Roots of Formal Power Series and New Theorems on Riordan Group Elements

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Abstract.
Elements of the Riordan group $\mathcal{R}$ over a field $\mathbb{F}$ of characteristic zero are infinite lower triangular matrices which are defined in terms of pairs of formal power series. We wish to bring to the forefront, as a tool in the theory of Riordan groups, the use of multiplicative roots $a(x)^{\frac{1}{n}}$ of elements $a(x)$ in the ring of formal power series over $\mathbb{F}$. Using roots, we give a Normal Form for non-constant formal power series, we prove a surprising simple Composition-Cancellation Theorem and apply this to show that, for a major class of Riordan elements (i.e., for non-constant $g(x)$ and appropriate $F(x)$), only one of the two basic conditions for checking that $\big(g(x), F(x)\big)$ has order n in the group $\mathcal{R}$ actually needs to be checked. Using all this, our main result is to generalize C. Marshall [6] and prove: Given non-constant $g(x)$ satisfying necessary conditions, there exists a unique $F(x)$, given by an explicit formula, such that $\big(g(x), F(x)\big)$ is an involution in $\mathcal{R}$. Finally, as examples, we apply this theorem to "aerated" series $h(x) = g(x^r)$ to find the unique $K(x)$ such that $\big(h(x), K(x)\big)$ is an involution.
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Key terms: Riordan group, formal power series, multiplicative roots of formal power series, involutions, group elements of order n.

1 Introduction

The Riordan group was introduced in [9] with applications to counting problems and combinatorial identities, and it has been of much interest in combinatorics. (See, for example, [1], [3], [5], [7], [9], [10].)

Riordan matrices

Elements of the Riordan group $\mathcal{R}$ are infinite lower triangular matrices which are generalizations of Pascal's triangle. Let $\mathbb{F}$ be a field of characteristic zero. A *Riordan matrix* $A \in \mathcal{R} = \mathcal{R}(\mathbb{F})$ is defined in terms of a pair of formal power series $g(x)$ and $F(x)$ and is denoted as $A = \big(g(x), F(x)\big)$ where

$$\begin{aligned} g(x) &= g_0 + g_1 x + \cdots + g_n x^n + \cdots, \quad g_0 \neq 0 \\ F(x) &= f_1 x + f_2 x^2 + \cdots + f_n x^n + \cdots, \quad f_1 \neq 1 \end{aligned}$$

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If $A = \left(a_{m,n}\right)_{m,n\geq 0}$, then the zeroth column of A has as generating function the formal power series $g(x)$ and, more generally, the n^{th} column $(n \geq 0)$ has as generating function the series $g(x) \cdot F(x)^n$. Thus,

$$A = (g(x), F(x)) = \begin{bmatrix} | & | & | & \cdots \\ g(x) & g(x)\cdot F(x) & g(x)\cdot\left(F(x)\right)^2 & \cdots \\ | & | & | & \cdots \end{bmatrix}$$

and

$$a_{m,n} = [x^m]\left(g(x)\cdot F(x)^n\right) \stackrel{\text{def}}{=} \text{coefficient of } x^m \text{ in } g(x)\cdot F(x)^n.$$

Example 1.1. ***Pascal's triangle*** $P = [p_{n,k}]_{n,k\geq 0} = \left[\binom{n}{k}\right]_{n,k\geq 0}$ *is given by*

$$P = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$$

The Riordan group

Matrix multiplication of elements of $\mathcal{R}(\mathbb{F})$ gives the rule

$$\left(g(x), F(x)\right)\cdot\left(h(x), K(x)\right) = \left(g(x)\cdot h(F(x)), K(F(x))\right).$$

This was introduced in [9] and is now called ***The Fundamental Theorem of Riordan Groups***. Under this multiplication $\mathcal{R}$ becomes a group with

- identity element equal to $(1, x)$,
- $\left(g(x), F(x)\right)^{-1} = \left(\frac{1}{g(\overline{F}(x))}, \overline{F}(x)\right)$, where $\overline{F}(x)$ is the compositional inverse of $F(x)$. (See Section 2.)

Involutions in $\mathcal{R}$

An ***involution*** in a group is an element of order two in that group. Thus $\left(g(x), F(x)\right) \neq (1, x)$ is an involution in $\mathcal{R}$

$$\iff \quad \left(g(x), F(x)\right)^2 = \left(g(x)\cdot g(F(x)),\ F(F(x))\right) = (1, x) \qquad (1)$$

$$\iff \quad \text{(a) } g(x)\cdot g(F(x)) = 1 \ \text{ and } \ \text{(b) } F(F(x)) = x. \qquad (2)$$

More generally, $\left(g(x), F(x)\right)$ has *order* n in the group $\mathcal{R}$ iff n is the least positive integer such that

$$\text{(a) } g(x)\cdot g(F(x))\cdots\cdot g(F^{(n-1)}(x)) = 1 \ \text{ and } \ \text{(b) } F^{(n)}(x) = x,$$

where $F^{(k)}(x) \stackrel{\text{def}}{=} F(F(\cdots F(x))\cdots)$, ($k$ times).

The goal of this paper

In this paper we study the existence and uniqueness, given $g(x)$, of an $F(x)$ such that $\left(g(x), F(x)\right)$ is an involution. Thus we are vitally interested in the *functional equations* (a) and (b) above, for power series. To study these, we highlight as a tool in the theory of Riordan groups the use of multiplicative roots $a(x)^{\frac{1}{n}}$ where $a(x) \in \mathbb{F}[[x]] =$ the ring of formal power series over $\mathbb{F}$.

In Section 2 we formulate basic facts concerning roots of formal power series, starting with [8], and we prove Theorem 2.5, a surprising compositional cancellation theorem.

In Section 3, we use this tool to prove (Theorem 3.1) that in major situations only condition (a) need be checked in proving that $\left(g(x), F(x)\right)$ has order n and then to generalize generalize C. Marshall [6] and prove (Theorem 3.3): *Given a non-constant series*

$$g(x) = g_0 + g_r x^r + g_{r+1}x^{r+1} + \cdots, \ \textit{with } g_0 = \pm 1, g_r \neq 0 \textit{ and } r \textit{ odd},$$

then there exists a unique $F(x) = -x + f_2x^2 + \cdots$, *given by an explicit formula, such that* $(g(x), F(x))$ *is an involution in* $\mathcal{R}$.

2 Formal Power Series

2.1 Background and Notation

Notation 2.1. *Given the field* $\mathbb{F}$ *of characteristic zero, we denote*

- $\mathbb{F}[[x]] = \{\, g_0 + g_1x + g_2x^2 + \cdots \mid g_i \in \mathbb{F}\,\}$.
- $\mathbb{F}_0[[x]] = \{\, g_0 + g_1x + g_2x^2 + \cdots \in \mathbb{F}[[x]] \mid g_0 \neq 0\,\}$.
- $\mathbb{F}_1[[x]] = \{\, f_1x + f_2x^2 + f_3x^3 + \cdots \in \mathbb{F}[[x]] \mid f_1 \neq 0\,\}$.
- $\mathbb{F}_+[[x]] = \{\, f_rx^n + f_{r+1}x^{r+1} + \cdots \in \mathbb{F}[[x]] \mid r > 0, f_r \neq 0\,\}$.

We start by summarizing well-known facts about the groups $\mathbb{F}_0[[x]]$, $\mathbb{F}_1[[x]]$. (See, for example, [8],[2], [4] .) The Riordan group $\mathcal{R} = \mathcal{R}(\mathbb{F})$ defined in the Introduction is very rich algebraically because it is isomorphic to the semi-direct product of $\mathbb{F}_0[[x]]$ and $\mathbb{F}_1[[x]]$, where these are given very different group structures.

1. $\mathbb{F}_0[[x]]$ is a group under multiplication:

If $g(x) = \sum_{n=0}^{\infty} g_nx^n \in \mathbb{F}_0[[x]$ and $h(x) = \sum_{j=0}^{\infty} h_jx^j \in \mathbb{F}_0[[x]]$, then

$$g(x)\cdot h(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} g_ih_{n-i}\right)x^n.$$

Moreover, if we write $g(x) = g_0 \cdot (1 + \frac{g_1}{g_0}x + \cdots) = g_0 \cdot (1 + G(x))$, then

$$\begin{aligned}(g(x))^{-1} &\stackrel{\text{def}}{=} \frac{1}{g(x)} = \frac{1}{g_0}\Big(1 - G(x) + G(x)^2 - \cdots\Big) \\ &= \frac{1}{g_0} - \frac{g_1}{g_0^2}x + \Big(-\frac{g_2}{g_0^2} + \frac{g_1^2}{g_0^3}\Big)x^2 + \cdots\end{aligned}$$

2. $\mathbb{F}_1[[x]]$ is a group under substitution (= composition):

If $F(x) = \sum_{n=1}^{\infty} f_n x^n \in \mathbb{F}_1[[x]]$ and $K(x) = \sum_{j=1}^{\infty} k_j x^j \in \mathbb{F}_1[[x]]$, then

$$(F \circ K)(x) \stackrel{\text{def}}{=} F(K(x)) \equiv f_1 K(x) + f_2 K(x)^2 + \cdots$$

The identity element is $\mathrm{id}(x) = x$.

The compositional inverse of $F(x)$ is denoted $\overline{F}(x)$ and is given by

$$\overline{F}(x) = \frac{1}{f_1}x - \frac{f_2}{f_1^3}x^2 + \cdots$$

(Solve $F(\overline{F}(x))$ term-by-term or use the LaGrange Inversion Formula([11, p. 38]:

$$[x^n]\,\overline{F}(x) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{F(x)}\right)^n.\,)$$

2.2 Roots in $\mathbb{F}_0[[x]]$

We state the foundational theorem on multiplicative roots of formal power series in Theorem 2.2 and then give very useful Theorems ropositions on roots which follow from this.

Theorem 2.2. (Root Theorem 1) *[8, Thm. 3, Thm. 17]:*

Suppose that $n \in \mathbb{N}$ and $a(x) = 1 + A(x)$, $(A(x) \in F_+[[x]])$. Then

(a) There exists a unique $b(x) = 1 + B(x)$, $(B(x) \in F_+[[x]])$ such that $(b(x))^n = a(x)$.

We denote: $b(x) \equiv (a(x))^{\frac{1}{n}}$.

(b) More precisely,

$$\begin{aligned}\text{If}\quad a(x) &= 1 + a_r x^r + a_{r+s}x^{r+s} + a_{r+s+1}x^{r+s+1} + \cdots, \\ &= 1 + A(x) \qquad \text{with } r, s > 0,\ a_r \neq 0, \\ \text{then } b(x) &= (1 + A(x))^{1/n} \\ &= \sum_{j=0}^{\infty}\binom{1/n}{j}A(x)^j \quad (\textit{Generalized Binomial Theorem})\end{aligned}$$

$$= \quad 1 + \frac{1}{n}a_r x^r + b_t x^t \ \textit{where } t = \min\{r+s, 2r\} \textit{ and}$$

$$b_t x^t = \begin{cases} \frac{1}{n}a_{r+s}x^{r+s} & \textit{if } r > s \\ \frac{1-n}{2n^2}\cdot a_r^2 \cdot x^{2r} & \textit{if } r < s \\ \left(\frac{1}{n}a_{2r} + \frac{1-n}{2n^2}\cdot a_r^2\right)x^{2r} & \textit{if } r = s \ \square \end{cases}$$

Corollary 2.3. (Normal form for non-constant $a(x)$)

If *$a(x) \in \mathbb{F}[[x]]$ is non-constant,*

$$a(x) = a_0 + a_r x^r + a_{r+1}x^{r+1} + \cdots, \quad (r > 0, a_r \neq 0)$$

Then *$a(x)$ may be* **uniquely** *written as*

$$a(x) = a_0 + a_r \cdot (A(x))^r \ \textit{with } A(x) \textit{ of the form } \ A(x) = x + \cdots.$$

Note: $A(x) \in \mathbb{F}_1[[x]]$ *with* $\overline{A}(x) = x + \cdots$.

Proof:

$$\begin{aligned} a(x) &= a_0 + a_r \cdot x^r \underbrace{\Big(1 + \frac{a_{r+1}}{a_r}x + \frac{a_{r+2}}{a_r}x^2 + \cdots\Big)}_{\equiv \widehat{a}(x)} \\ &= a_0 + \ a_r \cdot x^r \widehat{a}(x) \\ &= a_0 + a_r \cdot \left(x \cdot \widehat{a}^{\,1/r}(x)\right)^r \\ &= a_0 + \ a_r \cdot (A(x))^r \text{ where } A(x) = x \cdot \widehat{a}^{\,1/r}(x) \quad \square \end{aligned}$$

The following theorem for roots of series $a(x) \in \mathbb{F}_+[[x]]$ (i.e., with constant term $= 0$) follows immediately from Theorem 2.2.

Theorem 2.4. (Root Theorem 2)

If $a(x) = a_q x^q + a_{q+r}x^{q+r} + \cdots$, $(q, r > 0, a_q \neq 0)$
$= a_q \cdot (A(x))^q$ *with* $A(x) = x + \cdots$

then

1. *$B(x)$ is a solution of the equation $B(x)^q = a(x)$*

$$\iff \quad B(x) = b_1 \cdot A(x), \quad \textit{where } b_1^q = a_q.$$

Indeed, $\quad B(x) = b_1 x + b_r x^r + \cdots \quad$ *where* $b_1^q = a_q$.

2. *Given a q^{th} root b_1 of a_q, $B(x) = b_1 \cdot A(x)$ is the unique q^{th} root of $a(x)$ of the form $B(x) = b_1 x + \cdots$.* $\square$

Note: Our root $B(x)$ in the preceding theorem is an element of $\mathbb{F}_1[[x]]$ and has a compositional inverse $\overline{B}(x)$. It is this which often motivates us to take roots, as in the following Theorem.

2.3 A Composition-Cancellation Theorem

Theorem 2.5. **(The Composition-Cancellation Theorem)**

Suppose that $g(x) = g_0 + g_r x^r + g_r x^{r+1} + \cdots$ *with* $r \geq 1$ *and* $g_r \neq 0$.

Let $a(x) = a_s x^s + a_{s+1} x^{s+1} + \cdots$ *and* $b(x) = b_s x^s + b_{s+1} x^{s+1} + \cdots$, *where* $s \geq 1$ *and* $a_s = b_s \neq 0$. *Then*

$$g(a(x)) = g(b(x)) \implies a(x) = b(x).$$

Note: *There is no assumption that* $g(x)$ *has a compositional inverse.*

Proof: By Corollary 2.3, $g(x) = g_0 + g_r \cdot \big(G(x)\big)^r$ with $G(x) = x + \cdots \in \mathbb{F}_1[[x]]$. Thus

$$\begin{aligned} g(a(x)) &= g(b(x)) \\ \implies \big(G(a(x))\big)^r &= \big(G(b(x))\big)^r = (a_s)^r x^{rs} + \cdots \\ &= h_r \cdot (H(x))^r \text{ with} \\ & \quad h_r = (a_s)^r = (b_s)^r \text{ and } H(x) = x + h_2 x^2 + \cdots \end{aligned}$$

Thus, by Root Theorem 2 and the fact that $a_s = b_s$,

$$\begin{aligned} G(a(x)) &= a_s H(x) = b_s H(x) = G(b(x)) \\ \implies \overline{G}\big(G(a(x))\big) &= \overline{G}\big(G(b(x))\big)^{\cdot} \\ \implies a(x) &= b(x) \qquad \square \end{aligned}$$

3 Theorems on Riordan Group Elements

3.1 Shortening the Proof that $\big(g(x), F(x)\big)^n = (1, x)$

From the definition of multiplication in the Riordan group $\mathcal{R}$ one immediately gets by induction the fact that, for $n \in \mathbb{N}$, $\big(g(x), F(x)\big)^n = (1, x)$ (the identity matrix) if and only if

(a) $g(x) \cdot g\big(F(x)\big) \cdots g\big(F^{(n-1)}(x)\big) = 1$

and (b) $F^{(n)}(x) = x$

In proving that an element of the group $\mathcal{R}$ has finite order, the proofs of (a) and (b) can each be quite intricate. (See [2], [3].) We use the Composition-Cancellation Theorem (2.5) to show that in very common situations, (b) follows automatically from (a).

Theorem 3.1. *Suppose that* $\big(g(x), F(x)\big) \in \mathcal{R}$ *with* $g(x) = g_0 + g_r x^r + g_{r+1} x^{r+1} + \cdots$ *and* $F(x) = f_1 x + f_2 x^2 + \cdots$, *where* $g(x)$ *is non-constant (*$g_0 \neq 0$, $r > 0$ *and* $g_r \neq 0$*) and* $(f_1)^n = 1$.

Then: $\big(g(x), F(x)\big)^n = (1, x) \iff g(x) \cdot g\big(F(x)\big) \cdots g\big(F^{(n-1)}(x)\big) = 1.$

Proof: The necessity follows from the definition of multiplication in $\mathcal{R}$. To see sufficiency note that

$$\begin{aligned} g(x) \cdot g\big(F(x)\big) \cdots \cdots g\big(F^{(n-1)}(x)\big) &= 1 \\ \implies g\big(F(x)\big) \cdot g\big(F(F(x))\big) \cdots g\big(F^{(n-1)}(x)\big) \cdot g\big(F^{(n)}(x)\big) &= 1 \end{aligned}$$

But since $g\big(F^{(j)}(x)\big)$ has a multiplicative inverse and $F(x) = f_1 x + \cdots$, we then have

$$g\big(F^{(n)}(x)\big) = g(x) \text{ and } F^{(n)}(x) = x + \cdots.$$

By The Composition-Cancellation Theorem (2.5), $F^{(n)}(x) = x$. Thus

$$\big(g(x), F(x)\big)^n = \big(g(x) \cdot g\big(F(x)\big) \cdots g\big(F^{(n-1)}(x)\big), F^{(n)}(x)\big) = (1, x). \quad \square$$

3.2 The Unique Riordan Involution Determined by $g(x)$

The method of using roots to solve functional equations, which is given in Corollary 2.3 and Theorem 2.5, is used in this section to determine, for given $g(x) \in \mathbb{F}_0[[x]]$, exactly which (if any) $F(x)$ will make $\big(g(x), F(x)\big)$ an involution in the Riordan group. As a preliminary, the following Lemma gives necessary conditions for $\big(g(x), F(x)\big)$ to be an involution.

Lemma 3.2. *Suppose that* $\big(g(x), F(x)\big) \in \mathcal{R}$ *with*

$$g(x) = g_0 + g_r x^r + g_{r+1} x^{r+1} + \cdots, \quad F(x) = f_1 x + f_2 x^2 + \cdots.$$

Then

1. *If* $\big(g(x), F(x)\big)$ *is an involution then* $g_0^2 = 1$.
2. *If* $g(x) = g_0$ *then* $\big(g(x), F(x)\big)$ *is an involution* $\iff \big(g(x), F(x)\big) = (-1, x)$ *or* $\big(g(x), F(x)\big) = \big(\pm 1, F(x)\big)$, *where* $F(x) = -x + \cdots$ *with* $F(F(x)) = x$.
3. *If* $g(x) \neq g_0$ *and* $\big(g(x), F(x)\big)$ *is an involution , then* r *is odd,* $g_r \neq 0$ *and* $f_1 = -1$.

Proof: $\big(g(x), F(x)\big)$ is an involution iff

- $x = F(F(x)) = f_1[f_1 x + f_2 x^2 + \cdots] + f_2[f_1 x + f_2 x^2 + \cdots]^2 + \cdots$
  $= f_1^2 x +$ (higher powers)
- $1 = g(x) \cdot g\big(F(x)\big) = g_0^2 + g_0 g_r (1 + f_1^r) x^r + \cdots$.

Thus $f_1 = \pm 1$, $g_0 = \pm 1$ and either $g_r = 0$ or (r is odd and $f_1 = -1$). The results of the Lemma follow. $\square$

Theorem 3.3. *(Main Theorem, generalizing [6])*

Suppose that

$$\begin{aligned} g(x) &= g_0 + g_r x^r + g_{r+1} x^{r+1} + \cdots, \text{ with } g_0 = \pm 1, r \text{ odd}, \text{ and } g_r \neq 0 \\ &= g_0 + g_r \cdot (G(x))^r \text{ with} \\ &\quad G(x) = x + \cdots \ (\text{as in Corollary 2}) \\ &\quad = x \cdot \left(1 + \frac{g_{r+1}}{g_r} x + \frac{g_{r+2}}{g_r} x^2 + \cdots \right)^{1/r}. \end{aligned}$$

Then there exists a unique $F(x) = f_1 x + f_2 x^2 + \cdots$ such that $(g(x), F(x))$ is an involution. This is given by

$$F(x) = \overline{G}\left(\frac{-G(x)}{(g_0 g(x))^{1/r}} \right).$$

Proof:

$$\begin{aligned} & g(x) \cdot g(F(x)) = 1 \\ \iff & g(F(x)) = \frac{1}{g(x)} \\ \iff & g_r \cdot (G(F(x)))^r = \frac{1}{g(x)} - g_0 \\ \iff & g_r \cdot (G(F(x)))^r = \frac{1 - g_0 g(x)}{g(x)} \\ \iff & g_r \cdot (G(F(x)))^r = \frac{-g_0 g_r \cdot (G(x))^r}{g(x)} \quad (\text{since } g_0^2 = 1) \\ \iff & (G(F(x)))^r = \frac{-(G(x))^r}{g_0 g(x)} \end{aligned}$$

To solve the last functional equation, note that

- $(-1)^r = -1$,
- $\left(G(F(x))\right)^r$
  $= \left([-x + f_2 x^2 + \cdots] + G_2[-x + f_2 x^2 + \cdots]^2 + \cdots\right)^r$
  $= -\left(x + \cdots\right)^r$,
- $g_0 g(x) = 1 + g_0 g_r x^r + \cdots$

Thus $(g(x), F(x))$ is an involution

$$\begin{aligned} \iff & g(x) \cdot g(F(x)) = 1 && (\text{by Theorem 3.1}) \\ \iff & G(F(x)) = \frac{-(G(x))}{(g_0 g(x))^{1/r}} && (\text{by Theorem 2.4}) \\ \iff & F(x) = \overline{G}\left(\frac{-G(x)}{(g_0 g(x))^{1/r}} \right). \quad \square \end{aligned}$$

3.3 The Involution Determined by an Aerated $g(x)$

Perhaps the best known Riordan involution is $(g(x), F(x)) = \left(\frac{1}{1-x}, \frac{-x}{1-x}\right)$ – the Pascal matrix with negation of the odd-indexed columns. If we "aerate" $g(x) = \sum_{n=0}^{\infty} x^n$, adding blocks of zeroes to get

$$h(x) = \sum_{n=0}^{\infty} x^{qn} = \frac{1}{1 - x^q}, \quad q \in \mathbb{N},\ q \text{ odd},$$

what series $K(x)$ makes $(h(x), K(x))$ an involution? As an exercise in applying Theorem 3.3, one finds that

Example 3.4. *If $q \in \mathbb{N}$ is odd, the matrix*

$$(h(x), K(x)) = \left(\frac{1}{1 - x^q}, \frac{-x}{(1 - x^q)^{1/q}} \right) = \left(\sum_{n=0}^{\infty} x^{qn}, \sum_{n=0}^{\infty} (-1)^{n+1} \binom{-\frac{1}{q}}{n} x^{1+qn} \right)$$

is an involution.

Proof: Compute $h(x) \cdot h(K(x)) = 1$ and apply Theorem 3.1. $\square$

The general case of an aeration

More generally, suppose that $(g(x), F(x))$ is an involution and q is a positive odd integer. Let $h(x) = g(x^q)$. We use Theorem 3.3 to find $K(x)$ such that $(h(x), K(x))$ is an involution. For notational simplicity, we temporarily assume that $g(x) = g_0 + g_1 x + g_2 x^2 + \cdots$ with $g_1 \neq 1$ (*i.e.*, that $r = 1$ in Theorem 3.3) and sketch how Theorem 3.3 leads us to $K(x)$.

Notation (using Theorem 2.2): If $a(x) = x^q + a_{q+1} x^{q+1} + \cdots$, q odd, then $(a(x))^{1/q} \stackrel{\text{def}}{=} x\left(1 + a_{q+1} x + a_{q+1} x^2 + \cdots\right)^{1/q}$.

We have

- $g(x) = g_0 + g_1 \cdot G(x)$ where $G(x) = x + \frac{g_2}{g_1} x^2 + \cdots$
- $F(x) = \overline{G}\left(\frac{-G(x)}{g_0 g(x)}\right)$ as in [6] or Theorem 3.3, since we are assuming $r = 1$.
- $h(x) = g(x^q) = g_0 + g_1 \cdot G(x^q) \stackrel{\text{def}}{=} h_0 + h_q \cdot (H(x))^q$
  where $h_0 = g_0$, $h_q = g_1$ and $H(x) = [G(x^q)]^{1/q}$
- $\overline{H}(x) = \left(\overline{G}(x^q)\right)^{1/q}$ since

$$\begin{aligned} H\left((\overline{G}(x^q))^{1/q}\right) &= \left[G\left\{\left((\overline{G}(x^q))^{1/q}\right)^q\right\}\right]^{1/q} \\ &= \left[G\left\{\overline{G}(x^q)\right\}\right]^{1/q} = [x^q]^{1/q} = x \end{aligned}$$

- $K(x) = \overline{H}\left(\frac{-H(x)}{(h_0 h(x))^{1/q}}\right)$, by Theorem 3.3

$$= \left[\overline{G}\left(\frac{-G(x^q)}{(g_0 g(x^q))}\right)\right]^{1/q} \text{ (computation!)}, \quad = \left(F(x^q)\right)^{1/q} \text{ (since } g_1 \neq 0\text{)}.$$

Having used Theorem 3.3 to discover the formula for $K(x)$ in the special case $r = 1$, the direct proof of the validity of this formula in general is quite simple, given its uniqueness:

Theorem 3.5. (The Aeration Theorem)

Suppose that $(g(x), F(x))$ is an involution, where $g(x)$ is non-constant and suppose that $h(x) = g(x^q)$ for some positive odd integer q. Let $(h(x), K(x))$ be the unique involution given by Theorem 3.3. Then $K(x) = \left(F(x^q)\right)^{1/q}$.

Proof: $h(x) \cdot h\left[(F(x^q))^{1/q}\right] = g(x^q) \cdot g\left[\left(F(x^q)\right)^{1/q}\right]^q = g(x^q) \cdot g\left(F(x^q)\right) = 1.$

The last term equals one by substituting x^q into $g(x) \cdot g(F(x)) = 1$. Since $K(x)$ is unique, it must equal $\left(F(x^q)\right)^{1/q}$. □

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