SUPERUSER PANSOPHY

JEFFE BOATS, ARMANDO BRUCAJ, LAZAROS KIKAS

ABSTRACT. Given a graph G, we are interested in finding disjoint paths for a given set of distinct pairs of vertices. In 2017, we formally defined a new parameter, the pansophy of G, in the context of the disjoint path problem. In this paper, we investigate the pansophy of two classes of graphs that contain a vertex that we define as the superuser. The superuser of a graph is a vertex that is adjacent to every other vertex. We close with future research directions.

Keywords: Pansophy, interconnection networks, graphs, algorithms, vertex disjoint paths

1. INTRODUCTION

In 2017, Boats and Kikas introduced a new parameter, the pansophy of G, as a new measure of performance of graphical structures serving as communication networks. [2]. This idea was motivated by the $k-$Disjoint Path Problem: given k pairs of vertices (s_1, t_1), (s_2, t_2),...,(s_k, t_k), do there exist disjoint paths in the graph for each pair of vertices? If we can find $k-$disjoint paths for any collection of k pairs of vertices, then we say that G satisfies the **k-Disjoint Path Property**. Much has been studied about this problem. For example, in 2004 [1] [3] showed that the alternating group graph, A_n, has the $(n-2)$-Disjoint path property for $n \geq 5$.

More generally, suppose that G has the k-disjoint path property. If we randomly select more than k distinct pairs of vertices, there is no guarantee that we can successfully route the pairs disjointly, but we can consider the *probability* of being able to do so. This is the motivation behind pansophy. Given a random set of vertex pairs, the **pansophy** of a graph is the expected value of the number of disjoint paths that we can route. [2]

In a previous paper we formally defined the pansophy of a graph and derived a formula for computing it. We also computed the pansophy for several examples of classes of graphs, specifically the path graphs P_n and the cycles C_n. [2]

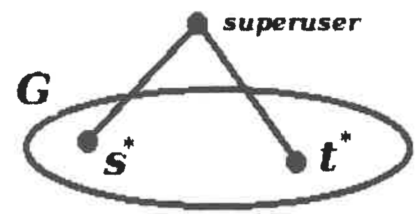

FIGURE 1. A 1-jaunt

2. Motivation and Definitions

We come to the motivation for this paper. Suppose we have a computer network modeled by some connected graph G. We now consider a computer represented by the graph K_1. This computer's purpose is to communicate and to assist processes of the processors in graph G. We model this situation by the graph $G^* = G + K_1$. That is, we form G^* by joining graph G with the graph K_1. We call the vertex of K_1 the **superuser** vertex. In this paper we derive formulas for the pansophy of the graphs P_n^* and C_n^*.

Before proceeding let us first introduce some terminology. Suppose that G is a connected graph and consider the graph G^*. Let s_1, t_1 be a pair of vertices in G. A **jaunt** from s_1 to t_1 is a two-edge path which routs through a vertex external to G (in this case the superuser). In our proof, we will denote ϕ_k^0 as the probability of completing the k disjoint paths with 0 jaunts, and with the superuser having been one of the assigned vertices. We similarly define ϕ_k^1 as the probability of completing the k disjoint paths using at most one jaunt through the initially unassigned superuser. See Figure 1.

Suppose that G is a connected graph with n vertices. Then G^* has $n+1$ vertices and so to compute the pansophy of of G^* we use the following formula derived in [2]

$$\psi(G^*) = \sum_{k=1}^{\Omega} p_k$$

where $\Omega = \lfloor \frac{n+1}{2} \rfloor$ and p_k is the probability of being able to route disjointly the randomly selected k pairs of vertices. Since G^* is connected, we see automatically that $p_1 = 1$.

To compute p_k for $k \geq 2$, we express p_k by:

$$p_k = \alpha \, \phi_k^0 + (1 - \alpha) \, \phi_k^1$$

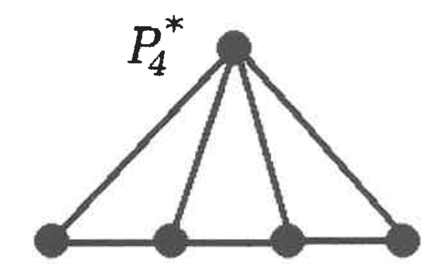

FIGURE 2. P_4^*

where α is the probability that the superuser is a vertex of a randomly selected pair.

3. Pansophy of P_n^*

We first work out a specific example. Consider the graph P_4^*. See Figure 2. Note that here we can only route up to two pairs. Since our graph is connected, we know that $p_1 = 1$, but we must compute p_2.

Let our pairs (s_1, t_1), (s_2, t_2) be given, and note there are $\binom{5}{4} = 5$ ways to assign the pairs to the vertices of the graph. Hence the probability that the superuser is being assigned is $\frac{4}{5}$; it is unassigned with probability $\frac{1}{5}$. Hence:

$$p_2 = \frac{4}{5} \, \phi_2^0 + \frac{1}{5} \, \phi_2^1.$$

Case 1: Suppose that s_1 is assigned to the superuser. Note there are $3! = 6$ ways to permute the other vertices within the remaining graph, which is isomorphic to P_4. In order to complete both paths, the vertices t_1, s_2, and t_2 must have t_2 in the first or third position. There are four such permutations, so the probability of completing the paths with the superuser assigned is $\phi_2^0 = \frac{4}{6} = \frac{2}{3}$.

Case 2: Suppose that the superuser is not assigned. Then the pairs are distrubuted the path P_4, covering it entirely. Note that there are $4! = 24$ ways to do this. To route both paths, whichever of the vertices s_1, t_1, s_2, or t_2 is on an end, its mate must be either adjacent or on the other end. This happens with probability $\frac{1}{3}$, so $\phi_2^1 = 1 - \frac{1}{3} = \frac{2}{3}$.

We conclude that

$$p_2 = \frac{4}{5}\,(\frac{2}{3}) + \frac{1}{5}\,(\frac{2}{3}) = \frac{2}{3}.$$

Therefore

$$\Psi(P_4^*) = p_1 + p_2 = 1 + \frac{2}{3} = \frac{5}{3} \approx 1.667\ . \ \square$$

Now consider the graph P_n^*. We seek a formula for p_k for $k \in \{1, 2, 3, ...\Omega\}$. If we randomly assign $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ to the vertices of P_n^*, then there is a

$$\frac{2k}{n+1}$$

probability that the superuser vertex is selected, so:

$$p_k = \frac{2k}{n+1}\ \phi_k^0 + (1 - \frac{2k}{n+1})\ \phi_k^1.$$

We consider two separate cases:

(1) **Case 1:** We compute ϕ_k^0. Suppose that s_1 is the superuser. We place the objects $t_1, s_2, t_2, s_3, t_3, ..., s_k, t_k$ on the path P_n. There are $(2k-1)!$ ways to arrange these vertices. To complete the paths the vertices must of the form where t_1 is placed somewhere on P_n along with the adjacent pairs (s_i, t_i) randomly placed. Note that there are k ways to place t_1. Once t_1 is placed, the pairs (s_i, t_i) fall into place, and there are $(k-1)!$ ways to arrange them. Within each of the $k-1$ adjacent pairs, there are two ways to arrange them. Hence:

$$\phi_k^0 = \frac{k(k-1)!2^{k-1}}{(2k-1)!} = \frac{k}{(2k-1)!!}.$$

(2) **Case 2:** Here we compute ϕ_k^1. In this case, the superuser has not been assigned, So the pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ are randomly distributed along P_n. Note that there are $(2k)!$ ways to permute these vertices. The superuser vertex can be used to complete a path if needed. If the permutation of the vertices is a permuation of the pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$, then the paths can be completed without the use of the superuser. Note that there are $k!\ 2^k$ such permutaitons. So the probability of completing the paths without the superuser is;

$$\frac{k!\ 2^k}{(2k)!}.$$

Now we consider permutations of the vertices $s_1, t_1, s_2, t_2, ..., s_k, t_k$ along P_n where the superuser must be used to complete the paths.

One example of such a permutation would be:

$$s_1(s_1, t_2)(s_4, t_4)(s_3, t_3)t_1(s_5, t_5)...(s_k, t_k).$$

How many such permutations are there? Let s_*, t_* be the pair where the vertices s_* and t_* are broken off from each other. There are $\binom{k}{2}$ ways to place the vertices s_*, t_* and two ways to permute them once placed. Note that there k ways to select which pair is broken up. Once the pair s_*, t_* is placed, the remaining adjacent pairs are automatically placed in the permutation. There are two ways to permute the vertices within each pair, and $(k-1)!$ ways to place these pairs within P_n. So in total there are:

$$2k\binom{k}{2}(k-1)!\ 2^{k-1}$$

such permutations. This simplifies to

$$2^k\ k!\ \binom{k}{2}.$$

The probability of needing the superuser to complete the paths is :

$$\frac{2^k\ k!\ \binom{k}{2}}{(2k)!}.$$

Hence

$$\phi_k^1 = \frac{k!\ 2^k}{(2k)!} + \frac{2^k\ k!\ \binom{k}{2}}{(2k)!}.$$

This simplifies to

$$\phi_k^1 = \frac{k^2-k+2}{2(2k-1)!!}.$$

So

$$p_k = \frac{2k}{n+1}\left(\frac{k}{(2k-1)!!}\right) + \left(1 - \frac{2k}{n+1}\right)\left(\frac{k^2-k+2}{2(2k-1)!!}\right).$$

We have thus proved the following theorem:

Theorem 3.1. *Consider the graph P_n^*. Let $\Omega = \lfloor\frac{n+1}{2}\rfloor$. Then the pansophy of P_n^* is*

$$\Psi(P_n^*) = \sum_{k=1}^{\Omega}\left\{\frac{2k}{n+1}\left(\frac{k}{(2k-1)!!}\right) + \left(1 - \frac{2k}{n+1}\right)\left(\frac{k^2-k+2}{2(2k-1)!!}\right)\right\}.$$

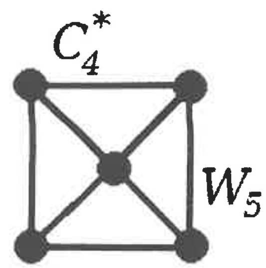

FIGURE 3. C_4^*

4. Pansophy of C_n^*

In this section we derive a formula for the pansophy of C_n^*. Note that C_n^* is the class of wheel graphs. We start with a specific example and work out a general formula. Consider C_4^*. See Figure 3. This graph has 5 vertices, and therefore we can route up to two disjoint paths. Since C_4^* is connected we have $p_1 = 1$, so again we focus on computing p_2. Let (s_1, t_1), (s_2, t_2) be our pairs. In computing p_2 we write

$$p_2 = \frac{4}{5}\,\phi_2^0 + \frac{1}{5}\,\phi_2^1.$$

(1) **Case 1:** We compute ϕ_2^0. Suppose that s_1 is the superuser. The vertices t_1, s_2 and s_3 are then randomly placed on the cycle C_4. Since s_1 and t_1 are adjacent we can complete the path from s_1 to t_1. This path never blocks any path from s_2 to t_2. Therefore, it is guaranteed that we can complete the paths, so $\phi_2^0 = 1$.

(2) **Case 2:** We compute ϕ_2^1. Here the superuser is not assigned, hence our pairs are distributed randomly on the cycle C_4, completely covering it. If we fix s_1 at a vertex of C_4 we notice that there are $3! = 6$ ways to arrange s_2, t_2, and t_1. Of these six, the four which allow completion of both paths are those for which t_2 is on an end. Hence $\phi_2^1 = \frac{4}{6} = \frac{2}{3}$.

So

$$\psi(C_4^*) = p_1 + p_2 = 1 + \frac{4}{5}\,(1) + \frac{1}{5}\,(\frac{2}{3}) = \frac{29}{15} \approx 1.9333\ .\ \square$$

Now consider the graph C_n^*. We have automatically that $p_1 = 1$; we must work out the general formula for p_k, $k \geq 2$.

Let the pairs $(s_1,t_1), (s_2,t_2), \ldots, (s_k,t_k)$ be randomly assigned to C_n^*. There is $\frac{2k}{n+1}$ chance that the superuser is assigned; the probability it is not assigned is thus $1 - \frac{2k}{n+1}$. As before we write p_k as:

$$p_k = \frac{2k}{n+1}\,\phi_k^0 + \left(1 - \frac{2k}{n+1}\right)\,\phi_k^1\ .$$

(1) **Case 1:** We compute ϕ_k^0. Assume that s_1 is assigned to the superuser. Notice that s_1 is directly adjacent to t_1. The remaining vertices are assigned to the remaining part of the graph, which is isomorphic to P_{n-1}. There are $(2k-2)!$ ways to arrange the remaining vertices. Of these permutations, only those for which all pairs are adjacent will allow the completion of all paths. There are $(k-1)!\,2^{k-1}$ such permutations, so:

$$\phi_k^0 = \frac{(k-1)!\,2^{k-1}}{(2k-2)!} = \frac{1}{(2k-3)!!}.$$

(2) **Case 2:** We compute ϕ_k^1. Here the superuser is not assigned, so we randomly place our pairs of vertices onto the cycle C_n. We fix s_1 and consider the $(2k-1)!$ permutations of the remaining vertices. Consider the permutations of the k symbols:

$$t_1, (s_2,t_2), (s_3,t_3), \ldots, (s_k,t_k)\ .$$

For any one of these permutations, either the pairs are all adjacent and one can complete the routes entirely within C_n, or there is one non-adjacent pair to be routed through the superuser. There are

$$k!\,2^{k-1}$$

such permutations. Hence,

$$\phi_k^1 = \frac{k!\,2^{k-1}}{(2k-1)!} = \frac{k}{(2k-1)!!}$$

So our formula for p_k is

$$p_k = \frac{2k}{n+1}\frac{1}{(2k-3)!!} + \left(1 - \frac{2k}{n+1}\right)\frac{k}{(2k-1)!!}.$$

Thus we have proven the following theorem:

Theorem 4.1. *Consider the graph C_n^*. Let $\Omega = \lfloor \frac{n+1}{2} \rfloor$. Then the pansophy of C_n^* is*

$$\Psi(C_n^*) = \sum_{k=1}^{\Omega} \left\{ \frac{2k}{n+1}\frac{1}{(2k-3)!!} + \left(1 - \frac{2k}{n+1}\right)\frac{k}{(2k-1)!!} \right\}\ .$$

5. Future Directions

In this paper we investigated the pansophy of two classes of graphs that include a superuser vertex. Future directions would include investigations into other classes of graphs. But a more general inquiry would involve the relationship between the pansophy of G and G^*. It is clear that the pansophy of G would not decrease with the addition of a superuser. So $\psi(G^*) \geq \psi(G)$. But can we say in all cases that $\psi(G^*) > \psi(G)$? If not, then under what conditions do we have equality?

Another investigation may include the joining of a graph with two or more superusers. More generally, suppose that we have two graphs G and H with known pansophy. Is there a relationship between the pansophies of G, H and $G + H$?

References

[1] E.Cheng, L.D. Kikas, and S.Kruk. A disjoint path problem in the alternating group graph. *Congressus Numerantium*, 175:117-159, 2005.

[2] J.Boats and L.D.Kikas. The pansophy of a graph. *Congressus Numerantium*, 229:125-134, 2017.

[3] Lazaros D. Kikas. *Interconnection Networks and the k-disjoint path property.* PhD thesis, Oakland University, 2004.

Lazaros Kikas, Corresponding Author, Department of Mathematics, University of Detroit Mercy, Detroit, MI, 48221-3038, USA
E-mail address: `kikasld@udmercy.edu`

Jeffe Boats, Department of Mathematics , University of Detroit Mercy, Detroit, MI, 48221-3038, USA
E-mail address: `boatsjj@udmercy.edu`

Armando Brucaj, Student, Deparment of Mathematics, University of Detroit Mercy, Detroit, MI,48221-3038, USA