Grid Domination on Hexagonal Boards

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Abstract

A grid on a cell of a game board attacks all neighboring cells. The domination number counts the minimum number of grids such that each cell of a board is occupied or attacked by a grid. For square boards (chess boards) the domination number has been determined in a series of papers. Here we start to consider grids on hexagon boards B_n as parts of the euclidean tessellation by congruent regular hexagons where B_1 is one hexagon, B_2 consists of the three hexagons around one vertex, and B_n for $n \geq 3$ consists of B_{n-2} together with all hexagons having at least one hexagon in common with B_{n-2}. An upper bound is presented for the grid domination number and exact values are determined by computer for small n.

1. Introduction

There are the three euclidean tessellations of the plane by equilateral triangles, by squares, and by regular hexagons as cells. On each of these tessallations game boards B_n can be defined as follows:
B_1 is one cell, B_2 consists of all cells surrounding one vertex, and B_n for $n \geq 3$ consists of B_{n-2} together with all cells having at least one vertex in common with B_{n-2} (see [1]). For squares this euclidean board is well known as the chess board. There are many results on combinatorical problems for chess boards, however, not much is known on corresponding combinatorical problems for hexagonal or triangle boards (see [1]).
A grid on a cell C attacks all neighboring cells having an edge in common with C. The domination number $\gamma(n)$ for a board B_n is the minimum number of grids such that each cell of B_n has a grid or is attacked by a grid (see [4]).
For chessboards the domination number is

$$\gamma_4(n) = \left\lfloor \frac{(n+2)^2}{5} \right\rfloor - 4$$

which was discussed in [2, 3, 6, 7] and finally proved in [5]. Here we will start to consider the domination number $\gamma_6(n)$ for hexagon boards.

2. First bounds

The number c_n of cells of a board B_n is

$$c_n = \left\lceil \frac{3n^2}{4} \right\rceil .$$

Since a grid occupies one cell and attacks at most six further cells the following trivial lower bound is implied.

Theorem 1

$$\gamma_6(n) \geq \left\lceil \frac{c_n}{7} \right\rceil = \begin{cases} \lceil \frac{3}{28} n^2 \rceil & \textit{for} \quad n \quad \textit{even}, \\ \lceil \frac{3}{28}(n^2+1) \rceil & \textit{for} \quad n \quad \textit{odd.} \end{cases}$$

A first upper bound is given in the following theorem.

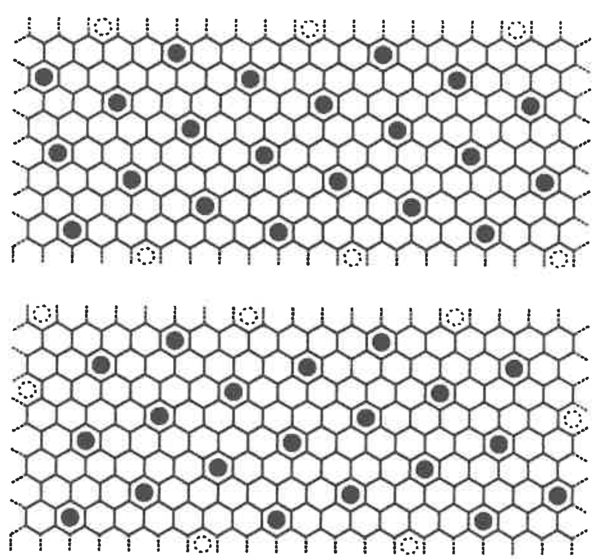

Figure 1: The two reflective orientations of the regular occupance.

Theorem 2

$$\gamma_6(n) \leq \left\lfloor \frac{1}{28}(3n^2+12n+15) \right\rfloor - \begin{cases} 1 & \textit{for} \quad n \equiv 3,7 \,(\textit{mod}\, 14), \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The euclidean hexagon tesselation of the plane is optimally dominated by that regular occupance where one grid is positioned in each seventh cell of each of the three linear directions of hexagons. There exist two reflective orientations (see Figure 1). If B_{n+2} is chosen from the regular occupance in all possible ways then the minimum number $g(n+2)$ of grids determines an upper bound for $\gamma_6(n)$ of B_n since those grids on a border hexagon of B_{n+2} can be moved to a neighboring hexagon of B_n so that B_n is dominated.

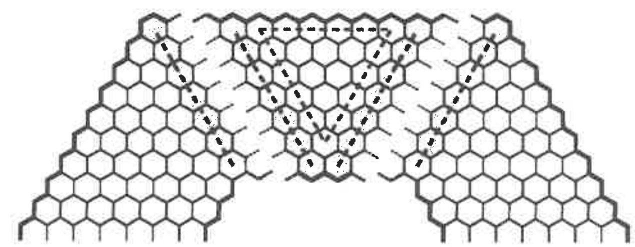

Figure 2: Covering by 7-sticks and 6-triangles.

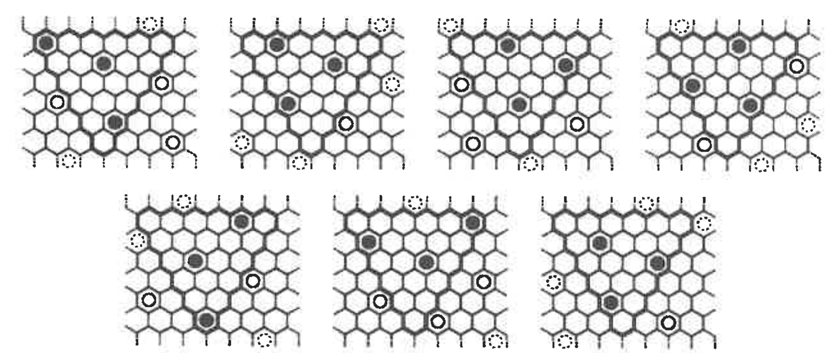

Figure 3: The seven possibilities for 6-triangles.

An induction from k to $k+14$ is possible for the minimum number $g(k)$. For $g(k+14)$ all hexagons are considered which are the first seven consecutive rings of hexagons starting from the border ring. These hexagons can be covered by 7-sticks (7 hexagons in a row) and six 6-triangles (with 6 hexagons at each triangle side), one 6-triangle at each side of B_{k+14} (see Figure 2). Each 7-stick contains exactly one grid and each 6-triangle contains exactly 3 grids (see the seven possibilities in Figure 3). For $k \equiv 1\,(\textit{mod}\, 2)$ there are six 7-sticks starting at each corner. At each side with $\frac{k+15}{2}$ hexagons there is one 6-triangle and then the num-

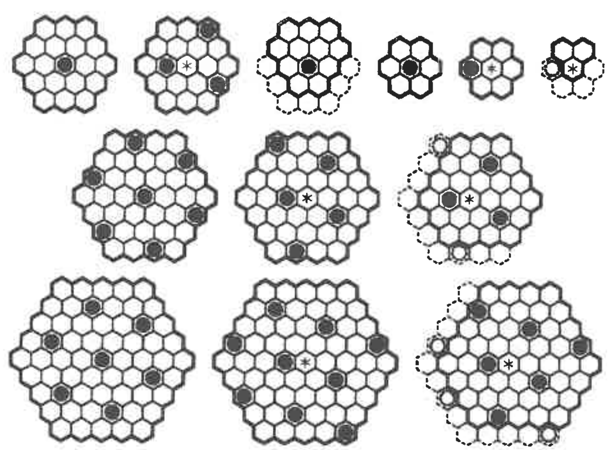

Figure 4: $g(5) = g(4) = g(3) = 1$, $g(7) = 5$, $g(6) = 3$, $g(9) = 7$, $g(8) = 6$.

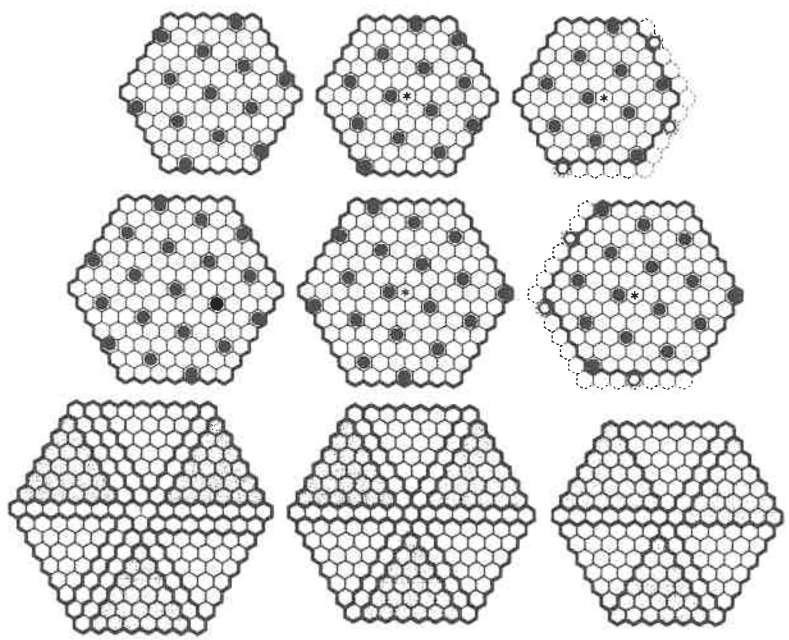

Figure 5: $g(11) = 13$, $g(10) = 10$, $g(13) = 18$, $g(12) = 15$, $g(16) = 6 \cdot 3 + 9 = 27$, $g(15) = 24$, $g(14) = 21$.

ber of remaining 7-sticks at each side is $\frac{k+15}{2} - 2 - 6 = \frac{k-1}{2}$ so that $g(k+14) = g(k)+6+6\cdot 3+\frac{k-1}{2}\cdot 6 = g(k)+3k+21$. For $k \equiv 0\,(mod\,2)$ there are alternating $\frac{k+14}{2}$ and $\frac{k+16}{2}$ hexagons on the sides of B_{k+14} and it follows correspondingly $g(k+14) = g(k)+6+6\cdot 3+3(\frac{k+14}{2}-2-6)+3(\frac{k+16}{2}-2-6) = g(k) + 3k + 21$.
From this the induction step is

$$\begin{aligned}\gamma_6(n+14) &\leq g(n+16) = g(n+2) + 3(n+2) + 21 \\ &= \left\lfloor \frac{1}{28}(3n^2 + 12n + 15) \right\rfloor + 3(n+2) + 21 - \eta \\ &= \left\lfloor \frac{1}{28}(3(n+14)^2 + 12(n+14) + 15) \right\rfloor - \eta,\end{aligned}$$

with $\eta = 1$ for $n \equiv 3, 7\,(mod\,14)$ and $\eta = 0$ otherwise. It remains the induction base, i.e. $g(n+2)$ for $1 \leq n \leq 14$.

These values are determined in Figures 4 and 5 since for odd n there are two possiblities only and for even n three consecutive sides are deleted from B_{n+1}. □

3. Slight improvements

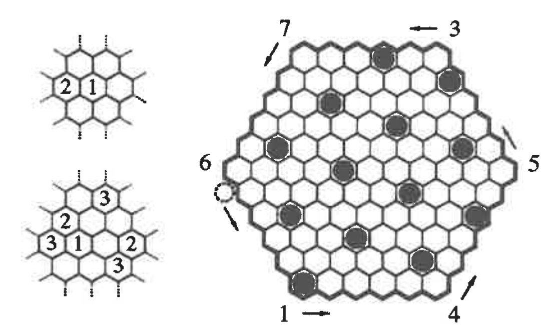

Figure 6: The types of B_n on the regular occupance for odd and even n and B_{11} of type 2 with the corner sequence 145376.

The upper bound in Theorem 2 seems to be nearly sharp. However, improvements up to 2 grids are possible.

Theorem 3

$$\gamma_6(n) \leq \left\lfloor \frac{1}{28}(3n^2 + 12n + 15) \right\rfloor - \begin{cases} 2 & \textit{for} \quad n \equiv 1, 11, 12, 13\,(mod\,14), \\ 1 & \textit{otherwise.} \end{cases}$$

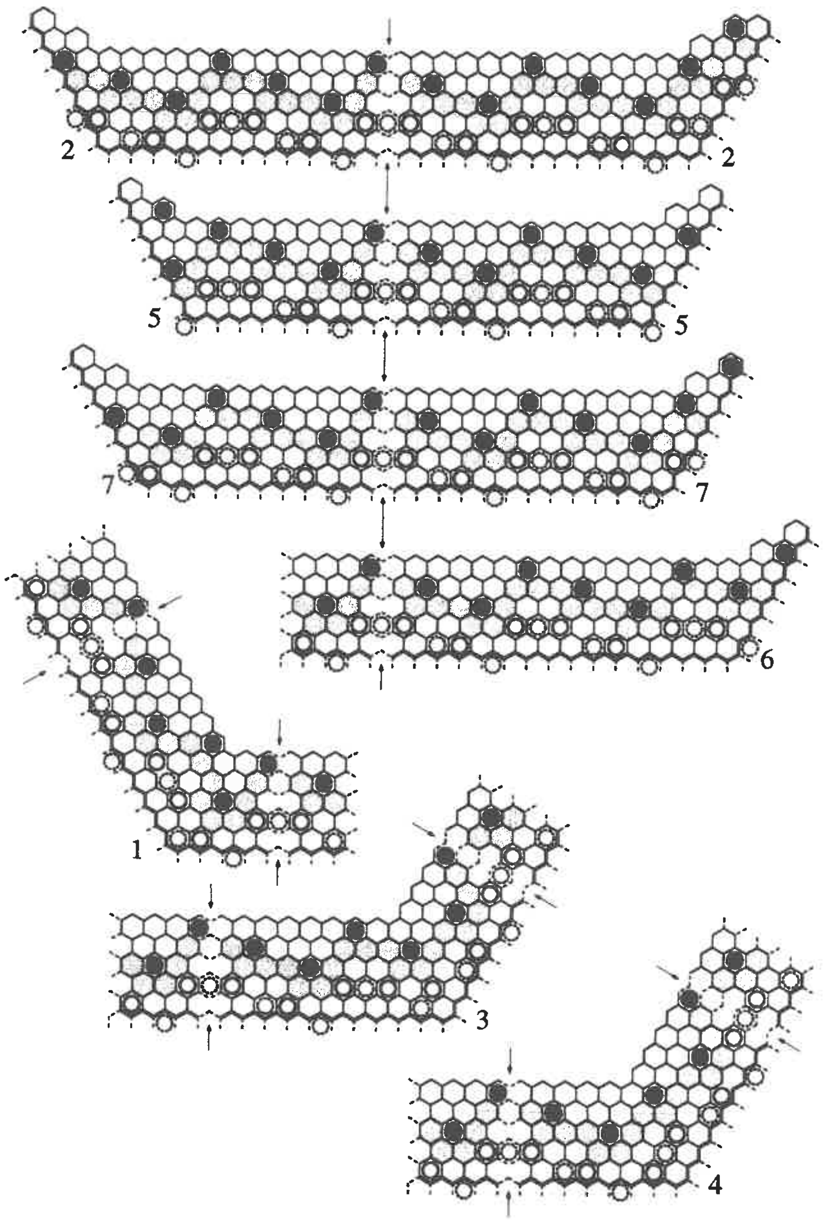

Figure 7: Three starting corners, four ending corners, and three corners for in between.

k	type 1	ε	ε^*	$n>$	type 2	ε	ε^*	$n>$	type 3	ε	ε^*	$n>$
0	165571	1	-1	0	634732	0	-1	0	242424	0	-1	0
1	111111	0	0		265734	0	-2	1				
2	421134	0	0		267365	1	-1	2	575757	2	-1	2
3	444444	-1	-1	0	521367	1	-1	3				
4	754467	1	-1	4	523621	1	-1	4	131313	0	0	
5	777777	2	-1	5	154623	0	-1	5				
6	317723	1	0		156254	1	-1		464646	0	0	6
7	333333	-1	-1	0	417256	1	-1	7				
8	643356	0	-1	0	412517	1	0		727272	2	-1	8
9	666666	0	0		743512	0	-1	0				
10	276612	1	-1	10	745143	0	-1	0	353535	0	-1	0
11	222222	1	-2	11	376145	0	-1					
12	532245	0	-2	0	371476	0	-1		616161	0	0	
13	555555	1	-2	0	632471	0	-1					

Table 1: For $n \equiv k \,(mod\, 14)$ all corner sequences of all types of B_n.

Proof. There are three and two types of B_n as parts of the regular occupance for even and odd n, respectively (see Figure 6). If at each corner of B_n the position of the first grid at the border in anti-clockwise direction is determined then these six position numbers describe a corner sequence of B_n dependent of its type (see B_{11} of type 2 in Figure 6). These corner sequences are repeated modulo 14 (see Figure 2) and all possibilities are gathered in Table 1.
For each type of B_n as part of the regular occupance the number G_i of grids ($i = 1$, 2 if n is odd and $i = 1$, 2, 3 if n is even) in the corresponding B_{n+2} gives an upper bound of $\gamma_6(n)$. The smallest value is determined in Theorem 2. In Table 1 the values of ε denote how much larger G_i is than the floor bracket term in Theorem 2.
Now in many cases the value of G_i can be reduced by up to three, that is, by $\varepsilon - \varepsilon^*$ (see Table 1). This reduction can be done as follows dependent on the corner sequences. A corner sequence implies a reduction of one grid if it contains a cyclical subsequence starting with the positions 2, 5, or 7 and ending with 2, 5, 7, or 6, and where possibly corners 1, 3, or 4 are inserted in between. This can be checked in Figure 7 where the dotted cycles are grids of the regular occupance which are substituted by the cycles. If there are up to three nonoverlapping subsequences of this kind then up to three grids are saved. Note, that blocks of seven hexagons can be inserted in Figure 7 to guarantee the equality mod 14. The smallest value of ε and of ε^* for $n \equiv k \,(mod\, 14)$ being listed in Table 1 are added to the floor brackets in Theorem 2 and 3, respectively. □

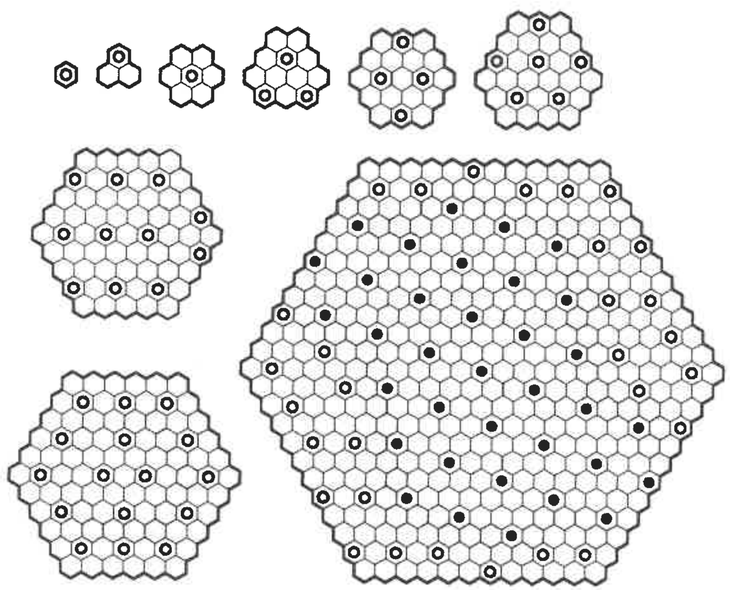

Figure 8: $\gamma_6(1) = 1$, $\gamma_6(2) = 1$, $\gamma_6(3) = 1$, $\gamma_6(4) = 3$, $\gamma_6(5) = 4$, $\gamma_6(6) = 6$, $\gamma_6(9) = 11$, $\gamma_6(11) = 16$, $\gamma_6(23) = 65$.

n	$\gamma_6(n)$	n	$\gamma_6(n)$	n	$\gamma_6(n)$	n	$\gamma_6(n)$
1	1	8	9	15	29	22	60
2	1	9	11	16	33	23	65
3	1	10	14	17	37	24	71
4	3	11	16	18	41	25	76
5	4	12	19	19	46	26	82
6	6	13	22	20	50	27	88
7	7	14	26	21	55	28	

Table 2: The exact values of $\gamma_6(n)$.

4. Exact values

For small values of n up to $n = 27$ we have checked by computer all possibilities of domination to obtain the exact values given in Table 2. For those small values of n for which $\gamma_6(n)$ does not attain the upper bound of Theorem 3 examples are presented in Figure 8. It may be conjectured that the bound of Theorem 3 is sharp in $n \geq 24$.

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