On the Likelihood of Symmetrical Cayley Maps for Certain Abelian Groups

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Abstract

Let Γ be a finite group and let Δ be a generating set for Γ. A Cayley map is an orientable 2-cell imbedding of the Cayley graph $G_\Delta(\Gamma)$ such that the rotation of arcs emanating from each vertex is determined by a unique cyclic permutation of generators and their inverses. A probability model for the set of all Cayley maps for a fixed group and generating set, where the distribution is uniform, is investigated for certain finite abelian groups with generating set chosen as the standard basis. A lower bound is provided for the probability that a Cayley map for such a group and generating set is symmetrical.

1 Random Topological Graph Theory

A *surface* is a closed orientable 2-manifold and the surface of genus k is denoted by S_k, where $k \geq 0$. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. It is well-known (see Edmonds [5], for example) that the labeled 2-cell imbeddings of a connected graph G are in one-to-one correspondence with the rotation schemes for G. The ordered pair $(G, \wp)$, where G is a graph and $\wp$ is a rotation scheme for G is called a *map*. The rotation schemes can be easily counted, and so, there are $\prod_{i=1}^{p} (\deg v_i - 1)!$ maps of G. In [17] White proposes five probability models for study in random topological graph theory and his Model I is implicitly used in much of the literature that focuses on the enumeration of graph imbeddings and calculations of average genus of graphs. This probability model consists

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of the set Ω containing all maps of G and the uniform distribution, that is, where the probability of a given map $(G, \wp)$ is $1/[\prod_{i=1}^{p} (\deg v_i - 1)!]$. With this model the *genus random variable* $g : \Omega \to \mathbb{N} \cup \{0\}$ is defined by $g(G, \wp) = k$ if the map $(G, \wp)$ describes a 2-cell imbedding of G in the surface S_k. Hence the *genus* $\gamma(G)$ and *maximum genus* $\gamma_M(G)$ of G are, respectively, the minimum and maximum of the genus random variable. Further, the expected value $E(g)$ of the genus random variable is the *average genus* $\bar{\gamma}(G)$ so that

$$\bar{\gamma}(G) = \frac{1}{|\Omega|} \sum_{\Omega} g(G, \wp),$$

where, of course, $|\Omega| = \prod_{v \in V(G)} (\deg v - 1)!$.

In general, it is difficult to obtain the average genus of a graph and this is known only for a few infinite classes of graphs and much work continues in this area. In this paper, we use a probability model for studying the special class of Cayley graph imbeddings that are known as Cayley maps and calculate the expected value of the associated genus random variable for a special class of groups, namely, the repeated direct product of a cyclic group. In the final section we consider the probability that such a Cayley map is symmetrical.

2 Random Cayley Maps

Let Γ be a finite group and let Δ be a generating set for Γ such that the identity $\epsilon \notin \Delta$. Also let $\Delta^{-1} = \{\delta^{-1} \mid \delta \in \Delta\}$ and $\Delta^* = \Delta \cup \Delta^{-1}$. In addition, we assume that Δ is chosen so that if $\delta \in \Delta \cap \Delta^{-1}$, then $\delta^2 = \epsilon$; in other words, if δ is chosen as a generator, then δ^{-1} is not chosen, unless, of course, δ is an involution. The *Cayley graph* $G_\Delta(\Gamma)$ is that graph whose vertex set is Γ and edge set is $\{ \{x, x\delta\} \mid x \in \Gamma, \delta \in \Delta^*\}$. Let $\rho : \Delta^* \to \Delta^*$ be a cyclic permutation. A *Cayley map* (Γ, Δ, ρ) is the map $(G_\Delta(\Gamma), \wp)$, where $\wp = \{\rho_x \mid x \in \Gamma\}$ is the rotation scheme for $G_\Delta(\Gamma)$ such that each vertex rotation ρ_x is given by $\rho_x(y) = x\rho(x^{-1}y)$ for $x \in \Gamma$ and $y \in N(x)$, the neighborhood of x. In other words, each vertex rotation ρ_x is uniquely determined by ρ, where the cyclic permutation of labels on the arcs emanating from x in the clockwise direction is precisely ρ.

A natural probability model for studying Cayley maps for a given group and generating set is defined in [13] and studied in [11] and [12]. Let Ω be the sample space consisting of all Cayley maps for a specified finite group Γ and generating set Δ for Γ, where the distribution is uniform, that is, where

$$P(\Gamma, \Delta, \rho) = \frac{1}{(|\Delta^*| - 1)!}$$

for each Cayley map (Γ, Δ, ρ). The *genus random variable* $g : \Omega \to \mathbb{N} \cup \{0\}$ is defined by $g(\Gamma, \Delta, \rho) = k$ if the Cayley map (Γ, Δ, ρ) describes a 2-cell imbedding of $G_\Delta(\Gamma)$ in the surface S_k. The *Cayley genus* $\gamma(\Gamma, \Delta)$ and *maximum Cayley genus* $\gamma_M(\Gamma, \Delta)$ are defined as

$$\gamma(\Gamma, \Delta) = \min_{\Omega} g(\Gamma, \Delta, \rho)$$

and

$$\gamma_M(\Gamma, \Delta) = \max_{\Omega} g(\Gamma, \Delta, \rho);$$

while the *average Cayley genus* $\bar{\gamma}(\Gamma, \Delta)$ is the expected value of the genus random variable and is given by

$$\bar{\gamma}(\Gamma, \Delta) = \frac{1}{(|\Delta^*| - 1)!} \sum_{\Omega} g(\Gamma, \Delta, \rho).$$

In [11], these parameters are determined for certain groups generated by involutions such as the symmetric group and dihedral group. The formula for the average Cayley genus of the dihedral group with generating set consisting of the reflections is improved in [8].

3 Random Cayley Maps for a Class of Finite Abelian Groups

Let $\Gamma = (\mathbb{Z}_m)^n = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{Z}_m\}$, where $m \geq 2$ and $n \geq 1$, with the generating set Δ chosen as the standard basis. Of course, when $m = p$ is prime, $(\mathbb{Z}_p)^n$ is the additive group in the Galois field $GF(p^n)$. In this section, we rely on the fact that a Cayley map is the covering of an index one voltage graph and refer the interested reader to Gross and Tucker [7] or White [16] for basic terminology and theory.

In case $m = 2$, the voltage graph has one region, and no matter the cyclic permutation of the generators, the order of the element on the boundary of the region is 2, so that the genus random variable is constant. It is interesting to note here that the Cayley graph $G_\Delta(\Gamma)$ is the n-cube and the average Cayley genus is the genus of the $(n+1)$-cube. (The genus of the n-cube is given in [10] and [2].)

Theorem 1 *Let $\Gamma = (\mathbb{Z}_2)^n$, where $n \geq 1$, and let Δ denote the standard basis for Γ. Then*

$$g(\Gamma, \Delta, \rho) = 1 + 2^{n-2}(n-3)$$

for every cyclic permutation $\rho : \Delta \to \Delta$.

For $m \geq 3$, the voltage graph is an index one voltage graph that has n directed loops, that is, it is a bouquet voltage graph of size n. Using the recurrence relation obtained by Gross, Robbins, and Tucker ([6]), we will be able to obtain the average Cayley genus for Γ and Δ. First, for integers n and r, let $S_{n,r}$ denote the number of 2-cell imbeddings of a bouquet voltage graph of size n having r regions. From the recurrence relation of [6] and Euler's identity, we obtain a recurrence relation for $S_{n,r}$.

Theorem 2 *The number $S_{n,r}$ of 2-cell imbeddings of a bouquet voltage graph of size n having r regions is given by the recurrence relation*

$$(n+1)S_{n,r} = 4(2n-1)(2n-3)(n-1)^2(n-2)S_{n-2,r}+4(2n-1)(n-1)S_{n-1,r-1}$$

for $n > 2$ and with boundary conditions

$$S_{n,r} = 0 \textit{ if } n < 0 \textit{ or } r < 1 \textit{ or } r > n+1,$$

$$S_{0,1} = S_{1,2} = 1, S_{0,r} = 0 \textit{ for } r \neq 1, \textit{ and } S_{1,r} = 0 \textit{ for } r \neq 2,$$
$$S_{2,1} = 2, S_{2,2} = 0, S_{2,3} = 4, \textit{ and } S_{2,r} = 0 \textit{ for } r > 3.$$

Next, since we are using the standard basis of $\Gamma = (\mathbb{Z}_m)^n$, it follows that a region R of a voltage graph for the Cayley map (Γ, Δ, ρ) will satisfy the KVL (i.e., the element on the boundary of a R is the identity) if and only if it is the unique region.

Lemma 3 *Let $\Gamma = (\mathbb{Z}_m)^n$, where $m \geq 3$ and $n \geq 1$, and let Δ denote the standard basis for Γ. Let K denote the voltage graph for the Cayley map (Γ, Δ, ρ). A region of K satisfies the KVL if and only if K has exactly one region, and in this case, n is even.*

Proof First assume that K is 2-cell imbedded in S_k with r regions. Euler's identity gives $1-n+r = 2-2k$. If R is a region of K and $r \geq 2$, then there exists a region R' distinct from R and such that R and R' share an edge e. Since not both e and e^{-1} are on the boundary of R, it follows that R cannot satisfy the KVL. Next, if K has one region, then every element of Δ^* is on the boundary of R and thus R satisfies the KVL. When this is the case, we see that $n = 2k$ is even. □

Let R be a region of the voltage graph for the Cayley map (Γ, Δ, ρ) and let s be the length of a closed walk bounding R. When R satisfies the KVL, then it lifts to m^n regions in the Cayley map; and when R does not satisfy the KVL, then it lifts to m^{n-1} regions in the Cayley map. Therefore, if r is the number of regions in the voltage graph, then (Γ, Δ, ρ) has m^n regions when $r = 1$ and has rm^{n-1} regions when $r \geq 2$. With this, we calculate the genus $g(\Gamma, \Delta, \rho)$.

Theorem 4 *Let* $\Gamma = (\mathbb{Z}_m)^n$*, where* $m \geq 3$ *and* $n \geq 1$*, and let* Δ *denote the standard basis for* Γ*. If the voltage graph for* (Γ, Δ, ρ) *has* r *regions, then*

$$g(\Gamma, \Delta, \rho) = \begin{cases} 1 + \frac{m^n}{2}(n-2) & \text{if } r = 1 \\ 1 + \frac{m^{n-1}}{2}(mn - m - r) & \text{if } r \geq 2. \end{cases}$$

The Cayley genus and maximum Cayley genus may now be obtained.

Corollary 5 *Let* $\Gamma = (\mathbb{Z}_m)^n$*, where* $m \geq 3$ *and* $n \geq 1$*, and let* Δ *denote the standard basis for* Γ*. Then*

$$\gamma(\Gamma, \Delta) = \begin{cases} 1 + \frac{m^n}{2}(n-2) & \text{if } n \text{ is even and } n < m-1 \\ 1 + \frac{m^{n-1}}{2}(mn - m - n - 1) & \text{otherwise,} \end{cases}$$

and

$$\gamma_M(\Gamma, \Delta) = \begin{cases} 1 + \frac{m^{n-1}}{2}(mn - m - 3) & \text{if } n \text{ is even} \\ 1 + \frac{m^{n-1}}{2}(mn - m - 2) & \text{if } n \text{ is odd.} \end{cases}$$

Using Theorem 2, we find the average Cayley genus.

Corollary 6 *Let* Δ *be the standard basis for* $(\mathbb{Z}_m)^n$*, where* $m \geq 3$ *and* $n \geq 1$*. Then the average Cayley genus is* $\bar{\gamma}((\mathbb{Z}_m)^n, \Delta) =$

$$\frac{1}{(2n-1)!}\left[(1 + \frac{m^n}{2}(n-2))S_{n,1} \quad + \sum_{r=2}^{n+1}(1 + \frac{m^{n-1}}{2}(mn - m - r))S_{n,r}\right].$$

4 Likelihood of Symmetrical Cayley Maps

A *map automorphism* of the Cayley map (Γ, Δ, ρ) is a bijection of Γ onto itself that preserves oriented region boundaries and the collection of all map automorphisms of (Γ, Δ, ρ) forms the *automorphism group* of (Γ, Δ, ρ). It is well known that the order of the automorphism group of (Γ, Δ, ρ) divides $|\Gamma| \cdot |\Delta^*|$ (see [4] for example). When the automorphism group has cardinality $|\Gamma| \cdot |\Delta^*|$, the Cayley map (Γ, Δ, ρ) is *symmetrical* [3]. The following result (also in [4]) is useful in showing a Cayley map is symmetrical.

Theorem 7 *Let* (Γ, Δ, ρ) *be a Cayley map. If there exists a group automorphism* $\alpha : \Gamma \to \Gamma$ *such that* $\alpha|_{\Delta^*} = \rho$*, then* (Γ, Δ, ρ) *is symmetrical.*

Škoviera and Širáň [15] show that there is a class of Cayley maps for which the converse holds. A Cayley map (Γ, Δ, ρ) is *balanced* if $\rho(x^{-1}) = \rho(x)^{-1}$ for every $x \in \Delta^*$.

Theorem 8 *Let (Γ, Δ, ρ) be a Cayley map. If there exists a group automorphism $\alpha : \Gamma \to \Gamma$ such that $\alpha|_{\Delta^*} = \rho$, then (Γ, Δ, ρ) is balanced and symmetrical. Conversely, if (Γ, Δ, ρ) is a symmetrical balanced Cayley map, then such an automorphism α exists.*

While the concept of balanced has been generalized, first to antibalanced (see [14]), and then, to t-balanced (see [9]), we focus here on the balanced case in order to be able to utilize group automorphisms.

Let $\Gamma = (\mathbb{Z}_m)^n$, where $m \geq 3$ and $n \geq 2$ and let Δ be the standard basis for Γ. We determine the probability that a given Cayley map (Γ, Δ, ρ) is balanced and symmetrical, which then gives a lower bound for the probability that it is symmetrical. (Note that when $n = 1$, then there is a unique Cayley map $(\mathbb{Z}_m, 1, m-1, (1, m-1))$ and it is symmetrical.) For $n \geq 2$, we view the group $(\mathbb{Z}_m)^n$ as the additive group in the free $\mathbb{Z}_m$-module $(\mathbb{Z}_m)^n$. See [1] for basic results for free modules. Of particular significance here is that every homomorphism of $(\mathbb{Z}_m)^n$ can be represented as left multiplication by a matrix whose entries belong to $(\mathbb{Z}_m)^n$.

Theorem 9 *Let $m \geq 3$ and $n \geq 2$ be integers and let Δ be the standard basis for $(\mathbb{Z}_m)^n$. Then the probability that $((\mathbb{Z}_m)^n, \Delta, \rho)$ is balanced and symmetrical is*

$$P\Big(((\mathbb{Z}_m)^n, \Delta, \rho) \text{ is balanced and symmetrical}\Big) = \frac{1}{1 \cdot 3 \cdot 5 \ldots \cdot (2n-1)}.$$

Proof Let $\Delta = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ be the standard basis for $\Gamma = (\mathbb{Z}_m)^n$, taking each to be a column vector. We count the automorphisms whose restriction to Δ^* is a cyclic permutation. An automorphism of $(\mathbb{Z}_m)^n$ can be represented as left multiplication by a matrix A, where $\det(A) \neq 0$ and the ith column of A is the image of ϵ_i. Since each ϵ_i must be mapped to an element of Δ^*, we may write A in terms of its columns as $A = [\tau(1)\epsilon_{\sigma(1)}, \tau(2)\epsilon_{\sigma(2)}, \ldots, \tau(n)\epsilon_{\sigma(n)}]$, where σ is a permutation of $\{1, 2, \ldots, n\}$ and $\tau : \{1, 2, \ldots, n\} \to \{-1, 1\}$ is a function. Thus $A\epsilon_i = \tau(i)\epsilon_{\sigma(i)}$ for each $i = 1, 2, \ldots, n$ and $A\epsilon_{\sigma^k(1)} = \tau(\sigma^k(1))\epsilon_{\sigma^{k+1}(1)}$ for each k with $1 \leq k \leq n-1$, where $\sigma^0(1) = 1$ and $\sigma^i(1) = \sigma(\sigma^{i-1}(1))$ for $i \geq 1$. Next, in order for A to induce a cyclic permutation of Δ^*, it is necessary that σ is a cyclic permutation and that τ has the property that $\prod_{i=1}^n \tau(i) = -1$.

We show that A, in fact, does induce a cyclic permutation of Δ^*. For that purpose, define the function $\theta : \{0, 1, \ldots, n-1\} \to \{0, 1\}$ so that

$$\prod_{i=0}^{k} \tau(\sigma^i(1)) = \tau(1)\tau(\sigma(1))\dots\tau(\sigma^k(1)) = (-1)^{\theta(k)}.$$

Now observe that $A\epsilon_1 = \tau(1)\epsilon_{\sigma(1)} = (-1)^{\theta(0)}\epsilon_{\sigma(1)}$, and for each k with $1 \le k \le n-1$,

$$A(-1)^{\theta(k-1)}\epsilon_{\sigma^k(1)} = (-1)^{\theta(k)}\epsilon_{\sigma^{k+1}(1)}.$$

In this way, A restricted to Δ^* produces the cyclic permutation

$$(\epsilon_1, (-1)^{\theta(0)}\epsilon_{\sigma(1)}, (-1)^{\theta(1)}\epsilon_{\sigma^2(1)}, \dots, (-1)^{\theta(n-2)}\epsilon_{\sigma^{n-1}(1)},$$
$$-\epsilon_1, (-1)^{\theta(0)+1}\epsilon_{\sigma(1)}, (-1)^{\theta(1)+1}\epsilon_{\sigma^2(1)}, \dots, (-1)^{\theta(n-2)+1}\epsilon_{\sigma^{n-1}(1)})$$

Finally we count these automorphisms by noticing there are $(n-1)!$ choices for the cyclic permutation σ and there are 2^{n-1} choices for τ. Hence the probability that a Cayley map $(\mathbb{Z}_m)^n, \Delta, \rho)$ is balanced and symmetrical is $\frac{2^{n-1}(n-1)!}{(2n-1)!} = \frac{1}{1\cdot 3\cdot 5\dots\cdot(2n-1)}$. □

And this leads to the bound.

Corollary 10 *Let $m \ge 3$ and $n \ge 2$ be integers and let Δ be the standard basis for $(\mathbb{Z}_m)^n$. Then the probability that $((\mathbb{Z}_m)^n, \Delta, \rho)$ is symmetrical is*

$$P\Big(((\mathbb{Z}_m)^n, \Delta, \rho) \text{ is symmetrical}\Big) \ge \frac{1}{1\cdot 3\cdot 5\dots\cdot(2n-1)}.$$

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