Minimally Non-Asymmetric Graphs and Balanced Incomplete Block Designs

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Abstract

A graph is asymmetric if the automorphism group of its set of vertices is trivial. A graph is called non-asymmetric if and only if it is not asymmetric. A graph G is called minimally non-asymmetric if G is non-asymmetric but $G - e$ is asymmetric for any e contained in G.

Given a finite set V (of elements called varieties) and integers k, r, and λ we define a balanced incomplete block design (BIBD) to be a family of k-element subsets of V, called blocks, such that any element contained in r blocks, and any pair of distinct varities u and w is contained in exactly λ blocks.

In this paper, we give examples of minimally non-asymmetric graphs constructed from balanced incomplete block designs.

1 Introduction

Recall that two simple graphs G and H are isomorphic if and only if there exists a bijection $f : V(G) \to V(H)$ such that uv is an edge of G if and only if $f(u)f(v)$ is an edge of H. In this case f is called an isomorphism of G onto H. An automorphism of a graph G is an isomorphism of G onto itself. A graph is asymmetric if the automorphism group of its set of vertices is trivial. Asymmetric graphs were introduced by Erdős and Rényi [2] in 1963. A graph is called non-asymmetric if and only if it is not asymmetric. A graph G is called *minimally non-asymmetric* if G is non-asymmetric but $G - e$ is asymmetric for any e contained in G. Given a finite set V (of elements called varieties) and integers k, r, and λ a balanced incomplete block design (BIBD) is a family of k-element subsets of V, called blocks, such that any element contained in r blocks, and any pair of distinct varities u and w is contained in exactly λ blocks. Block designs have been well studied, dating back to the work of Fisher in 1940 [3]. We use (v, b, r, k, λ)-BIBD to denote a block design with v varieties, b blocks, and r, k, and λ described above.

2 Block incidence graphs

A block design can be represented by a Levi graph, or incidence graph, which were introduced by Levi [4]. The Levi graph of a (v, b, r, k, λ)-BIBD is a bipartite graph G with parts X and Y with b and v vertices respectively, one for each block and one for each of the varieties. In the Levi graph, block b_i containing varieties v_i and v_j will be represented by vertex b_i in X that is adjacent to vertices v_i and v_j in Y.

We begin with an example of a $(9, 18, 4, 2, 1)$-block design, where $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The blocks are $b_1 = \{1, 8\}$; $b_2 = \{1, 9\}$; $b_3 = \{2, 9\}$; $b_4 = \{1, 2\}$; $b_5 = \{1, 3\}$; $b_6 = \{2, 3\}$; $b_7 = \{2, 4\}$; $b_8 = \{3, 4\}$; $b_9 = \{3, 5\}$; $b_{10} = \{4, 5\}$; $b_{11} = \{4, 6\}$; $b_{12} = \{5, 6\}$; $b_{13} = \{5, 7\}$; $b_{14} = \{6, 7\}$; $b_{15} = \{6, 8\}$; $b_{16} = \{7, 8\}$; $b_{17} = \{7, 8\}$; $b_{18} = \{8, 9\}$. We note that there are: 9 varieties; 18 blocks; each variety appears in exactly 4 blocks; at each block contains exactly two varieties; and each pair of varieties appears in exactly one block.

Next we present the Levi graph L_G for this example. Let the vertices of Y be $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and the vertices of X be $\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27\}$. Here vertex $x_i \in X$ is adjacent to $y_j \in Y$ if and only if y_j is in block b_i.

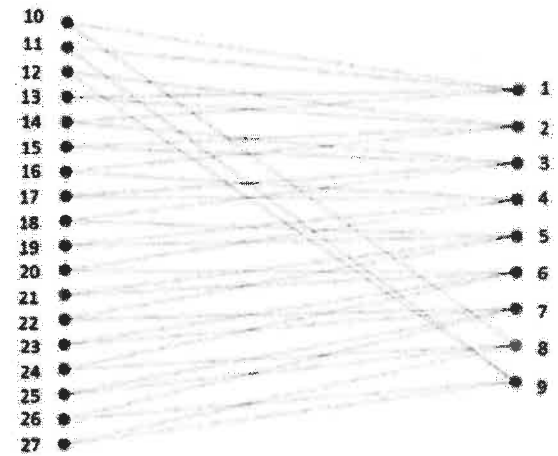

Figure 1. The graph L_G.

We see that the vertex automorphism group of L_G has three orbits: $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $\{10, 12, 14, 16, 18, 20, 22, 24, 26\}$, and $\{11, 13, 15, 17, 19, 21, 23, 25, 27\}$.

In Figure 2, we present a different drawing of the graph showing which vertices are in the different orbits, with the vertices in the different orbits being colored with white, black, or gray.

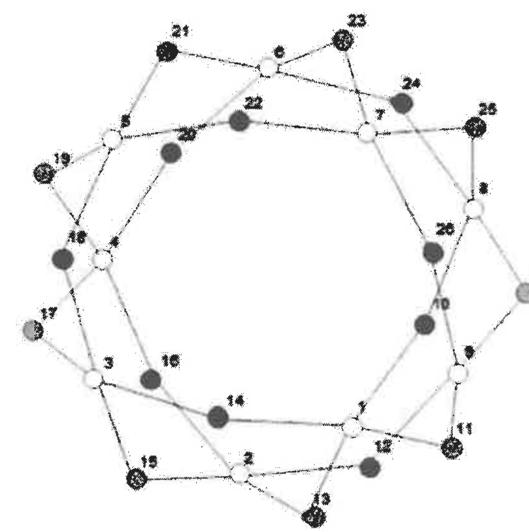

Figure 2. Another drawing of L_G

Theorem 1 *The graph L_G is minimally non-asymmetric.*

Proof. Since the graph L_G contains three orbits $\{1,2,3,4,5,6,7,8,9\}$, $\{10,12,14,16,18,20,22,24,26\}$, and $\{11,13,15,17,19,21,23,25,27\}$ it has at least one non-trivial automorphism. Next we will show that the removal of any edge will result in a graph that is asymmetric. It will be helpful to refer to Figure 2. There are two types of edges that can be removed: $\{w_i, b_j\} = \{i, 10 + 2i \pm 1\}$ (an edge between a white vertex and a black vertex), and $\{w_i, g_j\} = \{i, 10 + 2(i \pm 1)\}$ (an edge between a white vertex and a gray vertex). We note by symmetry the removal of each type of edge will result in two different isomorphism classes. We first show that the graph $L_G - \{w_i, b_j\}$ is asymmetric. Without loss of generality we remove the edge between vertex 1 and vertex 10. We will show that every vertex in the graph is distinct from any other vertex.

- 1 is distinct since it is the only vertex with degree 3 and 10 is distinct since it is the only vertex of degree 1.
- 2 is distinct since it is the only vertex that is distance 2 from 1 and distance 5 from 10.
- 3 is distinct since it is the only vertex that is distance 7 from 10.
- 4 is distinct since it is the only vertex that is distance 2 from 2.
- 5 is distinct since it is the only vertex that is distance 4 from 2.
- 6 is distinct since it is the only vertex that is distance 6 from 1.
- 7 is distinct since it is the only vertex that is distance 4 from 1.
- 8 is distinct since it is the only vertex that is distance 6 from a vertex other than 1.
- 9 is distinct since it is the only vertex that is distance 2 from 1 and distance 3 from 10.
- Since 1-9 are distinct, 11-27 are all distinct.

We next show that the graph $L_G - \{w_i, g_j\}$ is asymmetric. Without loss of generality we remove the edge between vertex 1 and vertex 11. We will show that every vertex in the graph is distinct from any other vertex:

- 1 and 11 are distinct.

- 3 is distinct since it is the only vertex that is distance 2 from 1 and is distance 5 from 11.
- 2 is distinct since it is the only vertex that is distance 2 from 1 and is distance 2 from 3.
- 4 is distinct since it is the only vertex that is distance 4 from 1 and is distance 2 from 2.
- 5 is distinct since it is the only vertex that is distance 4 from 1 and is distance 4 from 2.
- 8 is distinct since it is the only vertex that is distance 2 from 1 and is distance 4 from 3.
- 6 is distinct since it is the only vertex that is distance 2 from 8.
- 7 is distinct since it is the only vertex that is distance 4 from 1 and is distance 3 from 11.
- 9 is distinct since it is the only vertex that is adjacent to 11.
- Since 1-9 are distinct, 10,12-27 are all distinct.

Therefore, the graph L_G is minimally non-asymmetric. ■

Using *Mathematica* we found that of the block designs with pairs of block size 2, (a, b) and (c, d) where a, b, c, d are all between 0 and 9 inclusive these corresponded to only six non-isomorphic graphs. Two of these graphs are minimally non-asymmetric. The first is L_G and the second is shown in Figure 3.

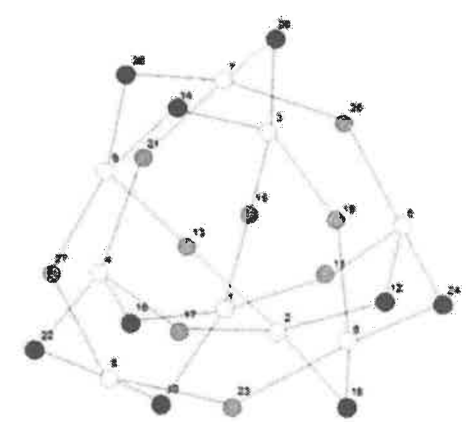

Figure 3. A second minimally non-asymmetric graph

(The argument why this graph is minimally non-asymmetric is similar to the approach used for the graph L_G).

The other four graphs are not minimally asymmetric. These are shown in Figure 4.

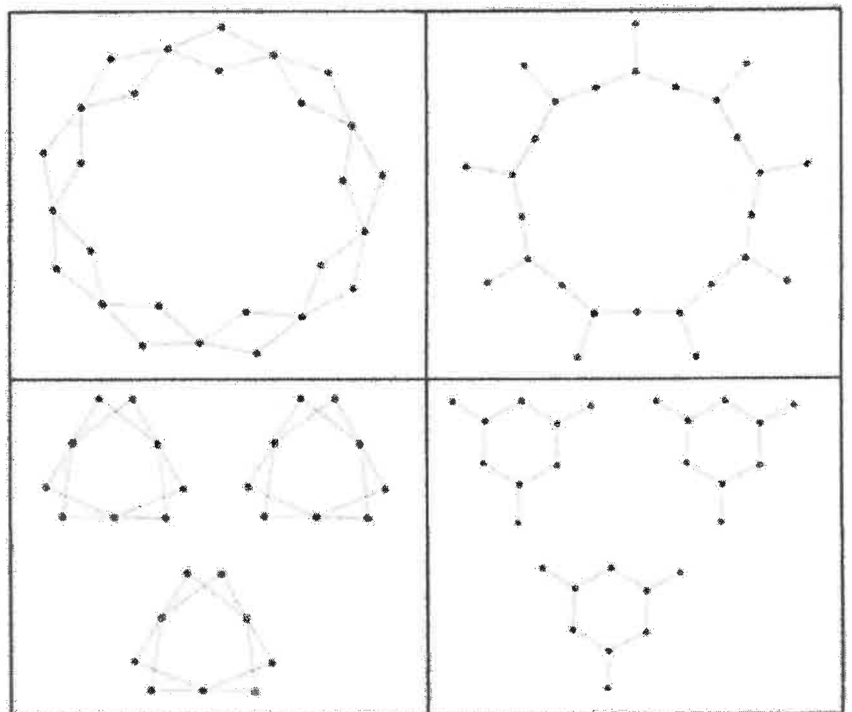

Figure 4. Four graphs that are not minimally asymmetric

2.1 Symmetrically repeated differences

In 1939, Bose introduced block designs with symmetrically repeated differences [1]. Here we consider the set of the differences between each ordered pair of elements. Each difference appears the same number of times over the entire set. We continue with an example of a $(9, 18, 8, 4, 3)$-BIBD which was given by R.C. Bose [1] with symmetrically repeated differences building from the blocks $(0, 1, 2, 4)$ and $(0, 3, 4, 7)$ where subsequent blocks were created by adding integers $1, 2, ..., 8$ taking the sums modulo 9. This creates a bipartite graph with parts X and Y where X contains the numbers of the 18 blocks and Y contains the integers $\{0, 1, 2, .., 8\}$. The blocks give the adjacencies of the vertices in Y. The blocks are: $b_1 = \{0, 1, 2, 4\}$;

$b_2 = \{1, 2, 3, 5\}$; $b_3 = \{2, 3, 4, 6\}$; $b_4 = \{3, 4, 5, 7\}$; $b_5 = \{4, 5, 6, 8\}$; $b_6 = \{5, 6, 7, 0\}$; $b_7 = \{6, 7, 8, 1\}$; $b_8 = \{7, 8, 0, 2\}$; $b_9 = \{8, 0, 1, 3\}$; $b_{10} = \{0, 3, 4, 7\}$; $b_{11} = \{1, 4, 5, 8\}$; $b_{12} = \{2, 5, 6, 0\}$; $b_{13} = \{3, 6, 7, 1\}$; $b_{14} = \{4, 7, 8, 2\}$; $b_{15} = \{5, 8, 0, 3\}$; $b_{16} = \{6, 0, 1, 4\}$; $b_{17} = \{7, 1, 2, 5\}$; and $b_{18} = \{8, 2, 3, 6\}$. We note that each difference appears exactly three times.

(0,1,2,4)	(0,1,2,4)	(0,3,4,7)
(1,2,3,5)		
(2,3,4,6)	0-1=8	0-3=6
(3,4,5,7)	0-2=7	0-4=5
(4,5,6,8)	0-4=5	0-7=2
(5,6,7,0)	1-0=1	3-0=3
(6,7,8,1)	1-2=8	3-4=8
(7,8,0,2)	1-4=6	3-7=5
(8,0,1,3)	2-0=2	4-0=4
(0,3,4,7)	2-1=1	4-3=1
(1,4,5,8)	2-4=7	4-7=6
(2,5,6,0)	4-0=4	7-0=7
(3,6,7,1)	4-1=3	7-3=4
(4,7,8,2)	4-2=2	7-4=3
(5,8,0,3)		
(6,0,1,4)		
(7,1,2,5)		
(8,2,3,6)		

Figure 5. A BIBD with symmetrically repeated differences

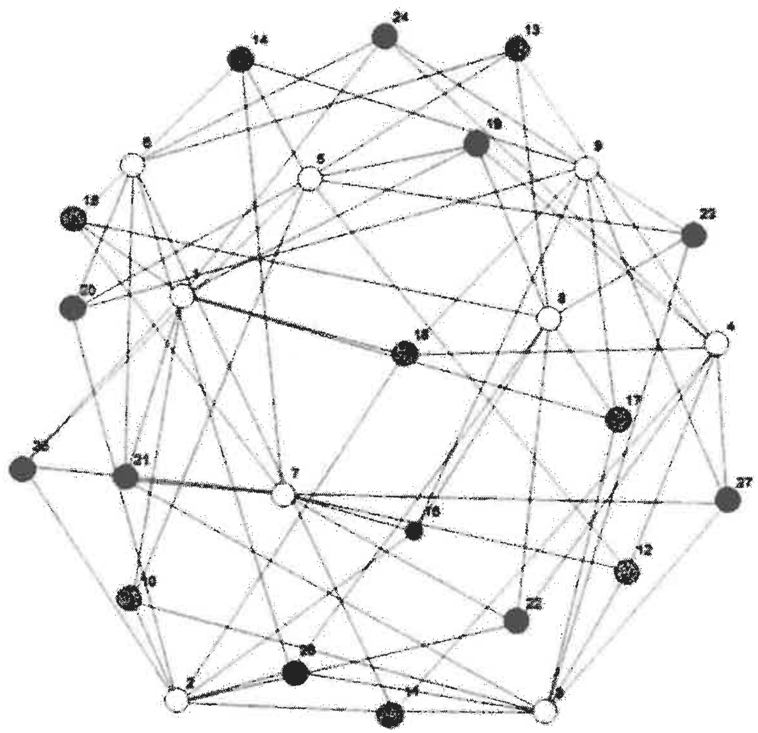

Figure 6. A graph realized from the BIBD in Figure 5

3 Conclusion

In this paper we considered designs with block sizes 2 and 4, and we have noted examples with block size 3 for various modular classes. It would be interesting to identify more families of block designs that correpsond to graphs that are minimally non-asymmetric. In some cases the graphs are also maximally non-asymmetric, where the graph is non-asymmetric, but the addition of any edge from the complement results in an asymmetric graphs. Another problem would be to determine which block designs lead to graphs that are both minimally non-asymmetric and maximally non-asymmetric.

References

[1] Bose, R. C., On the Construction of Balanced Incomplete Block Designs, Ann. Eugenics 9 (1939), 353-399.

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[3] Fisher, R. A., An Examination of the Different Possible Solutions of a Problem in Incomplete Blocks, Annals of Eugenics, volume 10, (1940), 52–75.

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