On Regal Colorings of Graphs

James Hallas
Western Michigan University
Kalamazoo, Michigan, USA

Mohra Zayed
King Khalid University
Abha, Saudi Arabia

Ping Zhang
Western Michigan University
Kalamazoo, Michigan, USA
ping.zhang@wmich.edu

Abstract

For a positive integer k, let $\mathcal{P}^*([k])$ be the set of nonempty subsets of the set $[k] = \{1, 2, \ldots, k\}$. For a connected graph G of order 3 or more, let $c : E(G) \to \mathcal{P}^*([k])$ be an edge coloring of G where adjacent edges may be colored the same. The induced vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ is defined by $c'(v) = \bigcap_{e \in E_v} c(e)$, where E_v is the set of edges incident with v. If c' is a proper vertex coloring of G, then c is called a regal k-edge coloring of G. The minimum positive integer k for which a graph G has a regal k-edge coloring is the regal index $\text{reg}(G)$ of G. If c' is vertex-distinguishing, then c is a strong regal k-edge coloring of G. The minimum positive integer k for which a graph G has a strong regal k-edge coloring is the strong regal index $\text{sreg}(G)$ of G. A brief survey of known results and conjectures on strong regal indexes of graphs is presented. The relationships between the regal index $\text{reg}(G)$ and the chromatic number $\chi(G)$ of a connected graph G are investigated and results and problems on $\text{reg}(G)$ are presented.

Key Words: set-defined coloring, edge coloring, regal coloring, strong coloring, regal index, strong regal index.

AMS Subject Classification: 05C15, 05C05, 05C35.

1 Introduction

For a graph G without isolated vertices, an edge coloring of G is an assignment of colors to the edges of G. An edge coloring c is *unrestricted* if no condition is placed on how the edges may be colored; in particular, adjacent edges may be colored the same by c. If every two adjacent edges of G are colored differently, then c is a *proper edge coloring* and the minimum number of colors required of a proper edge coloring of G is its *chromatic index* $\chi'(G)$. A vertex coloring c' of a graph G is an assignment of colors to the vertices of G. A vertex coloring c' of G is *neighbor-distinguishing* or *proper* if adjacent vertices are colored differently. The minimum number of colors required of a proper vertex coloring of G is its *chromatic number* $\chi(G)$. A vertex coloring c' of a graph G is *vertex-distinguishing* or *rainbow* if no two vertices are colored the same by c'.

During the past three decades, a number of different types of edge colorings (or edge labelings) of graphs have been described that give rise to vertex colorings defined in a variety of manners (see [1, 2, 3, 7, 8, 13, 14, 15] for example). Among vertex colorings c' of a graph G obtained from an edge coloring c of G where the colors are selected from a set $[k] = \{1, 2, \ldots, k\}$ for some positive integer k, the most commonly studied are those for which the color $c'(v)$ of a vertex v of G is either (1) the set of colors of those edges incident with v, (2) the multiset of colors of the edges incident with v, or (3) the sum of the colors of the edges incident with v. In most cases, the induced vertex coloring c' is required to be proper or rainbow.

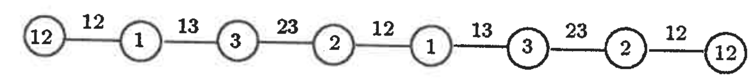
While an edge coloring c of a graph G typically uses colors from the set $[k]$ for some positive integer k resulting in $c(e) = i \in [k]$ for $e \in E(G)$, we might define $c(e) = \{i\}$ instead. That is, in (1) above, both the edge coloring c and the induced vertex coloring c' assign subsets of $[k]$ to the edges and vertices of G. This suggests the idea of studying edge colorings c where both c and its derived vertex coloring c' assign subsets of $[k]$ to the elements (edges and vertices) of a graph G. A number of unrestricted edge colorings of a graph have been studied that use subsets of $[k]$ as colors and give rise to proper or rainbow vertex colorings by means of set union (see [4, 9, 10, 11, 12, 13, 14]).

In [5], an edge coloring of a graph was introduced that use subsets of $[k]$ as colors and give rise to proper or rainbow vertex colorings by means of set intersection. For a positive integer k, we write $\mathcal{P}([k])$ for the power set of $[k]$ and $\mathcal{P}^*([k]) = \mathcal{P}([k]) - \{\emptyset\}$ for the set of nonempty subsets of $[k]$. For a graph G without isolated vertices, let $c : E(G) \to \mathcal{P}^*([k])$ be an unrestricted edge coloring of G, where then adjacent edges may be colored the same. The vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ is defined by

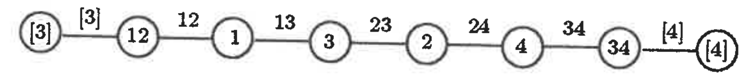
$$c'(v) = \bigcap_{e \in E_v} c(e),$$

where E_v is the set of edges incident with a vertex v of G. That is, $c'(v)$ is the intersection of the sets of colors of those edges incident with v. Furthermore, the coloring c has the property that requires $c'(v) \neq \emptyset$ for every vertex v of G. If c' is a proper vertex coloring of G, then c is called a *regal k-edge coloring* of G. An edge coloring of G is a *regal coloring* if it is a regal k-edge coloring for some positive integer k. The minimum positive integer k for which a graph G has a regal k-edge coloring is called the *regal index* $\text{reg}(G)$ of G. If c' is vertex-distinguishing, then c is called a *strong regal k-edge coloring* of G. An edge coloring of G is a strong regal coloring if it is a strong regal k-edge coloring for some positive integer k. The minimum positive integer k for which a graph G has a strong regal k-edge coloring is called the *strong regal index* $\text{sreg}(G)$ of G. The concepts of regal and strong regal colorings were introduced by Chartrand and studied in [5]. Since every strong regal coloring is also a regal coloring, it follows that $\text{reg}(G) \leq \text{sreg}(G)$ for every connected graph G of order at least 3.

For example, Figure 1 shows a regal 3-edge coloring and a strong regal 4-edge coloring of the path P_8 of order 8. For simplicity, we write the set $\{a\}$ as a, $\{a, b\}$ as ab, and $\{a, b, c\}$ as abc. In fact, $\text{reg}(P_8) = 3$ and $\text{sreg}(P_8) = 4$. Thus, $\text{reg}(P_8) \neq \text{sreg}(P_8)$.



A regal 3-edge coloring of P_8



A strong regal 4-edge coloring of P_8

Figure 1: A regal 3-edge coloring and a strong regal 4-edge coloring of P_8

While no regal coloring exists for the graph K_2, it was shown in [5] that such a coloring exists for every connected graph of order at least 3. In fact, more can be said.

Theorem 1.1 *If G is a graph each of whose components has order at least 3, then G has a strong regal coloring and therefore a regal coloring.*

Proof. We proceed by induction on the number of components of a graph. Since every connected graph of order at least 3 has a strong regal coloring, the base case is true. Suppose that if H is a graph consisting of $p - 1 \geq 1$ components, each of which has order at least 3, then H has a strong regal coloring. Let $G = G_1 + G_2 + \cdots + G_p$ be a graph consisting of p components, each of which has order at least 3. Let $H = G_1 + G_2 + \cdots + G_{p-1}$. By the induction hypothesis, each of H and G_p has a strong regal coloring. Suppose that $\text{sreg}(H) = k_1$ and $\text{sreg}(G_p) = k_2$. Let $k = \max\{k_1, k_2\} + 1$. Let $c_1 : E(H) \to \mathcal{P}^*([k_1])$ be a strong regal k_1-edge coloring of H and let

$c_2 : E(G_p) \to \mathcal{P}^*([k_2])$ be a strong regal k_2-edge coloring of G. Define an edge coloring $c : E(G) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} c_1(e) & \text{if } e \in E(H) \\ c_2(e) \cup \{k\} & \text{if } e \in E(G_p) \end{cases}$$

Then the induced coloring $c' : V(G) \to \mathcal{P}^*([k])$ is given by

$$c'(v) = \begin{cases} c_1'(v) & \text{if } v \in V(H) \\ c_2'(v) \cup \{k\} & \text{if } v \in V(G_p). \end{cases}$$

Since c_1' and c_2' are rainbow and $c'(x) \neq c'(y)$ if $x \in V(H)$ and $y \in V(G_p)$, it follows that c' is a rainbow vertex coloring of G. Hence, c is a strong regal k-edge coloring of G and the result follows. ■

In this work, we continue the study of regal colorings of graphs. In Section 2, we present a brief survey of known results, problems and conjectures on strong regal indexes of graphs obtained in [5]. In Section 3, we investigate the relationship between the regal index reg(G) and the chromatic number $\chi(G)$ of a connected graph G. We refer to the book [6] for graph theory notation and terminology not described in this paper.

2 Some Known Results on Regal Colorings

In [5] the primary emphasis is on strong regal colorings of connected graphs. The following result gives a lower bound for the regal index (and the strong regal index) of a graph in terms of its chromatic number.

Theorem 2.1 [5] *If G is a connected graph of order 3 or more, then*

$$\max\{3, \lceil \log_2(\chi(G)+1) \rceil\} \leq \text{reg}(G) \leq \text{sreg}(G).$$

If G is a connected graph of order 3, then $G = P_3$ or $G = K_3$. In each case, reg(G) = sreg$(G) = 3$. Strong regal 3-edge coloring s of P_3 and K_3 are shown in Figure 2. Thus, we consider connected graphs of order 4 or more.

If G is a connected graph of order $n \geq 4$ with $n \geq 2^{k-1} = |\mathcal{P}^*([k-1])|+1$ for some integer $k \geq 3$, then sreg$(G) \geq k$. Thus, if $2^{k-1} \leq n \leq 2^k - 1$, then sreg$(G) \geq 1 + \lfloor \log_2 n \rfloor$. This observation provides a lower bound for the strong regal index of a connected graph in terms of its order.

Proposition 2.2 [5] *If G is a connected graph of order $n \geq 4$, then*

$$\text{sreg}(G) \geq 1 + \lfloor \log_2 n \rfloor.$$

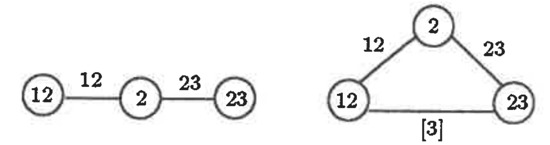


Figure 2: Strong regal 3-edge coloring s of P_3 and K_3

The lower bounds for the strong regal index of a graph in Theorem 2.1 and Proposition 2.2 are sharp, as we see in the next result.

Theorem 2.3 [5] *For each integer $n \geq 4$,*

$$\text{reg}(K_n) = \text{sreg}(K_n) = 1 + \lfloor \log_2 n \rfloor$$

$$\text{sreg}(C_n) = \text{sreg}(P_n) = \begin{cases} 1 + \lceil \log_2 n \rceil & \text{if } n = 7 \\ 1 + \lfloor \log_2 n \rfloor & \text{if } n \neq 7. \end{cases}$$

The following result presents an upper bound for the strong regal index of a connected graph G in terms of the strong regal index of a connected spanning subgraph of G. This bound shows the value of determining the strong regal indexes of trees.

Lemma 2.4 [5] *Let H be a connected spanning subgraph of a graph G of order at least 3. If H has a strong regal k-edge coloring for some positive integer k, then so does G. Consequently,* sreg$(G) \leq$ sreg(H). *In particular, if T is a spanning tree of G, then* sreg$(G) \leq$ sreg(T).

Since reg$(P_n) = 3$ and reg$(K_n) = \lceil \log_2(n+1) \rceil \geq 4$ for $n \geq 8$ by Theorem 2.3, it follows that Lemma 2.4 is not true in general for regal colorings of graphs. By Lemma 2.4, the strong regal indexes of trees play an important role in studying strong regal colorings of connected graphs. It was conjectured in [5] that if T is a tree of order $n \geq 4$, then there are only two possible values for sreg(T) in terms of n.

Conjecture 2.5 [5] *For every tree T of order $n \geq 4$,*

$$1 + \lfloor \log_2 n \rfloor \leq \text{sreg}(T) \leq 1 + \lceil \log_2 n \rceil.$$

Conjecture 2.5 was verified for several classes of trees, including stars and double stars.

Theorem 2.6 [5] *For every integer $n \geq 4$,* sreg$(K_{1,n-1}) = 1 + \lceil \log_2 n \rceil$.

Theorem 2.7 [5] *If T is a double star of order $n \geq 4$, then*

$$1 + \lfloor \log_2 n \rfloor \leq \text{sreg}(T) \leq 1 + \lceil \log_2 n \rceil.$$

There are infinitely many double stars T of order $n \geq 4$ with $\text{sreg}(T) = 1 + \lfloor \log_2 n \rfloor$ and there are infinitely many double stars T of order $n \geq 4$ with $\text{sreg}(T) = 1 + \lceil \log_2 n \rceil$. In fact, if T is any tree of order n with $4 \leq n \leq 6$ that is not a star, then $\text{sreg}(T) = 3 = 1 + \lfloor \log_2 n \rfloor$ and if T is any tree of order $n = 7$, then $\text{sreg}(T) = 4 = 1 + \lceil \log_2 n \rceil$. Furthermore, Conjecture 2.5, Proposition 2.2 and Lemma 2.4 give rise to the following conjecture.

Conjecture 2.8 *For every connected graph G of order $n \geq 4$,*

$$1 + \lfloor \log_2 n \rfloor \leq \text{sreg}(G) \leq 1 + \lceil \log_2 n \rceil.$$

3 On Graphs Having Regal Index 3

By Theorem 2.1, if G is a connected graph of order at least 3, then $\text{reg}(G) \geq 3$. This lower bound is sharp. Indeed, there are many graphs with regal index 3. The following is a consequence of the proof of the fact that $\text{sreg}(K_n) = 1 + \lfloor \log_2 n \rfloor$ for $n \geq 4$.

Corollary 3.1 [5] *If G is a complete ℓ-partite graph of order 3 or more for some integer $\ell \geq 2$, then $\text{reg}(G) = \max\{3, \lceil \log_2(\ell + 1) \rceil\}$.*

By Corollary 3.1, every complete ℓ-partite graph with $2 \leq \ell \leq 7$ has regal index 3. This therefore includes every complete bipartite graph. Not only does every complete bipartite graph have regal index 3 but every connected bipartite graph of order 3 or more has regal index 3, which we show next. To prove this, it is useful to describe some definitions and notation. The *distance* $d(u, v)$ between vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. The *eccentricity* $e(v)$ of a vertex v of G is the distance between v and a vertex farthest from v in G. The *diameter* $\text{diam}(G)$ of G is the largest eccentricity among the vertices of G. A vertex v with $e(v) = \text{diam}(G)$ is called a *peripheral vertex* of G. For two disjoint sets U and W of vertices of a graph, $[U, W]$ denotes the set of edges joining a vertex of U and a vertex of W.

Theorem 3.2 *Every connected bipartite graph of order at least 3 has regal index 3.*

Proof. Let G be a connected bipartite graph of order at least 3. By Theorem 2.1, it suffices to show that there exists a regal 3-edge coloring $c : E(G) \to \mathcal{P}^*([3])$ of G. Let v be a peripheral vertex of G. Thus, either v is an end-vertex of G or $\deg v \geq 2$ and v is not adjacent to any end-vertex of G. For each integer i with $0 \leq i \leq d = \text{diam}(G) \geq 2$, let

$$V_i = \{u \in V(G) : d_G(u, v) = i\}.$$

Thus, $V_0 = \{v\}$ and $V_1 = N_G(v)$ is the neighborhood of v. Since G is bipartite, each set V_i $(0 \leq i \leq d)$ is an independent set of vertices of G. For $0 \leq i \leq d-1$ and for each edge $v_i v_{i+1} \in [V_i, V_{i+1}]$, where $v_i \in V_i$ and $v_{i+1} \in V_{i+1}$, define

$$c(v_i v_{i+1}) = \begin{cases} 12 & \text{if } i \equiv 0 \pmod 3 \\ 13 & \text{if } i \equiv 1 \pmod 3 \\ 23 & \text{if } i \equiv 2 \pmod 3. \end{cases}$$

Thus, $c'(v) = 12$. For $1 \leq i \leq d-1$, each vertex $v_i \in V_i$ is adjacent to a vertex in V_{i-1}. If v_i is also adjacent to a vertex in V_{i+1}, then

$$c'(v_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 3 \\ 3 & \text{if } i \equiv 2 \pmod 3 \\ 2 & \text{if } i \equiv 0 \pmod 3; \end{cases}$$

while if v_i is adjacent only to vertices in V_{i-1}, then

$$c'(v_i) = \begin{cases} 12 & \text{if } i \equiv 1 \pmod 3 \\ 13 & \text{if } i \equiv 2 \pmod 3 \\ 23 & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

This coloring is illustrated Figure 3 for a bipartite graph.

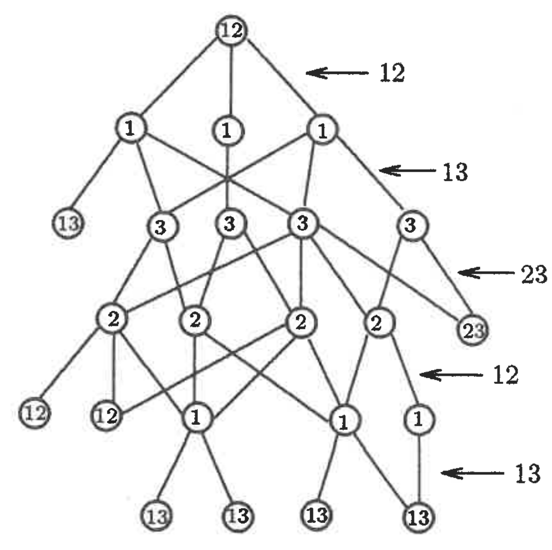


Figure 3: A regal 3-edge coloring of a bipartite graph

Consequently, $|c'(u)| \leq 2$ for every vertex of G. Since $c'(u) \neq c'(w)$ for every two adjacent vertices u and w of G, it follows that c' is a proper vertex coloring of G and so c is a regal 3-edge coloring of G. Therefore, $\text{reg}(G) = 3$. ■

By Theorem 3.2, it follows that if G is a connected graph of order 3 or more such that $\chi(G) = 2$, then $\text{reg}(G) = 3$. With the aid of the proof of Theorem 3.2, we now show that the same conclusion follows if $\chi(G) = 3$.

Theorem 3.3 *Every connected graph with chromatic number 3 has regal index 3.*

Proof. Let G be a connected graph with $\chi(G) = 3$. Since $\text{reg}(K_3) = 3$, we may assume that the order of G is at least 4. By Theorem 2.1, it suffices to show that there exists a regal 3-edge coloring of G. Since $\chi(G) = 3$, the vertex set $V(G)$ of G can be partitioned into three independent sets. Among all partitions of $V(G)$ into three independent sets, let $\{V_1, V_2, V_3\}$ be one such that the subgraph $H = G[V_1 \cup V_2]$ induced by the set $V_1 \cup V_2$ is a connected graph of order 3 or more. Since H is a connected bipartite graph of order at least 3, it follows by Theorem 3.2 that $\text{reg}(H) = 3$. Furthermore, as indicated in the proof of Theorem 3.2, H has a regal 3-edge coloring c_H such that $|c'_H(u)| \le 2$ for each vertex u of H. We now extend the coloring c_H of H to a regal 3-edge coloring $c_G : E(G) \to \mathcal{P}^*([3])$ of G by assigning [3] to each edge joining a vertex of V_3 to a vertex of H. Then $c'_G(u) = c'_H(u)$ for each $u \in V(H) = V_1 \cup V_2$ and $c'_G(u) = [3]$ for each $u \in V_3$. Let x and y be two adjacent vertices of G. Then at least one of x and y belongs to H, say $y \in V(H)$. If $x \in V(H)$, then $c'_G(x) = c'_H(x) \ne c'_H(y) = c'_G(y)$; while if $x \in V_3$, then $c'_G(x) = [3]$ and $|c'_G(y)| \le 2$ and so $c'_G(x) \ne c'_G(x)$. Thus, c'_G is a proper vertex coloring of G and so c_G is a regal 3-edge coloring of G. Therefore, $\text{reg}(G) = 3$. ■

With the aid of the proof of Theorem 3.3, we now see that if G is a connected graph G with $\chi(G) = 4$, then $\text{reg}(G) \le 4$.

Proposition 3.4 *Every connected graph with chromatic number 4 has regal index at most 4.*

Proof. Let G be a connected graph with $\chi(G) = 4$. We show that there exists a regal 4-coloring of G. Since $\chi(G) = 4$, the vertex set $V(G)$ can be partitioned into four independent sets V_1, V_2, V_3, V_4 of vertices of G such that the subgraph $H = G[V_1 \cup V_2 \cup V_3]$ induced by $V_1 \cup V_2 \cup V_3$ is a connected graph. Since $\chi(H) = 3$, it follows that $\text{reg}(H) = 3$ by Theorem 3.3. Let $c_H : E(H) \to \mathcal{P}^*([3])$ be a regal 3-edge coloring of H. We now extend the coloring c_H of H to a regal 4-edge coloring $c_G : E(G) \to \mathcal{P}^*([4])$ of G by assigning [4] to each edge joining a vertex of V_4 to a vertex of H. Then $c'_G(v) = c'_H(v) \subseteq [3]$ for each $v \in V(H) = V_1 \cup V_2 \cup V_3$ and $c'_G(v) = [4]$ for each $v \in V_4$. An argument similar to that employed in the proof of Theorem 3.3 shows that c'_G is a proper vertex coloring of G and so c_G is a regal 4-edge coloring of G. Therefore, $\text{reg}(G) \le 4$. ■

The statement and the proof of Proposition 3.4 suggest the following result dealing with regal indexes.

Proposition 3.5 *Let G be a connected graph containing an independent set U of vertices of G such that $H = G - U$ is a connected graph of order at least 3. If $\text{reg}(H) = k$ for some integer $k \ge 3$, then $\text{reg}(G) \le k + 1$. Furthermore, if H has a regal k-edge coloring c_H such that $|c'_H(v)| < k$ for every $v \in V(H)$, then $\text{reg}(G) \le k$.*

Proof. Let a regal k-edge coloring c_H of H be given. Then $c'_H(x) \ne c'_H(y)$ for every two adjacent vertices x and y of H. The edge coloring $c_G : E(G) \to \mathcal{P}^*([k+1])$ of G is defined as

$$c_G(e) = \begin{cases} c_H(e) & \text{if } e \in E(H) \\ [k+1] & \text{if } e \text{ is incident with a vertex of } U. \end{cases}$$

Let $xy \in E(G)$. If $xy \in E(H)$, then $c'_G(x) = c'_H(x) \ne c'_H(y) = c'_G(y)$. If $xy \notin E(H)$, then we may assume that $x \in U$ and $y \in V(H)$. Hence, $c'_G(x) = [k+1]$ and $c'_G(y) = c'_H(y) \subseteq [k]$. Thus, $c'_G(x) \ne c'_G(y)$. Hence, c'_G is a proper vertex coloring of G and so c_G is a regal $(k+1)$-edge coloring of G. Therefore, $\text{reg}(G) \le k + 1$. Furthermore, if $|c'_H(v)| < k$ for every $v \in V(H)$, then we define the edge coloring $c_G : E(G) \to \mathcal{P}^*([k])$ as

$$c_G(e) = \begin{cases} c_H(e) & \text{if } e \in E(H) \\ [k] & \text{if } e \text{ is incident with a vertex of } U. \end{cases}$$

We may then proceed as above to show that c_G is a regal k-edge coloring of G and so $\text{reg}(G) \le k$. ■

For a graph G, the join $G \vee K_1$ of G and K_1 is the graph obtained from G by adding a new vertex and joining this vertex to every vertex of G. It is well known that if G is a connected graph, then $\chi(G \vee K_1) = \chi(G) + 1$. In particular, if $\chi(G) = 3$, then $\chi(G \vee K_1) = 4$. This, however, need not be the case for the regal indexes of graphs. For example, by Theorem 3.3, $\text{reg}(C_n) = 3$ for each odd integer $n \ge 3$. Next, we show that $\text{reg}(C_n \vee K_1) = 3$ for all odd integers $n \ge 3$.

Proposition 3.6 *For every odd integer $n \ge 3$, $\text{reg}(C_n \vee K_1) = 3$.*

Proof. By Theorem 2.1 and Proposition 3.5, it suffices to show that for each odd integer $n \geq 3$, there is a regal 3-edge coloring c of the n-cycle C_n such that $|c'(v)| \leq 2$ for each vertex v of C_n. We show this fact for all integers $n \geq 3$. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ where $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$ and $e_n = v_1 v_n$. For an edge coloring c of C_n and a vertex coloring c' of C_n, let

$$\begin{aligned} \mathcal{S}_c(C_n) &= (c(e_1), c(e_2), \ldots, c(e_n)) \\ \mathcal{S}_{c'}(C_n) &= (c'(v_1), c'(v_2), \ldots, c'(v_n)). \end{aligned}$$

Write $n = 3a + r$ for some positive integer a where $r = 0, 1, 2$.

⋆ For $n = 3a$, define c such that

$$\mathcal{S}_c(C_n) = (\underline{123, 12, 23}, \underline{123, 12, 23}, \ldots, \underline{123, 12, 23}),$$

where the triple $(123, 12, 23)$ repeats a times. The induced vertex coloring c' satisfies

$$\mathcal{S}_{c'}(C_n) = (\underline{23, 12, 2}, \underline{23, 12, 2}, \ldots, \underline{23, 12, 2}),$$

where the triple $(23, 12, 2)$ repeats a times.

⋆ For $n = 3a + 1$, define c such that

$$\mathcal{S}_c(C_n) = (\underline{123, 12, 23}, \underline{123, 12, 23}, \ldots, \underline{123, 12, 23}, 13),$$

where the triple $(123, 12, 23)$ repeats a times. Then c' satisfies

$$\mathcal{S}_{c'}(C_n) = (13, \underline{12, 2, 23}, \underline{12, 2, 23}, \ldots, \underline{12, 2, 23}, 12, 2, 3),$$

where the triple $(12, 2, 23)$ repeats $a - 1$ times if $n \geq 7$.

⋆ For $n = 3a + 2$, define c such that

$$\mathcal{S}_c(C_n) = (\underline{123, 12, 23}, \ldots, \underline{123, 12, 23}, 23, 13),$$

where the triple $(123, 12, 23)$ repeats a times. Then c' satisfies

$$\mathcal{S}_{c'}(C_n) = (13, \underline{12, 2, 23}, \underline{12, 2, 23}, \ldots, \underline{12, 2, 23}, 3),$$

where the triple $(12, 2, 23)$ repeats a times.

In each case, c' is a proper vertex coloring of C_n and $|c'(v)| \leq 2$ for each vertex v of C_n. Hence, c is a regal 3-edge coloring of C_n, as desired. Therefore, $\text{reg}(C_n \vee K_1) = 3$ by Proposition 3.5 and Theorem 2.1. ■

Proposition 3.4 brings up a natural question, namely: Does there exist a 4-chromatic graph whose regal index is 4? Of course, we know that there

are 4-chromatic graphs whose regal index is 3. Indeed, by Corollary 3.1, every complete 4-partite graph has regal index 3. Also, by Corollary 3.1, if G is any complete ℓ-partite graph with $2 \leq \ell \leq 7$, then $\text{reg}(G) = 3$. Based on the results obtained in Theorems 3.2 and 3.3, one might expect every graph with chromatic number ℓ with $2 \leq \ell \leq 7$ to have regal index 3. This is not true, however. In fact, there exists a connected graph G with $\chi(G) = 7$ and $\text{reg}(G) = 4$. In general, we have the following result.

Proposition 3.7 *For each integer $k \geq 3$, there is a connected graph G such that $\chi(G) = 2^k - 1$ and $\text{reg}(G) = k + 1$.*

Proof. Let $n = 2^k - 1$ where $k \geq 3$ and let $G = \text{cor}(K_n)$ be the corona of K_n (that is, the graph G is obtained from K_n by adding a new vertex v for each vertex u of K_n and joining v to u). Then $\chi(G) = \chi(K_n) = n = 2^k - 1$.

It remains to show that $\text{reg}(G) = k + 1$. Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and $u_i v_i$ is the pendant edge of G at u_i for $1 \leq i \leq n$. Since $\chi(G) = 2^k - 1$, it follows by Theorem 2.1 that $\text{reg}(G) \geq k$. Assume, to the contrary, that $\text{reg}(G) = k$. Then there is a regal k-edge coloring $c : E(G) \to \mathcal{P}^*([k])$. Let $X_1, X_2, \ldots, X_n$ be the $n = 2^k - 1$ elements of $\mathcal{P}^*([k])$ such that $1 = |X_1| \leq |X_2| \leq \cdots \leq |X_n| = k$. Since c' is a proper vertex coloring of G, it follows that $c'(u_i) \neq c'(u_j)$ for each pair $i, j \in [n]$ and $i \neq j$. We may assume that $c'(u_i) = X_i$ for $1 \leq i \leq n$. In particular, $c'(u_n) = [k]$. Hence, $c(e) = [k]$ for each edge e incident with u_n. Since $c(u_n v_n) = [k]$ and v_n is an end-vertex of G, it follows that $c'(v_n) = [k] = c'(u_n)$, which is a contradiction. Thus, $\text{reg}(G) \geq k + 1$. Next, we show that $\text{reg}(G) \leq k + 1$. Since $\text{reg}(K_n) = k$, there is a regal k-edge coloring $c_0 : E(K_n) \to \mathcal{P}^*([k])$ of K_n. We now extend the coloring c_0 to a regal $(k+1)$-edge coloring $c : E(G) \to \mathcal{P}^*([k+1])$ of G by assigning $[k+1]$ to each edge $u_i v_i$ for $1 \leq i \leq n$. Then $c'(v) = c_0'(v) \subseteq [k]$ for each $v \in V(K_n)$ and $c'(v_i) = [k+1]$ for $1 \leq i \leq n$. This implies that c' is a proper vertex coloring of G and so c is a regal $(k+1)$-edge coloring of G. Therefore, $\text{reg}(G) \leq k + 1$ and so $\text{reg}(G) = k + 1$. ■

If G is the corona of K_7, then the proof of Proposition 3.7 states that $\chi(G) = 7$ and $\text{reg}(G) = 4$. In fact, with the aid of the complete graph K_7, we can construct other connected graphs G for which $\chi(G) = 7$ and $\text{reg}(G) = 4$. More generally, for each integer $k \geq 3$, we can construct other connected graphs G from the complete graph of order $2^k - 1$ such that $\chi(G) = 2^k - 1$ and $\text{reg}(G) = k + 1$. As an illustration, we consider the case where $k = 3$. First, we introduce an additional definition. For a graph H, the *shadow graph* $S(H)$ of H is obtained from H by adding, for each vertex v of H, a new vertex v', called the *shadow vertex* of v, and joining v' to the neighbors of v in H. Observe that (1) a vertex of H and its shadow vertex are not adjacent in $S(H)$ and (2) no two shadow vertices are adjacent in $S(H)$. We start with the complete graph K_7 of order 7 with $V(K_7) = \{u_1, u_2, \ldots, u_7\}$. Let F be the shadow graph $S(K_7)$ with $V(F) = V(K_7) \cup \{v_1, v_2, \ldots, v_7\}$ where v_i is the shadow vertex of u_i for $1 \le i \le 7$. The graph G is obtained from F by adding a pendant edge v_iw_i at each shadow vertex v_i of u_i for $1 \le i \le 7$. The graph G is shown in Figure 4.

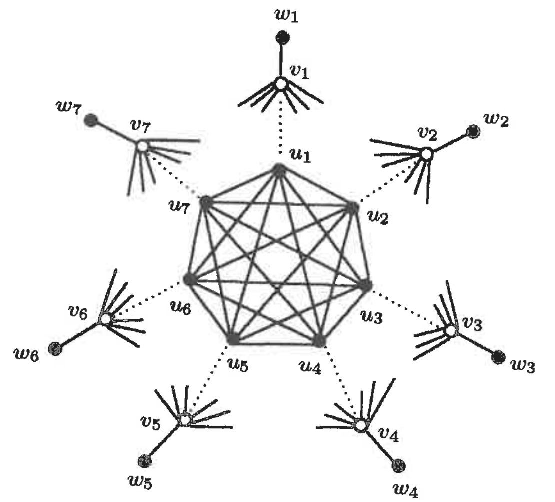


Figure 4: A graph G with $\chi(G) = 7$ and $\text{reg}(G) = 4$

Example 3.8 *For the graph G of Figure 4, $\chi(G) = 7$ and $\text{reg}(G) = 4$.*

Proof. Since $\chi(G) = 7$, it remains to show that $\text{reg}(G) = 4$. First, we show that $\text{reg}(G) \ge 4$. Assume, to the contrary, that there is a regal 3-edge coloring $c : E(G) \to \mathcal{P}^*([3])$. Let $X_1, X_2, \ldots, X_7$ be the seven elements of $\mathcal{P}^*([3])$ such that $1 = |X_1| \le |X_2| \le \cdots \le |X_7| = 3$. Since c' is a proper vertex coloring of G, it follows that $c'(u_i) \ne c'(u_j)$ for each pair $i, j \in [7]$ and $i \ne j$. We may assume that $c'(u_i) = X_i$ for $1 \le i \le 7$. For each integer i with $1 \le i \le 7$, since v_i is adjacent to u_j for each $j \in [7] - \{i\}$, it follows that $c'(v_i) \ne c'(u_j)$ and so $c'(v_i) = c'(u_i) = X_i$. In particular, $c'(u_7) = c'(v_7) = [3]$. This implies that $c(e) = [3]$ for each edge e incident with u_7 and so $c(u_7w_7) = [3]$. However then, $c'(w_7) = c'(u_7) = [3]$, which is impossible. Thus, $\text{reg}(G) \ge 4$.

Next, we show that $\text{reg}(G) \le 4$. Again, let $X_1, X_2, \ldots, X_7$ be the seven elements of $\mathcal{P}^*([3])$ such that $1 = |X_1| \le |X_2| \le \cdots \le |X_7| = 3$ and $X_i = \{i\}$ for $i = 1, 2, 3$. First, we define a labeling f of the vertices of G by $f(u_i) = f(v_i) = X_i$ and $f(w_i) = X_i \cup \{4\}$ for $1 \le i \le 7$. We now use the vertex labeling f to define an edge coloring of G. In particular, we define $c : E(G) \to \mathcal{P}^*([4])$ by $c(xy) = f(x) \cup f(y)$ for each edge $e = xy$ of G. From the manner in which the edge coloring c is defined, it follows that $f(x) \subseteq c'(x)$ for each vertex x of G. We claim that $c'(x) = f(x)$ for each vertex x of G. Since w_i ($1 \le i \le 7$) is an end-vertex of G, it follows that $c'(w_i) = c(w_iu_i) = f(w_i) \cup f(u_i) = (X_i \cup \{4\}) \cup X_i = X_i \cup \{4\} = f(w_i)$. Thus, we may assume that $x = u_i$ or $x = v_i$ for some integer i with $1 \le i \le 7$.

- ⋆ First, suppose that $x = u_i$ where $1 \le i \le 7$. Then $f(u_i) \subseteq [3]$. If $f(u_i) = [3]$, then $c(u_iu_j) = c(u_iv_j) = [3]$ for all integers j with $1 \le j \le 7$ and $j \ne i$ and so $c'(u_i) = [3]$. Next, suppose that $f(u_i) \subset [3]$. For each integer $\ell \in [3] - f(u_i)$, let $t \in [3] - \{i, \ell\}$. Then $f(u_t) = \{t\}$. It follows from the definition of $c'(u_i)$ that $c'(u_i) \subseteq c(u_iu_t) = f(u_i) \cup f(u_t) = f(u_i) \cup \{t\}$. Since $\ell \notin f(u_i) \cup \{t\}$, it follows that $\ell \notin c'(u_i)$. Because $f(u_i) \subseteq c'(u_i)$ and, for each $\ell \in [3] - f(u_i)$, we have $\ell \notin c'(u_i)$, it follows that $c'(u_i) = f(u_i)$ for $1 \le i \le 7$.
- ⋆ Next, suppose that $x = v_i$ where $1 \le i \le 7$. Then $f(v_i) \subseteq [3]$. If $f(v_i) = [3]$, then $c(v_iu_j) = [3]$ for all integers j with $1 \le j \le 7$ and $j \ne i$. Since $c(v_iw_i) = f(v_i) \cup \{4\} = [4]$, it follows that $c'(v_i) = [3]$. Next, suppose that $f(v_i) \subset [3]$. For each integer $\ell \in [3] - f(v_i)$, let $t \in [3] - \{i, \ell\}$. Then $f(u_t) = \{t\}$. It follows from the definition of $c'(v_i)$ that $c'(v_i) \subseteq c(v_iu_t) = f(v_i) \cup f(u_t) = f(v_i) \cup \{t\}$. Since $\ell \notin f(v_i) \cup \{t\}$, it follows that $\ell \notin c'(v_i)$. Because $f(v_i) \subseteq c'(v_i)$ and, for each $\ell \in [3] - f(v_i)$, we have $\ell \notin c'(v_i)$, it follows that $c'(v_i) = f(v_i)$ for $1 \le i \le 7$.

Hence, c' is proper and so c is a regal 4-edge coloring of G, implying that $\text{reg}(G) \le 4$. Therefore, $\text{reg}(G) = 4$. ■

From the results obtained in this section, we are left with the following question.

Problem 3.9 *Does there exist a connected graph G having chromatic number 4, 5, or 6 such that $\text{reg}(G) \ne 3$?*

References

[1] G. Chartrand, C. Egan, and P. Zhang, *How to Label a Graph.* Springer, New York (2019).

[2] C. Bazgan, A. Harkat-Benhamdine, H. Li and M. Woźniak, On the vertex-distinguishing proper edge-colorings of graphs. *J. Combin. Theory Ser. B.* **75** (1999) 288-301.

[3] A. C. Burris and R. H. Schelp, Vertex-distinguishing proper edge colorings. *J. Graph Theory.* **26** (1997) 73-82.

[4] G. Chartrand, J. Hallas and P. Zhang, Royal colorings in graphs. Preprint.

[5] G. Chartrand, J. Hallas and P. Zhang, Color-induced graph colorings. Preprint.

[6] G. Chartrand and P. Zhang, *Chromatic Graph Theory.* Chapman & Hall/CRC Press, Boca Raton (2009).

[7] J. Hallas, I. Hart and P. Zhang, Sum-defined colorings in graphs. *J. Combin. Math. Combin. Comput.* **106** (2018) 291-308.

[8] F. Harary and M. Plantholt, The point-distinguishing chromatic index. *Graphs and Applications.* Wiley, New York (1985) 147-162.

[9] I. Hart, *Induced Graph Colorings.* Ph.D. Dissertation, Western Michigan University (2018).

[10] I. Hart and P. Zhang, Majestic 2-tone colorings of graphs. *Ars Combin.* To appear.

[11] I. Hart and P. Zhang, Majestic t-tone colorings of graphs. *Congr. Numer.* **228** (2017) 141-154.

[12] I. Hart and P. Zhang, Majestic t-tone colorings of bipartite graphs with large cycles. Preprint. *J. Combin. Math. Combin. Comput.* To appear.

[13] P. Zhang, *Color-Induced Graph Colorings.* Springer, New York (2015).

[14] P. Zhang, *A Kaleidoscopic View of Graph Colorings.* Springer, New York (2016).

[15] Z. Zhang, L. Liu and J. Wang, Adjacent strong edge coloring of graphs. *Appl. Math. Lett.* **15** (2002), 623–626.