes, recolor vertices x_{n-5} and x_{n-6} with 0. We now have a proper coloring using $\frac{n-1}{2} - 2$ colors. We continue alternating in this fashion until we have 3 color classes. We thus get $F(H) = \{2, 3, 4, ..., \frac{n-1}{2}\}$.

□

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