CONSTRUCTING CLIFFORD GRAPH ALGEBRAS FROM CLASSICAL CLIFFORD ALGEBRAS

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ABSTRACT. There is a special case of a generalized Clifford algebra, known as a Clifford graph algebra, which is useful for studying a simple graph G_n with n vertices. We will discuss how this algebra $GA(G_n)$ can represent G_n and prove that it exists in general by defining it as an appropriate sub-algebra of a classical Clifford algebra. We will then refine this process of "construction by inclusion" for the path graph P_n and the complete star graph $K_{1,n}$ by choosing from a parent classical Clifford algebra as many bi-vectors as possible for the generators which define $GA(P_n)$ and $GA(K_{1,n})$.

1. INTRODUCTION

In 1878 William Clifford published a work wherein he presented a class of algebras, called Clifford algebras in is honor, which included both the exterior product in Grassman's algebra and generators that possess the squaring and anti-commutativity properties of Hamilton's quaternions [4]. Although originally useful primarily for physics applications, in 2008 T. Khovanova [3] extended their application to algebraic graph theory, which is the topic of this article. Equipped with a quadratic form, a Clifford algebra with signature is defined as follows [1], [2].

Definition 1.1. A (real) Clifford (geometric) algebra of signature (p, q), denoted $\mathbb{G}^{(p,q)}$, where $p + q = n$, is an $\mathbb{R}$-algebra which is generated by the set $S = \{e_1, \dots, e_n\}$ where the elements in S satisfy the fundamental conditions

(i) $e_j e_k = -e_k e_j$ for $k \neq j$

(ii) $$e_k^2 = \begin{cases} 1 & \text{if } 1 \leq k \leq p \\ -1 & \text{if } p+1 \leq k \leq p+q = n\ . \end{cases}$$

In particular, $\mathbb{G}^{(0,n)}$ denotes a geometric algebra where each generator squares to -1, and $\mathbb{G}^n = \mathbb{G}^{(n,0)}$ denotes a geometric algebra where each generator squares to 1.

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Remark 1.1. Throughout this work, we will omit the base field $\mathbb{R}$ in $\mathbb{G}^{(p,q)}(\mathbb{R})$ and simply denote this algebra as $\mathbb{G}^{(p,q)}$.

After Clifford developed his algebras, mathematicians and physicists; namely J. Sylvester [6], E. Cartan [7], H. Weil [5], and J. Schwinger [8] progressively extended Clifford's algebra into a generalized Clifford algebra, defined as follows (see [9], [10]).

Definition 1.2. A generalized Clifford algebra is a $\mathbb{C}$-algebra which is generated by the set $S = \{e_1, \dots, e_n\}$ where the elements in S satisfy the following relations for all $j, k, \ell, m = 1, 2, \dots, n$

(i) $e_j e_k = \omega_{jk} e_k e_j \ , \ \omega_{jk} e_\ell = e_\ell \omega_{jk} \ , \ \omega_{jk} \omega_{\ell m} \omega_{jk}$

(ii) $e_j^{N_j} = 1 \ , \ \omega_{jk}^{N_j} = \omega_{jk}^{N_k} = 1$ for some $N_j, N_k \in \mathbb{N}$.

In [3] T. Khovanova explains how the special case where $\omega_{jk} = \pm 1$ can be used to depict the the connectivity between vertices in a finite, simple graph. Hence, Khovanova refers to such a generalized Clifford algebra as a Clifford graph algebra. Although in [3] Khovanova defines a Clifford graph algebra over $\mathbb{C}$, we will alter our definition here from that in [3] by instead defining a Clifford graph algebra over $\mathbb{R}$ with signature (p, q) where $p + q = n$. We will tacitly assume that any graph in this article is simple (no multiple edges between any pair of vertices) and finite (finitely many edges and vertices).

Definition 1.3. Given a graph G_n with n vertices $v_1 \ , \ v_2 \ , \ \dots \ , \ v_n$, a Clifford graph algebra for G_n, denoted $GA(G_n)$, is an $\mathbb{R}$-algebra with n generators $e'_1 \ , \ e'_2 \ , \ \dots \ , \ e'_n$ such that each generator e'_i is paired with one vertex v_i so that the following rules hold

(i) $e'_i e'_j = -e'_j e'_i$ if v_i and v_j are adjacent
$e'_i e'_j = e'_j e'_i$ if v_i and v_j share no edge

(ii)

$$(e'_k)^2 = \begin{cases} 1 & \text{if } 1 \le k \le p \\ -1 & \text{if } p+1 \le k \le p+q = n \ . \end{cases}$$

The main objective of this article is to construct any Clifford graph algebra for a given graph G_n in a way that is simpler than in the general case where ω_{jk} is an arbitrary complex number. As an additional advantage, the constructive proof presented here is motivated primarily by the the connectivity of G_n.

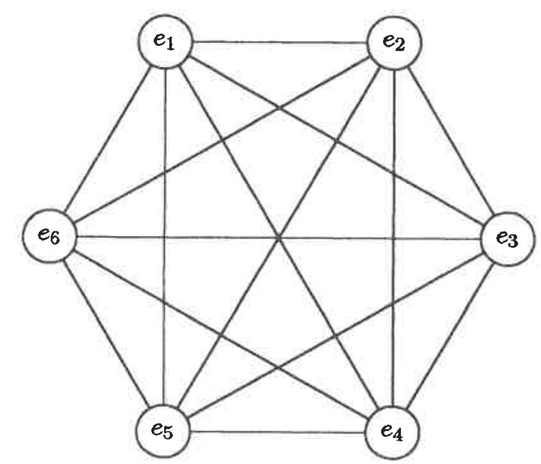
As a first example, a classical Clifford algebra itself can serve as the Clifford graph algebra for the complete graph.

Example 1.1. Consider the Clifford graph algebra $GA(K_n)$ for the complete graph K_n. Since by definition each pair of vertices in K_n are adjacent (see,

for instance, K_6 in figure 1 below), then each pair of distinct generators in the corresponding Clifford graph algebra commute; so in this case the algebra $\mathbb{G}^n$ or $\mathbb{G}^{0,n}$ itself can serve as the Clifford graph algebra for K_n. Occasionally in this article we will use the underlying graph G_n to illustrate a specific Clifford graph algebra $GA(G_n)$ schematically by labeling each vertex in G_n with its corresponding generator. Such a depiction is accurate because the commutativity and anti-commutativity of the generators defines the algebra. In particular, the diagram below shows such a schematic representation for $GA(K_6) = \mathbb{G}^6$ or $GA(K_6) = \mathbb{G}^{(0,6)}$.

Figure 1

Schematic depiction of $GA(K_6)$



If the graph G_n is not complete, the generators in a Classical Clifford algebra will not be able to provide the needed property of commutativity for pairs of vertices that share no edge. As an alternative to constructing a generalized Clifford algebra to serve this purpose for G_n by the process explained in [9], in our case where each $\omega_{jk} = 1$ or $\omega_{jk} = -1$ we will more efficiently prove that we may obtain $GA(G_n)$ directly from a classical Clifford algebra with signature. For convenience, we will choose this underlying algebra to be either $\mathbb{G}^m$ or $\mathbb{G}^{(0,m)}$, where $m > n$. We will construct $GA(G_n)$ by selecting from the basis for either $\mathbb{G}^m$ or $\mathbb{G}^{(0,m)}$ a subset of monomials which satisfies the connectivity conditions of G_n as prescribed in Definition 1.3; thereby establishing $GA(G_n)$ as a sub-algebra of $\mathbb{G}^m$ or $\mathbb{G}^{(0,m)}$. This method of selection works because every pair of such monomials either commutes or anti-commutes.

Remark 1.2. Most of the results in this work will hold equally for either $\mathbb{G}^m$ or $\mathbb{G}^{(0,m)}$. Although using $\mathbb{G}^m$ as the parent algebra for $GA(G_n)$, where $m > n$, would seem simpler since each of its generators square to 1, we will also use $\mathbb{G}^{(0,m)}$, where each generator squares to -1, in order to incorporate classical isomorphisms such as the following into any subalgebras of $\mathbb{G}^{(0,m)}$ which can define an algebra $GA(G_n)$. For instance, as A. Macdonald discusses in [1],

$$\mathbb{G}^{(0,1)} \cong \mathbb{C} \qquad \text{(the complex numbers)} \qquad (1)$$

$$\mathbb{G}^{(0,2)} \cong \mathbb{H} \qquad \text{(the quaternions)}$$

In particular, note that (1) underscores the fact that any generator in $\mathbb{G}^{(0,m)}$ behaves like the imaginary unit i.

As an example of selecting generators for a Clifford graph algebra from a parent algebra, we will construct $GA(G_3)$ from $\mathbb{G}^{(0,3)}$ for each of the four different configurations for G_3.

Example 1.2. If $n = 3$, the basis B_3 for $\mathbb{G}^{(0,3)}$ contains monomials which can serve as generators for any graph G_3, where

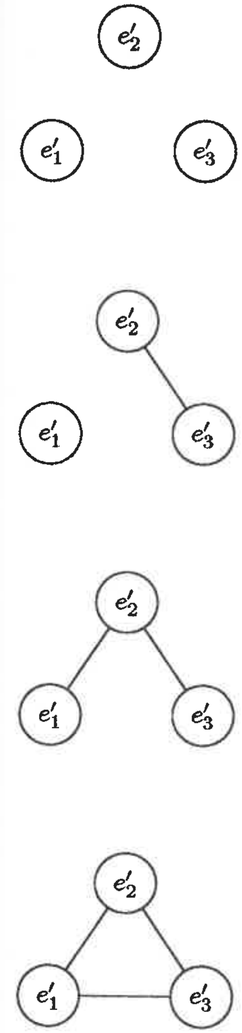
$$B_3 = \{ \mathbf{1} , e_1 , e_2 , e_3 , e_1e_2 , e_1e_3 , e_2e_3 , e_1e_2e_3 \}.$$

The following table lists the possible commutativity (c) and anti-commutativity (a) relations between the monomials in B_3.

e_2	a					
e_3	a	a				
e_1e_2	a	a	c			
e_1e_3	a	c	a	a		
e_2e_3	c	a	a	a	a	
$e_1e_2e_3$	c	c	c	c	c	c
	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3

This table shows that the choice of the generators in each of the following possible graphs for G_3 accurately depicts the connectivity of their associated vertices by conforming to property (i) in Definition 1.3. Also note that the choice of generators satisfies property (ii) in Definition 1.3. For instance, in the first in figure 2 we have that $(e_1')^2 = 1 , (e_2')^2 = (e_3')^2 = -1$; hence the signature for this graph is $p = 1 , \ q = 2$.

Figure 2



Generators :	Signature :
$e_1' = e_1e_2e_3$, $e_2' = e_2e_3$, $e_3' = e_1$	$p = 1 , \ q = 2$
$e_1' = e_1e_2e_3$, $e_2' = e_1e_2$, $e_3' = e_1e_3$	$p = 1 , \ q = 2$
$e_1' = e_1$, $e_2' = e_2$, $e_3' = e_2e_3$	$p = 0 , \ q = 3$
$e_1' = e_1e_2$, $e_2' = e_1e_3$, $e_3' = e_2e_3$	$p = 0 , \ q = 3$

2. Preliminaries

Throughout this work we will assume that the following notations satisfy the stated conditions.

(i) $n \in \mathbb{N}$ and $n > 1$, and will denote the number of generators in $\mathbb{G}^n$ or $\mathbb{G}^{(0,n)}$. Thus, to avoid trivialities we will always assume that every graph and Clifford algebra considered has at least two vertices or generators.

(ii) The symbols $e_1, \ldots, e_n$ will denote the generators for $\mathbb{G}^n$ or $\mathbb{G}^{(0,n)}$. At times we will indicate this by the notation $\mathbb{G}^n = \langle e_1, \ldots, e_n \rangle$ or $\mathbb{G}^{(0,n)} = \langle e_1, \ldots, e_n \rangle$.

(iii) Indices for vertices of G_n and generators of $GA(G_n)$ are natural numbers, denoted as $i_1, i_2, \ldots, i_n$, which we will assume to satisfy $1 \leq i_1 < \ldots < i_r \leq n$ where $r \in \mathbb{N}$ and $1 < r \leq n$.

(iv) A monomial of the form $e_{i_1} e_{i_2} \cdots e_{i_r}$ where $1 \leq i_1 < \ldots < i_r \leq n$ is said to have grade r. We will tacitly assume that the symbol $e_{i_1} e_{i_2} \cdots e_{i_r}$ denotes monomial of grade r. we will use the convention that **1** has grade 0.

(v) For convenience, the symbol **1** will denote the multiplicative unit for any classical Clifford algebra.

The following property for any vector space will be useful in proving the main theorem and some additional results in this article.

Proposition 2.1. *Let V be a real vector space with basis $\{v_1, \ldots, v_n\}$. Then $\{k_1 v_1, \ldots, k_r v_r\}$ where $1 \leq r \leq n$ is a linearly independent set for any $k_1, \ldots, k_r \in \mathbb{R} \setminus \{0\}$.*

Proof. Let $c_1, \ldots, c_r \in \mathbb{R}$. Then

$$c_1(k_1 v_1) + \cdots + c_r(k_r v_r) = 0 \text{ implies that}$$
$$(c_1 k_1) v_1 + \cdots + (c_r k_r) v_r = 0.$$

Since $r \leq n$ and n is the dimension of V, then $c_1 k_1 = \cdots = c_r k_r = 0$, which implies $c_1 = \cdots = c_r = 0$, and $\{k_1 v_1, \ldots, k_r v_r\}$ is a linearly independent set. □

The following propositions pertaining to the commutative and anti-commutative behavior between monomials in a Clifford algebra $\mathbb{G}^n$ or $\mathbb{G}^{(0,n)}$ will be useful for studying Clifford graph algebras.

Proposition 2.2. *If two monomials $e_{j_1} \cdots e_{j_s}$ and $e_{i_1} \cdots e_{i_r}$ share no factor in common, where either r or s is even, they commute.*

Proof.

$$\begin{aligned}(e_{j_1} \cdots e_{j_s})(e_{i_1} \cdots e_{i_r}) &= (-1)^{rs}(e_{i_1} \cdots e_{i_r})(e_{j_1} \cdots e_{j_s}) \\ &= (e_{i_1} \cdots e_{i_r})(e_{j_1} \cdots e_{j_s})\end{aligned}$$

□

On the other hand, if two monomials of the same size agree in all factors except for the last factor they anti-commute.

Proposition 2.3. *Let $1 \leq i_1 < \ldots < i_r < j < k \leq n$. Then*

$$(e_{i_1} \cdots e_{i_r} e_j)(e_{i_1} \cdots e_{i_r} e_k) = -(e_{i_1} \cdots e_{i_r} e_k)(e_{i_1} \cdots e_{i_r} e_j) \tag{2}$$

Proof.

$$\begin{aligned}&(e_{i_1} \cdots e_{i_r} e_j)(e_{i_1} \cdots e_{i_r} e_k) \\ &\quad = (-1)^r (e_{i_1} \cdots e_{i_r})(e_{i_1} \cdots e_{i_r}) e_j e_k \\ &\quad = (-1)^{r+1} (e_{i_1} \cdots e_{i_r})(e_{i_1} \cdots e_{i_r}) e_k e_j \\ &\quad = (-1)^{r+1}(-1)^r (e_{i_1} \cdots e_{i_r} e_k)(e_{i_1} \cdots e_{i_r} e_j) \\ &\quad = -(e_{i_1} \cdots e_{i_r} e_k)(e_{i_1} \cdots e_{i_r} e_j)\end{aligned}$$

□

Note that in the case where $r = 1$ Proposition 2.3 also states that two monomials with exactly one factor in common anti-commute. This observation generalizes to the following result.

Proposition 2.4. *A bi-vector $e_i e_j$ and a monomial $e_{i_1} e_{i_2} \cdots e_{i_r}$ of grade r with exactly one factor in common anti-commute; i.e.*

$$(e_i e_j)(e_{i_1} e_{i_2} \cdots e_{i_r}) = -(e_{i_1} e_{i_2} \cdots e_{i_r})(e_i e_j). \tag{3}$$

Proof. In the case where $r = 2$ we have the following if i is the common index

$$(e_i e_j)(e_i e_{i_2}) = -e_i e_i e_j e_{i_2} = e_i e_i e_{i_2} e_j = -(e_i e_{i_2})(e_i e_j).$$

Likewise $(e_i e_j)(e_{i_1} e_i) = -(e_{i_1} e_i)(e_i e_j)$. Similarly, the following hold if j is the common index.

$$\begin{aligned}(e_i e_j)(e_{i_1} e_j) &= -(e_{i_1} e_j)(e_i e_j) \\ (e_i e_j)(e_j e_{i_2}) &= -(e_j e_{i_2})(e_i e_j)\end{aligned}$$

We now proceed by induction on r. Assume that (3) holds for $2 \leq r < n$, and choose i_{r+1} so that $i_2 \leq i_r < i_{r+1} \leq n$. First suppose that i is the index in common. If $i \in \{i_1, i_2, \ldots, i_r\}$ then by the inductive assumption (4) implies (5)

$$\begin{aligned}&(e_i e_j)(e_{i_1} e_{i_2} \cdots e_{i_r} e_{i_{r+1}}) \\ &\quad = (e_i e_j)(e_{i_1} e_{i_2} \cdots e_{i_r}) e_{i_{r+1}} &(4) \\ &\quad = -(e_{i_1} e_{i_2} \cdots e_{i_r})(e_i e_j) e_{i_{r+1}} &(5) \\ &\quad = (-1)^3 (e_{i_1} e_{i_2} \cdots e_{i_r} e_{i_{r+1}})(e_i e_j). \\ &\quad = -(e_{i_1} e_{i_2} \cdots e_{i_r} e_{i_{r+1}})(e_i e_j).\end{aligned}$$

If $i = i_{r+1}$, interchange i_r and i_{r+1} to obtain the following (where the circumflex indicates that factor is omitted). Note that (6) implies (7) by the inductive assumption.

$$\begin{aligned}
&(e_ie_j)(e_{i_1}e_{i_2}\cdots e_{i_r}e_{i_{r+1}}) \\
&\quad= -(e_ie_j)(e_{i_1}e_{i_2}\cdots \widehat{e_{i_r}}e_{i_{r+1}})e_{i_r} && (6)\\
&\quad= (e_{i_1}e_{i_2}\cdots \widehat{e_{i_r}}e_{i_{r+1}})(e_ie_j)e_{i_r} && (7)\\
&\quad= (-1)^3(e_{i_1}e_{i_2}\cdots e_{i_r}e_{i_{r+1}})(e_ie_j). \\
&\quad= -(e_{i_1}e_{i_2}\cdots e_{i_r}e_{i_{r+1}})(e_ie_j).
\end{aligned}$$

By the principle of mathematical induction, (3) is true for the case where i is the shared index. If j is the common index, we can use the preceding part of this proof to show that (8) implies (9) as follows.

$$\begin{aligned}
&(e_ie_j)(e_{i_1}\cdots e_{i_r}) \\
&\quad= -(e_je_i)(e_{i_1}\cdots e_{i_r}) && (8)\\
&\quad= (-1)^2(e_{i_1}\cdots e_{i_r})(e_je_i) && (9)\\
&\quad= (-1)^3(e_{i_1}\cdots e_{i_r})(e_ie_j) \\
&\quad= -(e_{i_1}\cdots e_{i_r})(e_ie_j)
\end{aligned}$$

□

This result naturally generalizes to the following.

Proposition 2.5. *A monomial of even grade $e_{j_1}e_{j_2}\cdots e_{j_{2m-1}}e_{j_{2m}}$ and a monomial $e_{i_1}e_{i_2}\cdots e_{i_r}$ of grade r with exactly one factor in common anti-commute; i.e.*

$$\begin{aligned}
&(e_{j_1}\cdots e_{j_{2m-1}}e_{j_{2m}})(e_{i_1}\cdots e_{i_r}) \\
&\quad= -(e_{i_1}e_{i_2}\cdots e_{i_r})(e_{j_1}e_{j_2}\cdots e_{j_{2m-1}}e_{j_{2m}}). && (10)
\end{aligned}$$

Proof. Without loss of generality, assume that the common factor is the one labeled as e_{j_1}. Since each of the bi-vectors $e_{j_3}e_{j_4},\ldots,(e_{j_{2m-1}}e_{j_{2m}})$ share no common factor with $e_{i_1}e_{i_2}\cdots e_{i_r}$ we may apply Proposition to each of these in turn, and obtain the first equation in (11). The second equation follows by applying Proposition to the first bi-vector and $e_{i_1}e_{i_2}\cdots e_{i_r}$.

$$\begin{aligned}
&(\,(e_{j_1}e_{j_2})(e_{j_3}e_{j_4})\cdots(e_{j_{2m-1}}e_{j_{2m}})\,)(e_{i_1}e_{i_2}\cdots e_{i_r}) \\
&\quad= (e_{j_1}e_{j_2})(e_{i_1}e_{i_2}\cdots e_{i_r})(\,(e_{j_3}e_{j_4})\cdots(e_{j_{2m-1}}e_{j_{2m}})\,) && (11)\\
&\quad= -(e_{i_1}e_{i_2}\cdots e_{i_r})(\,(e_{j_1}e_{j_2})(e_{j_3}e_{j_4})\cdots(e_{j_{2m-1}}e_{j_{2m}})\,).
\end{aligned}$$

□

Remark 2.1. Note that Proposition 2.5 need not be true if neither of monomials in this proposition are of even grade. For instance,

$$\begin{aligned}
(e_1e_2e_3)(e_1e_4e_5) &= (e_1e_2)(e_1e_4)(e_3e_5) \\
&= -(e_1e_4)(e_1e_2)(e_3e_5) \\
&= (e_1e_4e_5)(e_1e_2e_3).
\end{aligned}$$

The fundamental property (ii) of a Clifford graph algebra in Definition 1.3 extends by induction to the following.

Proposition 2.6. Let $n, w \in \mathbb{N}$ such that $2w \le n$.

If $\mathbb{G}^{(0,n)} = \langle e_1, \ldots, e_n\rangle$ then

$$(e_{i_1}e_{i_2}\cdots e_{i_{2w-1}})^2 = \begin{cases} 1 & \text{if } w \text{ is even} \\ -1 & \text{if } w \text{ is odd}\end{cases} \qquad (12)$$

If $\mathbb{G}^n = \langle e_1, \ldots, e_n\rangle$ then

$$(e_{i_1}e_{i_2}\cdots e_{i_{2w-1}})^2 = \begin{cases} -1 & \text{if } w \text{ is even} \\ 1 & \text{if } w \text{ is odd}\end{cases} \qquad (13)$$

If $\mathbb{G}^{(0,n)} = \langle e_1, \ldots, e_n\rangle$ or $\mathbb{G}^n = \langle e_1, \ldots, e_n\rangle$ then

$$(e_{i_1}e_{i_2}\cdots e_{i_{2w}})^2 = \begin{cases} 1 & \text{if } w \text{ is even} \\ -1 & \text{if } w \text{ is odd}\end{cases} \qquad (14)$$

Proof. We will assume that $\mathbb{G}^{(0,n)} = \langle e_1, \ldots, e_n\rangle$ and prove (12) and (14). The proofs of (13) and (14) in the case where $\mathbb{G}^n = \langle e_1, \ldots, e_n\rangle$ are similar. Note that (14) is automatically true if (12) is, since

$$\begin{aligned}
&(e_{i_1}e_{i_2}\cdots e_{i_{2w-1}}e_{i_{2w}})^2 \\
&\quad= (e_{i_1}e_{i_2}\cdots e_{i_{2w-1}})e_{i_{2w}}(e_{i_1}e_{i_2}\cdots e_{i_{2w-1}})e_{i_{2w}} \\
&\quad= (-1)^{2w-1}(e_{i_1}e_{i_2}\cdots e_{i_{2w-1}})(e_{i_1}e_{i_2}\cdots e_{i_{2w-1}})e_{i_{2w}}e_{i_{2w}} \\
&\quad= (-1)^{2w}(e_{i_1}e_{i_2}\cdots e_{i_{2w-1}})^2 && (15)\\
&\quad= (e_{i_1}e_{i_2}\cdots e_{i_{2w-1}})^2 \\
&\quad= \begin{cases} 1 & \text{if } w \text{ is even} \\ -1 & \text{if } w \text{ is odd}\end{cases}
\end{aligned}$$

Thus it suffices to prove (12). Observe that this result is true if $w = 1$, establishing the case $2\cdot 1 - 1 = 1$ since then $(e_{i_1})^2 = -1$. By the equations in (15) it automatically follows for the case where $w = 1$ that

$$(e_{i_1}e_{i_2})^{(2\cdot 1)} = (e_{i_1}e_{i_2})^2 = -1$$

If $w = 2$ the case $2 \cdot 2 - 1 = 3$ holds in turn since then

$$(e_{i_1} e_{i_2} e_{i_3})^2 = (-1)^2 (e_{i_1} e_{i_2}) e_{i_3} e_{i_3} (e_{i_1} e_{i_2}) = (-1)^3 (e_{i_1} e_{i_2})^2 = (-1)^4 = 1$$

Assume now that (12) holds for any $w \in \mathbb{N}$. Equations (15) immediately imply that (14) holds for w. In a manner similar to (15) we obtain

$$\begin{aligned}
&(e_{i_1} e_{i_2} \cdots e_{i_{2w}} e_{i_{2(w+1)-1}})^2 \\
&= (e_{i_1} e_{i_2} \cdots e_{i_{2w}}) e_{i_{2w+1}} (e_{i_1} e_{i_2} \cdots e_{i_{2w}}) e_{i_{2w+1}} \\
&= (-1)^{2w} (e_{i_1} e_{i_2} \cdots e_{i_{2w}})^2 e_{i_{2w+1}} e_{i_{2w+1}} \\
&= (-1)^{2w+1} (e_{i_1} e_{i_2} \cdots e_{i_{2w}})^2 \\
&= -(e_{i_1} e_{i_2} \cdots e_{i_{2w}})^2 \\
&= -1 \cdot \begin{cases} 1 & \text{if } w \text{ is even} \\ -1 & \text{if } w \text{ is odd} \end{cases} \\
&= \begin{cases} -1 & \text{if } w \text{ is even} \\ 1 & \text{if } w \text{ is odd} \end{cases} \\
&= \begin{cases} -1 & \text{if } w+1 \text{ is odd} \\ 1 & \text{if } w+1 \text{ is even} \end{cases}
\end{aligned}$$

Thus, (12) and, by (15), (14) hold for $w + 1$, so this proposition is true by the principle of mathematical induction. □

3. Main results

The first objective of this article is to use inclusion to prove that a Clifford graph algebra exists for any given graph G_n.

3.1. An Existence Theorem for Clifford Graph Algebras.

Theorem 3.1. *Let G_n be a simple graph with n vertices $v_1, \ldots, v_n$. Then*

(i) *There is a Clifford graph algebra $GA(G_n)$ for G_n, whose generators $\{e'_1, \ldots, e'_n\}$ satisfy the two fundamental properties in Definition 1.3, which is a sub-algebra of either $\mathbb{G}^{2^n}$ or $\mathbb{G}^{(0,2^n)}$.*

(ii) *The following set of 2^n monomials form a basis for $GA(G_n)$* (i.e. the dimension of $GA(G_n)$ is 2^n).

$$\{1\} \cup \{e'_{i_1} \cdots e'_{i_r} \mid 1 \le i_1 \le \ldots \le i_r \le n, \text{ for } r = 1, \ldots, n\} \tag{16}$$

Proof. We will prove (i) by induction on n for $GA(G_n)$, starting with $n = 2$. Let G_2 be a graph with two vertices v_1, v_2. Let e_1, e_2, e_3, and e_4 be generators for either $\mathbb{G}^{2^2}$ or $\mathbb{G}^{(0,2^2)}$. If there is no edge between v_1 and v_2 assign $e(1,1) = e_1 e_3$ to v_1 and assign $e(2,1) = e_2 e_4$ to v_2. Observe that :

$$e(1,1)e(2,1) = e(2,1)e(1,1) \tag{17}$$

The even grade monomials for $e(1,1)$ and $e(2,1)$ have no factor in common, and Proposition 2 insures that condition (17) holds.

If there is an edge between v_1 and v_2, assign $e(1,1) = e_1 e_3$ and $e(2,1) = e_2 e_3$. This time we have

$$e(1,1)e(2,1) = -e(2,1)e(1,1), \text{ there is an edge between } v_1 \text{ and } v_2 \tag{18}$$

where in this case Proposition 2.3 respectively insures that (18) holds.

Since in either case the set $B_2 = \{1, e(1,1), e(2,1), e(2,1)e(2,2)\}$ is a subset of the basis for $\mathbb{G}^{2^2}$ or $\mathbb{G}^{(0,2^2)}$, then the monomials in B_2 are linearly independent, and we may define the Clifford graph algebra for G_2 as

$$GA(G_2) = \text{span}(B_2)$$

which is a sub-algebra of $\mathbb{G}^{2^2}$ or $\mathbb{G}^{(0,2^2)}$. In either of these cases we have shown the following.

There is a Clifford graph algebra $GA(G_2)$ for any graph G_2, with generators $e(1,1), e(2,1)$ and basis $\{1, e(1,1), e(2,1), e(2,1)e(2,2)\}$.
$GA(G_2)$ is a sub-algebra of $\mathbb{G}^{(0,2^2)}$.
Each of the defining monomials for $e(1,1)$ and $e(2,1)$ are of even grade.
If two vertices share an edge, their monomials have exactly one factor in common. If two vertices share no edge, they have no factor in common.

Now choose any $n \ge 2$, and let G_n be any graph with n vertices $v_1, \ldots, v_n$, and assume that the following are true for $GA(G_n)$, which is depicted in figure 3.

(iii) There is a Clifford graph algebra $GA(G_n)$ for G_n, with generators $e(1,n), \ldots, e(n,n)$, wherein each $e(k,n)$ is paired with v_k for each k, and B_n is a basis for $GA(G_n)$ where

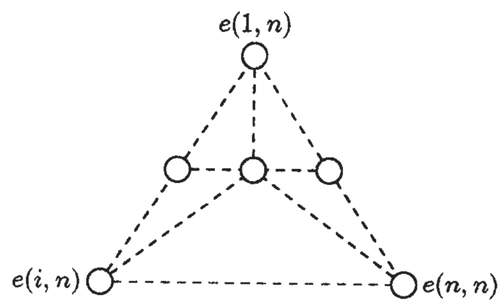
$$B_n = \{1\} \cup \{e(i_1, n), \ldots, e(i_r, n) \mid 1 \le i_1 \le \ldots \le i_r \le n, \ r = 1, \ldots, n\}.$$

(iv) $GA(G_n)$ is a sub-algebra of $\mathbb{G}^{2^n}$ or $\mathbb{G}^{(0,2^n)}$.

(v) Each of the defining monomials for $e(1,n), \ldots, e(n,n)$ are of even grade.

(vi) If two vertices share an edge, their monomials have exactly one factor in common. If two vertices share no edge, they have no factor in common.

Figure 3
$GA(G_n)$ with generators labeled



Let r be the largest index among all of the indices for the factors e_k that occur in all the monomials associated with the generators for the vertices $v_1, ..., v_n$; so that $r \leq 2^n$. Adjoin one vertex, v_{n+1}, to G_n, such that there are no or m edges between v_{n+1} and the vertices in G_n, with $1 \leq m \leq n$; thereby establishing an arbitrary graph G_{n+1} with $n+1$ vertices. Since the basis for $\mathbb{G}^{2^n}$ or $\mathbb{G}^{(0,2^n)}$ is a subset of the basis for $\mathbb{G}^{2^{n+1}}$ or $\mathbb{G}^{(0,2^{n+1})}$, we may assume that each of the monomials in B_n are in the basis for $\mathbb{G}^{2^{n+1}}$ or $\mathbb{G}^{(0,2^{n+1})}$.

In the following discussion, an index as large as $i = r + 2m + 1$ can be assigned to a factor e_i in a monomial in the case where n is odd, which in this case implies $n \geq 3$. Note that this is within the allowable bounds for indices of the generators for $\mathbb{G}^{2^{n+1}}$ or $\mathbb{G}^{(0,2^{n+1})}$ since 2^{n+1} is the largest index allowed. Using the inequality $2n + 1 \leq 2^n$ for $n \geq 3$, we have that

$$r + (2m + 1) \leq 2^n + (2n + 1) \leq 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}.$$

If there is no edge between v_{n+1} and any of $v_1, ..., v_n$, assign to v_{n+1} the bi-vector $e(n+1, n+1) = e_{r+1}e_{r+2}$, and in this case set $e(k, n+1) = e(k, n)$ for $k = 1, ..., n$. Since this bi-vector shares no factor in common with any other monomial for $e(1, n+1)$ through $e(n, n+1)$, Proposition 2 insures that $e(n+1, n+1)$ will commute with each of $e(1, n+1), ..., e(n, n+1)$.

Now suppose there is one edge between v_{n+1} and each of the vertices denoted $v_{i_1}, ..., v_{i_m}$. As shown in figure 4 below, assign to v_{n+1} the monomial defined by

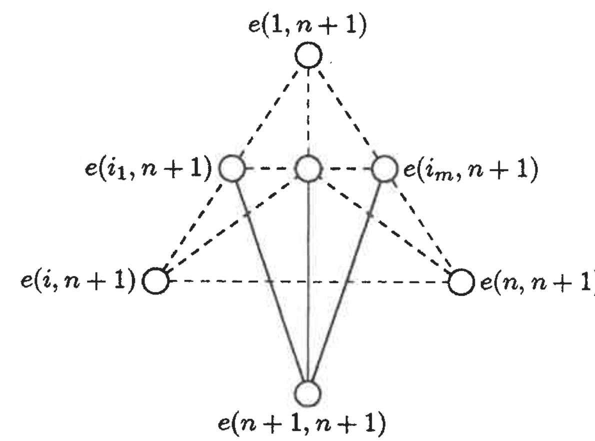
$$e(n+1, n+1) = \begin{cases} e_{r+2}e_{r+4} \cdots e_{r+2m} & \text{if } m \text{ is even} \\ e_{r+2}e_{r+4} \cdots e_{r+2m}e_{r+(2m+1)} & \text{if } m \text{ is odd} \end{cases}$$

and note that in either case the monomial for $e(n+1, n+1)$ has even grade. Also, set

$$e(i_1, n+1) = e(i_1, n)e_{r+1}e_{r+2}$$
$$e(i_2, n+1) = e(i_2, n)e_{r+3}e_{r+4}$$
$$\vdots$$
$$e(i_m, n+1) = e(i_m, n)e_{r+(2m-1)}e_{r+2m}$$

$$e(j, n+1) = e(j, n) \quad \text{for } j \neq i_1, ..., i_m, n+1$$

Figure 4
$GA(G_{n+1})$ as $GA(G_n)$ with v_{n+1} adjoined to G_n



Since the monomial for $e(n+1, n+1)$ is of even grade and has exactly one factor, specifically e_{r+2k} in common with the monomial for $e(i_k, n+1)$ for each $k = 1, ..., m$, then each of these monomials anti-commute with $e(n+1, n+1)$ by Proposition 2.5. Since the monomials for the remaining vertices in G_n have no factor in common with the even grade monomial $e(n+1, n+1)$, then by Proposition 2, $e(n+1, n+1)$ commutes with each of these monomials for which their associated vertices share no common edge with v_{n+1}.

Note as well that the commutativity and/or anti-commutativity between each $e(i_k, n+1)$ for $k = 1, ..., m$ and any monomial $e(j, n+1)$ for which $j \neq i_1, ..., i_m, n+1$ remains unchanged by the multiplication of

$e_{r+(2k-1)}e_{r+2k}$, so that the connectivity between the vertices $v_1, \dots, v_n$ in G_{n+1} is as it was for G_n. Precisely, if v_{i_k} and $e(i_k, n)$ share an edge, then $e(i_k, n+1)$ and $e(j, n+1)$ anti-commute since

$$\begin{aligned} e(i_k, n+1)e(j, n+1) &= \left(e(i_k, n)e_{r+(2k-1)}e_{r+2k}\right)e(j, n) \\ &= e(i_k, n)e(j, n)\left(e_{r+(2k-1)}e_{r+2k}\right) \\ &= -e(j, n)e(i_k, n)\left(e_{r+(2k-1)}e_{r+2k}\right) \\ &= -e(j, n)\left(e(i_k, n)e_{r+(2k-1)}e_{r+2k}\right) \\ &= -e(j, n+1)e(i_k, n+1) \end{aligned}$$

Similarly, $e(i_k, n+1)$ and $e(j, n+1)$ commute if v_{i_k} and v_j share no edge in common.

We may now replace n with $n+1$ in each of the statements (iii) through (vi) since these properties all hold for $GA(G_{n+1})$, and the principle of mathematical induction therefore insures that they hold for any $n \in \mathbb{N}$. Finally, enumerate the generators selected for $GA(G_n)$ as $e'_1, \dots, e'_n$ so that property (ii) in Definition 1.3 holds.

To prove (ii), note that since each e'_k is a distinct monomial in the basis for $\mathbb{G}^{2^n}$ or $\mathbb{G}^{(0,2^n)}$, then each product $e'_{i_1} \cdots e'_{i_r}$ for $1 \le r \le n$ is a signed monomial in the basis for $\mathbb{G}^{2^n}$ or $\mathbb{G}^{(0,2^n)}$. Thus, the set in (16) consists of scalar multiples of elements from $\mathbb{G}^{2^n}$ or $\mathbb{G}^{(0,2^n)}$, and its elements are thus linearly independent by Proposition 2.1; therefore the collection (16) is a basis for

$$S = \operatorname{span}\{\mathbf{1} \ , \ e'_{i_1} \cdots e'_{i_r} \mid 1 \le i_1 \le \dots \le i_r \le n, \ \text{ for } r = 1, \dots, n\}.$$

We will conclude this proof by showing that $\{e'_1, e'_2, \dots, e'_n\}$ generates S. Let $\mathbb{A}$ be any sub-algebra of S that contains $\{e'_1, e'_2, \dots, e'_n\}$. Let $u \in S$ have basis representation

$$\begin{aligned} u = &a_0\mathbf{1} + a_1 e'_1 + \dots + a_n e'_n \\ &+ a_{12}e'_1e'_2 + \dots + a_{(n-1)n}e'_{n-1}e'_n + \dots + a_{(1\cdots n)}e'_1e'_2 \cdots e'_n. \end{aligned}$$

Since multiplication by elements of S and scalars is closed in $\mathbb{A}$ and $\{e'_1, e'_2, \dots, e'_n\} \subset \mathbb{A}$, then $u \in \mathbb{A}$. Therefore $\mathbb{A} = GA(G_n)$ and so the only sub-algebra of S that contains $\{e'_1, e'_2, \dots, e'_n\}$ is S itself; thus $\{e'_1, e'_2, \dots, e'_n\}$ generates S; and so by Definition 1.3 we have that

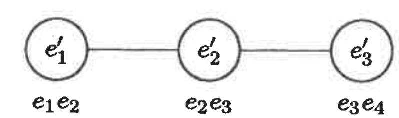
$$\begin{aligned} GA(G_n) &= \langle e'_1 \ , \ \dots \ , \ e'_n \rangle \\ &= \operatorname{span}\{\mathbf{1} \ , \ e'_{i_1} \cdots e'_{i_r} \mid 1 \le i_1 \le \dots \le i_r \le n, \ \text{ for } r = 1, \dots, n\}; \end{aligned}$$

and (16) is a basis for $GA(G_n)$. □

3.2. **Clifford Graph Algebras for Some Classical Graphs.**
In order to facilitate the previous general proof that a Clifford graph algebra exists for any graph G_n, it was helpful to select the n generators from $\mathbb{G}^{(0,2^n)}$, each of which was of even length. However, for some classical graphs, it is possible to choose the generators to be mostly bi-vectors from a much smaller classical Clifford algebra.

3.2.1. *Path Graphs.* The Clifford graph algebra $GA(P_n)$ for any path graph P_n with n vertices can be obtained entirely from bi-vectors in $\mathbb{G}^{n+1}$ or $\mathbb{G}^{(0,n+1)}$. As an example, consider $GA(P_3)$.

Figure 5
The path graph P_3



If we choose the generators for $GA(P_3)$ from either $\mathbb{G}^4$ or $\mathbb{G}^{0,4}$ as

$$e'_1 = e_1e_2 \ , \ e'_2 = e_2e_3 \ , \ \text{ and } \ e'_3 = e_3e_4, \tag{19}$$

then their commutative and anti-commutative behavior will correctly depict the connectivity in P_3, since then the generators at the adjacent pairs of vertices satisfy

$$e'_1e'_2 = -e'_2e'_1 \ \text{ and } \ e'_2e'_3 = -e'_3e'_2$$

and the generators at non-adjacent pairs of vertices satisfy

$$e'_1e'_3 = e'_3e'_1 \ .$$

As a sub-algebra of $\mathbb{G}^4$ or $\mathbb{G}^{0,4}$, the basis for $GA(P_3)$ consists of 2^3 monomials of all possible even grades in a basis for $\mathbb{G}^4$ or $\mathbb{G}^{0,4}$, which is obtained by multiplying each pair of generators for $GA(P_3)$ listed in (19). Specifically, if the parent algebra is $\mathbb{G}^4$ then

$$\begin{aligned} GA(P_3) &= \operatorname{span}\{\mathbf{1} \ , \ e'_1 \ , \ e'_1e'_2 \ , \ e'_1e'_2e'_3 \ , \ e'_2 \ , \ e'_2e'_3 \ , \ e'_3 \ , \ e'_1e'_3\} \\ &= \operatorname{span}\{\mathbf{1} \ , \ e_1e_2 \ , \ e_1e_3 \ , \ e_1e_4 \ , \ e_2e_3 \ , \ e_2e_4 \ , \ e_3e_4 \ , \ e_1e_2e_3e_4\} \\ &\subsetneq \mathbb{G}^4 \end{aligned}$$

and if the parent algebra is $\mathbb{G}^{0,4}$ then

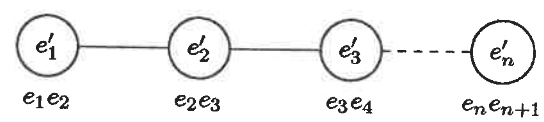
$$GA(P_3) = \text{span}\{1\,,\ e_1'\,,\ e_1'e_2'\,,\ e_1'e_2'e_3'\,,\ e_2'\,,\ e_2'e_3'\,,\ e_3'\,,\ e_1'e_3'\}$$
$$= \text{span}\{1\,,\ e_1e_2\,,\ -e_1e_3\,,\ e_1e_4\,,\ e_2e_3\,,\ -e_2e_4\,,\ e_3e_4\,,\ e_1e_2e_3e_4\}.$$
$$\subsetneq \mathbb{G}^{(0,4)}$$

The choice of generators for $GA(P_4)$ suggests that we may generally define the n generators for $GA(P_n)$ entirely as bi-vectors from either $\mathbb{G}^{n+1}$ or $\mathbb{G}^{(0,n+1)}$ as

$$e_i' = e_ie_{i+1} \quad \text{for} \quad i = 1, \ldots, n \tag{20}$$

The selection (20) provides the connectivity in P_n as depicted in the figure below as follows.

Figure 6
The path graph P_n



If the parent algebra is either $\mathbb{G}^{n+1}$ or $\mathbb{G}^{(0,n+1)}$ we obtain the following properties for $i = 1, \ldots, n-1$. First, note that the generators for adjacent vertices anti-commute since

$$e_i'e_{i+1}' = (e_ie_{i+1})(e_{i+1}e_{i+2}) = (-1)^3(e_{i+1}e_{i+2})(e_ie_{i+1}) = -e_{i+1}'e_i'\ .$$

Otherwise, if $j > i+1$, the vertices v_i and v_j share no common edge as shown in figure 6, and their generators should commute, which is the case since $e_i' = e_ie_{i+1}$ and $e_j' = e_je_{j+1}$ share no factor in common, hence $e_i'e_j' = e_j'e_i'$ by Proposition 2. Thus, the selection of generators agrees with the connectivity of P_n.

By equation (14) in Proposition 2.6 we have that $(e_i')^2 = (e_ie_{i+1})^2 = -1$ for all $i = 1, \ldots, n$, hence $GA(P_n)$ has signature $(0, n)$ or $p = 0, q = n$ by this construction. Therefore, the selection of generators in 20 defines $GA(P_n)$ as in Definition 1.3.

Suppose that the parent algebra for $GA(P_n)$ is $\mathbb{G}^{(0,n+1)}$. Then in this case each product $e_i'e_{i+1}' = (e_ie_{i+1})(e_{i+1}e_{i+2}) = -e_ie_{i+2}$ is another bi-vector, and if $j > i+1$ then $e_i'e_j' = (e_ie_{i+1})(e_je_{j+1})$ is a monomial of even grade 4 in the basis for $\mathbb{G}^{(0,n)}$. Thus, any product $e_{i_1}' \cdots e_{i_r}'$ in the basis for $GA(G_n)$ is a signed multiple of a monomial of even grade in the basis for $\mathbb{G}^{(0,n+1)}$. Furthermore, exactly half, that is 2^n of the 2^{n+1} monomials in the basis for $\mathbb{G}^{(0,n+1)}$are of even grade. Since $GA(P_n)$ has dimension 2^n,

then as a sub-algebra of $\mathbb{G}^{(0,n+1)}$ its basis consists of signed monomials of all possible even grades from the basis for $\mathbb{G}^{0,n+1}$, specifically

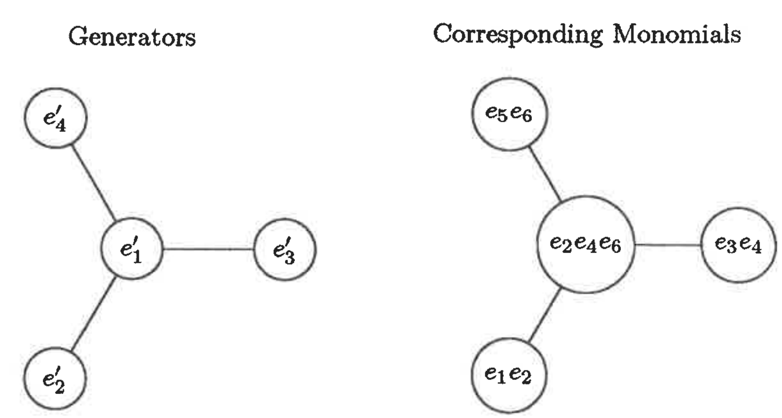
$$GA(P_n) = \text{span}\left\{ 1\,,\ \prod_{r=1}^{m} \omega_r \cdot \prod_{r=1}^{m} e_{i_r}e_{i_{r+1}} \,\middle|\, \omega_r = (-1)^{i_{r+1}-i_r+1}, m = 1, \ldots, n \right\}.$$

If $\mathbb{G}^{n+1}$ is the parent algebra we obtain the simpler representation for $GA(P_n)$:

$$GA(P_n) = \text{span}\left\{ 1\,,\ \prod_{r=1}^{m} e_{i_r}e_{i_{r+1}} \,\middle|\, m = 1, \ldots, n \right\}.$$

3.2.2. *Star Graphs.* We will construct the Clifford graph algebra $GA(K_{1,n})$ for any complete star graph $K_{1,n}$ in a manner similar to the construction of $GA(P_n)$ by choosing as many bivectors as possible for its generators. The only vertex of $K_{1,n}$ which will not have a corresponding bivector will be the internal node; we will associate a bivector with all of the exterior vertices. For motivating examples we will consider $GA(K_{1,3})$ and $GA(K_{1,4})$.

Figure 7
Schematic depiction of $GA(K_{1,3})$



As figure 7 illustrates, we will choose the generators for $GA(K_{1,3})$ from either $\mathbb{G}^6$ or $\mathbb{G}^{0,6}$ as follows.

We shall define the central node as

$$e'_1 = e_2e_4e_6 \tag{21}$$

and the external nodes as

$$e'_2 = e_1e_2\ ,\ e'_3 = e_3e_4\ ,\ \text{and}\ \ e'_4 = e_5e_6. \tag{22}$$

As was the case for $GA(P_n)$, the commutative and anti-commutative behavior of the generators in (21) and (22) will correctly depict the connectivity in $K_{1,3}$, because each of the bivectors at the external vertices anti-commute with the trivector at the adjacent internal node by Proposition 2 since they share exactly one factor in common:

$$e'_2e'_1 = -e'_1e'_2\ ,\ e'_3e'_1 = -e'_1e'_3\ ,\ \text{and}\ \ e'_4e'_1 = -e'_1e'_4\ .$$

Likewise, the generators at the external nodes all commute by Proposition 2 since their defining monomials are of even grade and have no factor in common :

$$e'_2e'_3 = e'_3e'_2\ ,\ e'_2e'_4 = e'_4e'_2\ ,\ e'_3e'_4 = e'_4e'_3\ .$$

As a sub-algebra of $\mathbb{G}^6$ or $\mathbb{G}^{(0,6)}$ $GA(K_{1,3})$ consists of **1**, bi-vectors, tri-vectors, vectors of grade 4, and 1 vector of grade 6.

Figure 8

Schematic depiction of $GA(K_{1,4})$

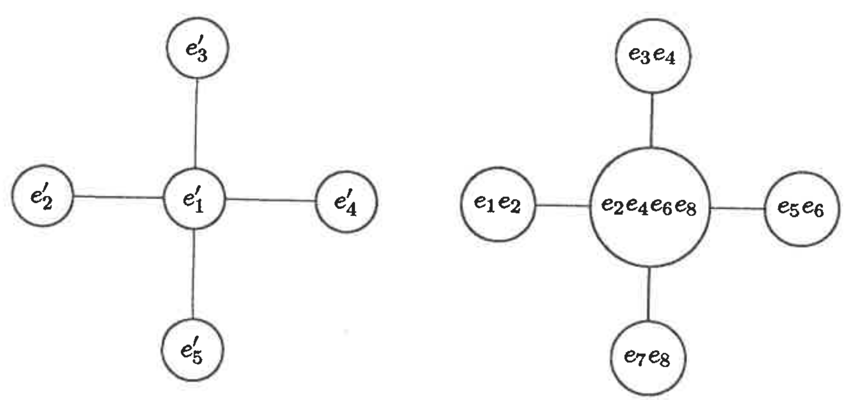


Figure 8 likewise shows that we can choose the generators for $GA(K_{1,4})$ from either $\mathbb{G}^8$ or $\mathbb{G}^{0,8}$ by defining the central node as

$$e'_1 = e_2e_4e_6e_8 \tag{23}$$

and the external nodes as

$$e'_2 = e_1e_2\ ,\ e'_3 = e_3e_4\ ,\ e'_4 = e_5e_6\ \text{and}\ \ e'_5 = e_7e_8. \tag{24}$$

Again, the commutative and anti-commutative behavior of the generators in (23) and (24) likewise correctly depicts the connectivity in $K_{1,4}$. As a sub-algebra of $\mathbb{G}^8$ or $\mathbb{G}^{(0,8)}$ $GA(K_{1,4})$ consists of **1**, bi-vectors, vectors of grade 4, vectors of grade 6, and 1 vector of grade 8.

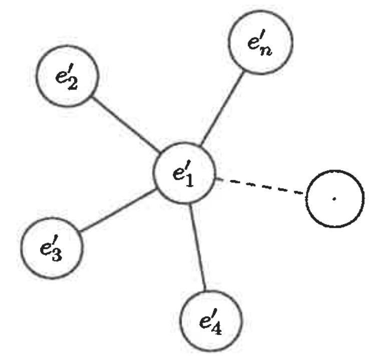
These examples suffice for generalizing to the construction of $GA(K_{1,n})$. We will select the generators for $GA(K_{1,n})$ from either $\mathbb{G}^{2n}$ or $\mathbb{G}^{(0,2n)}$, and define them as

$$\begin{aligned} &e'_1 = e_2e_4\cdots e_{2n} \quad \text{(at the central vertex)} \\ &e'_k = e_{2k-3}e_{2k-2} \quad \text{for } k = 2, ..., n+1 \quad \text{(at each external vertex)} \end{aligned} \tag{25}$$

Figure 9 below shows the connectivity of $K_{1,n}$.

Figure 9

The star graph $K_{1,n}$



Lets consider the commutativity and anti-commutativity of these generators. We will start with the generators for each pair of external vertices. If we choose $n+1 \geq j > i \geq 2$, then $e'_i = e_{2i-3}e_{2i-2}$ and $e'_j = e_{2j-3}e_{2j-2}$. Since $j \geq i+1$ we have that $2j-3 \geq 2i-1$; hence we may order the subscripts of the factors in the defining monomials for e'_i and e'_j as

$$2i-3 \ < \ 2i-2 \ < \ 2i-1 \ \leq \ 2j-3 \ < \ 2j-2\ ,$$

so the subscripts for $e_{2i-3}e_{2i-2}$ are distinct from those in $e_{2j-3}e_{2j-2}$. Since the bi-vectors for e'_i and e'_j share no factor in common, Proposition 2 implies that these bi-vectors commute. Therefore

$$e_ie'_j = (e_{2i-3}e_{2i-2})(e_{2j-3}e_{2j-2}) = (e_{2j-3}e_{2j-2})(e_{2i-3}e_{2i-2}) = e'_je'_i\ ,$$

and so the generators for each pair of external vertices commute.

Now lets consider the generator at any external vertex $e'_k = e_{2k-3}e_{2k-2}$ for

$2 \leq k \leq n+1$ and the generator for the internal vertex e_1'. Since

$$e_1' = e_2 \cdots e_{2k-2} e_{2k} e_{2k+2} \cdots e_{2n}$$

then the bi-vector for e_k' and the monomial for e_1' have the single factor e_{2k-2} in common, and Proposition 2 insures that this bi-vector and monomial anti-commute. Therefore

$$\begin{aligned} e_k' e_1' &= (e_{2k-3} e_{2k-2})(e_2 \cdots e_{2k-2} e_{2k} e_{2k+2} \cdots e_{2n}) \\ &= -(e_2 \cdots e_{2k-2} e_{2k} e_{2k+2} \cdots e_{2n})(e_{2k-3} e_{2k-2}) = -e_k' e_1' \end{aligned}$$

Thus, the selection of generators (25) agrees with the connectivity of $K_{1,n}$.

The signature of $GA(K_{1,n})$ depends on the parity of n. First, note that

$$(e_k')^2 = (e_{2k-3} e_{2k-2})^2 = -1 \quad \text{for } k = 2, ..., n+1$$

if either of $\mathbb{G}^{2n}$ or $\mathbb{G}^{(0,2n)}$ is the parent algebra. If n is even, then $n = 2m$ for some $m \in \mathbb{N}$. When m is even we may use either $\mathbb{G}^{2n}$ or $\mathbb{G}^{(0,2n)}$ for the parent algebra because in this case Proposition 2.6 insures that $(e_1')^2 = (e_2 e_4 \cdots e_{2n})^2 = 1$, and the signature for $GA(K_{1,n})$ is $(1, n)$ or $p = 1, q = n$ in this case. If m is odd, then Proposition 2.6 implies $(e_1')^2 = (e_2 e_4 \cdots e_{2n})^2 = -1$, and the signature of $GA(K_{1,n})$ is $(0, n+1)$ or $p = 0, q = n+1$.

If n is odd, then $n = 2m - 1$ for some $m \in \mathbb{N}$. If $\mathbb{G}^{(0,2n)}$ is the parent algebra and m is even then $(e_1')^2 = (e_2 e_4 \cdots e_{2n})^2 = 1$ and $GA(K_{1,n})$ has a signature of $(1, n)$. If m is odd the relation $(e_1')^2 = (e_2 e_4 \cdots e_{2n})^2 = -1$ means that the signature of $GA(K_{1,n})$ is $(0, n+1)$.

Finally, if $\mathbb{G}^{2n}$ is the parent algebra and m is even then $(e_1')^2 = (e_2 e_4 \cdots e_{2n})^2 = -1$ and $GA(K_{1,n})$ has a signature of $(0, n+1)$, whereas an odd m results in $(e_1')^2 = (e_2 e_4 \cdots e_{2n})^2 = 1$ and $GA(K_{1,n})$ has a signature of $(1, n)$. Thus, the selection of generators in (25) defines $GA(K_{1,n})$ as in Definition 1.3.

By generalizing $GA(K_{1,3})$ and $GA(K_{1,4})$ to the possibilities for $GA(K_{1,n})$ as a sub-algebra of $\mathbb{G}^{2n}$ or $\mathbb{G}^{(0,2n)}$ we obtain the following. If n is even, then vectors of grade $0, 2, ..., 2n$ span $GA(K_{1,n})$, whereas vectors of grades $0, 2, ..., n-1, n, n+1, ..., 2n$ span $GA(K_{1,n})$ if n is odd.

4. Conclusion : areas for future research

The characterization of $GA(P_n)$ and $GA(K_{1,n})$ might be helpful to studying the Clifford graph algebras for trees, which are composed of path and star graphs. In so far as the commutativity and anti-commutativity properties for generators are conceived of as decisions, the notion of Clifford

graph algebra may be facilitative in constructing a decision tree, which is useful as a predictive tool in data mining and machine learning.

References

[1] A. Macdonald, *An Elementary Construction of the Geometric Algebra*, Advances in Applied Clifford Algebras, **12** No.1, 2002.

[2] ———, *Linear and Geometric Algebra*, Createspace Independent Publishing Platform, 2011.

[3] T. Khovanova, *Clifford Algebras and Graphs*, retrieved from arXiv:0810.3322v1[math.CO], Oct. 2008.

[4] W. Clifford, *Applications of Grassman's Extensive Algebra*, American Journal of Mathematics, 1 No.4, 1878.

[5] H. Weil, *Quantenmachanik und Gruppentheorie*, Zeitschrift für Physik, **46**, 1927.

[6] J. Sylvester, *The Collected Mathematics Papers of James Joseph Sylvester*, Cambridge University Press, **5** No.3, 1909.

[7] E. Cartan, *Les groupes bilinéaires et les Systèmes de nombers complexes*, Annes de la faculté des sciences de Toulouse, **12** No.1, 1898.

[8] J. Schwinger, *Unity Operator Bases, Proc. Natl Acad Sci USA* Apr **46** No. 41, 1960.

[9] R. Jagannathan, *On Generalized Clifford Algebras and their Physical Applications*, retrieved from arXiv:1005.4300v1[math.ph], May 2010.

[10] A. Vourdas, *Quantum Systems with Finite Hilbert Space*, Reports on Progress in Physics **67** No. 267., 2004.

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