EDGE-NIM ON THE $K_{2,n}$

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ABSTRACT. Edge-Nim is a combinatorial game played on finite regular graphs with positive, integrally weighted edges. Two players alternately move from an initialized vertex to an adjacent vertex, decreasing the weight of the incident edge to a strictly non-negative integer as they travel across it. The game ends when no incident edge has a nonzero weight and a player is unable to move, in which case, this player loses. We characterize the winner of edge-Nim on the complete bipartite graphs, $K_{2,n}$ for all positive integers, n, giving the solution and complete strategy for the player able to win.

Nim is an ancient game that has grown vastly in popularity since Fukuyama extended game play to graphs [7], opening the door to an immense area of study. The number of variants of game play and the many classes of graphs alone give rise to a wealth of problems in the area. Here, we focus on extending previous work to completely characterize the positional values and winner of edge-Nim on complete bipartite graphs with arbitrary weight.

Within this analysis, we first describe game theory and the general game of Nim in the Background section. Next, we move to an outline of the basic definitions of graph theory before introducing the game of Nim on graphs. Finally, we provide a set of seven lemmas, theorems, and corollaries that collectively give the complete solution to edge-Nim on the $K_{2,n}$ graph.

1. BACKGROUND

Before getting into the solution to complete bipartite graphs, we first give a brief introduction to game theory, including relevant definitions and basic assumptions for all combinatorial games of perfect information. We then recall the history of the general game of Nim and positional values. Following that, we move into the game of Nim on graphs.

1.1. Game Theory.

Nim is a combinatorial game of perfect information. Combinatorial games involve players who make decision problems. In this section, we break down all of the necessary definitions.

Definition 1.1. *A* ***player*** *in a game makes decisions and gains reward* [9].

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A reward in this context can be as simple as the satisfaction of winning the game and is not necessarily monetary. Furthermore, in games of perfect information, there are no unexpected occurrences. Every possible change of state is known.

Definition 1.2. *A* ***decision problem*** *is a choice between different sets of possible alternatives* [9].

Decision problems have two prerequisites. The player must have an adequate understanding of the decision's ramifications, and the player must have a preference, as the choice's significance is established in the preferences of the player. Decision problems also assume rationality. The player will select a winning strategy, provided one exists.

Combinatorial game theory models conditions with the following characteristics:

(1) A game consists of two or more players.
(2) Games start by at least one player selecting an option from a given set.
(3) The first choice causes a new situation to arise.
(4) Depending on the conditions of the game, it is possible that the decisions made by the players may not become known.
(5) Each choice can be defined as a move, and a termination move exists.
(6) Each game ends in a situation.

Definition 1.3. *A* ***strategy*** *is the set of choices for all possible situations that could occur during the course of a game* [3].

At the most fundamental level, a strategy is just a set of decisions. Furthermore, by the aforementioned assumption of rationality, a player will proceed with the best strategy available.

Definition 1.4. *A game in which every player knows the total move history is one of* ***perfect information*** [10].

An example of such a game is chess. On the contrary, a card game where players hide hands from each other is imperfect. The following characteristics define two-player games of perfect information.

(1) The game contains two players, P_1 and P_2.
(2) Moves are changes in position and follow a fixed beginning position.
(3) The game contains restrictions that define the possible moves a player can make from a given position.
(4) The players make successive moves.
(5) The player unable to move loses. Note that there is also the *misère* form of games where the player unable to move wins.
(6) Any game must end in a state in which some player is unable to move as to prevent a tie.

(7) There are no chance moves. [1]

1.2. **General Nim.** Nim is a two-player game where players take turns making moves from three or more piles of stones. In Nim, a move consists of first choosing a pile of stones and then selecting a positive number of stones for removal. During a turn, a player can only take stones from an individual pile. To win, one must take the last stone or stones from play [2].

The game of Nim has been traced to ancient China. However, its name appears to have roots in Germany; *nimm* means "to take" in German. Analyzing mathematically, Charles Bouton was the first to study Nim, publishing *Nim, a game with complete mathematical theory* at the beginning of the twentieth century [2]. In this study, Bouton defined Nim addition, 0-positions, p-positions, and three-pile "safe combinations."

Definition 1.5. *If P_1 can win from a given position, regardless of any move made by the second player, the position is called a* ***p-position.*** *If the second player can win from a given position, regardless of any move made by the first player, the position is called a* ***0-position*** [7].

An important characteristic of p- and 0-positions is that if a player is on a p-position, there is at least one move to a 0-position. If a player is on a 0-position, all moves are to p-positions. In this way, a player with a winning strategy is guaranteed to be able to keep the winning strategy, provided the player knows the 0-positions of the game.

2. The Game of Nim on Graphs

The interested reader can find basic graph theory definitions in [4], among others. We reiterate some terms used commonly throughout the paper.

For $k \geq 1$, a graph G is called *k-partite* if $V(G)$ can be subdivided into k subsets V_1, V_2, V_3, ..., V_k. These partite sets satisfy the condition that every element of $E(G)$ connects V_i to V_j where $i \neq j$. A graph is *bipartite* if $k = 2$.

Definition 2.1. *A* ***complete bipartite graph****, with partite sets V_1 and V_2, such that $|V_1| = r$ and $|V_2| = s$, is denoted $K_{r,s}$, and also satisfies the condition if $u \in V_1$ and $v \in V_2$ then $uv \in E(K_{r,s})$* [11].

Definition 2.2. *A uv-**walk** is a finite sequence of vertices and edges that starts with vertex u and terminates with vertex v and is such that*

$$u = u_0, e_1, u_1, e_2, u_2, u_{k-1}, e_k u_k = v.$$

*A uv-**trail** is a walk in which no edge is traversed twice or more. A uv-**path** is a trail that does not traverse any vertex more than once* [4].

We call a path *even* (respectively, *odd*) if the number of edges in the path is even (respectively, odd).

Definition 2.3. *A uv-path is a* **cycle** *when* $u = v$ [4]. *A cycle on n vertices is denoted by* C_n.

For the purpose of this paper, all graphs are simple, finite, and undirected. Vertices are named v_1, v_2, v_3, ... with the edges between these vertices called e_{12}, e_{23}, ... Paths will be declared even or odd according to the number of edges present.

2.1. **Playing Nim on Graphs.** Nim can be extended to graphs in a few different ways by assigning weight to either each edge or vertex and defining moves therein. The focus of this paper is edge-Nim, so that the piles of stones of the general game of Nim correspond to "weight" on the edges of a predetermined graph. Here we describe the general definitions for playing the game of edge-Nim on graphs before describing solutions to basic situations such as paths and even cycles.

Definition 2.4. *The* **weight** *of each edge is defined by assigning a positive, integral value to each* $e \in E(G)$. *The weight of edge* e_{ij} *is denoted* $\omega(e_{ij})$ [11].

A graph with *unit weight* has $\omega(e_{ij}) = 1$ for all $e \in E(G)$.
Assume that $\omega(e_{ij}) \neq 0$ for all $e \in E(G)$. Let Δ denote the game's current position. The game begins when P_1 selects an edge incident with the vertex upon which Δ resides to move across, removes a positive, integral amount from the weight of this edge, and transfers the position of Δ to the incident edge. After this, P_2 proceeds similarly. Once an edge's weight has reached zero, the edge is no longer in play, analogous to an empty pile of stones. To depict this, the edge is simply be deleted from the graph, symbolizing that it is no longer playable. The game continues until Δ's position is transferred to a vertex that has no incident edges of nonzero weight after the move has been completed. The player unable to move loses.

Definition 2.5. *A* **uniform** n-**path** *is a path in which the n edges of the path have same weight.*

A nonuniform path is one that must have at least one edge weighted differently from the rest.

Definition 2.6. *For either player, the set of vertices adjacent to the position* Δ *is called the set of* **options**. *For* P_i, *the set of options at vertex* v_j *is denoted* $O(P_i, v_j)$ [5].

Definition 2.7. *The* **choice** *of the player is the selection of how much weight to remove from any edge* e_{ij} *when at the position* Δ *at a vertex* v_j [5].

Recall that the neighborhood of a vertex v_i is defined to be the set of all adjacent vertices to v_i.

Definition 2.8. P_i *has a pair of* **isomorphic** *options* $v_j, v_k \in O(P_i, v_i)$ *if, between* v_j *and its neighbors and* v_k *and its neighbors, there exists a graph isomorphism* [6].

In other words, there exists a bijection between the neighborhoods of v_j and v_k. Additionally, two options are **identical** if they are both isomorphic and the edges from v_i to every $v_j \in O(P_i, v_i)$ are weighted the same.
Here, the term *option* is only used when referring to the destination vertices. *Choice* will signify the weight removed from an option's incident edge during P_i's move. Thus, each move consists of both an option and a nontrivial choice (assuming the $\omega(e) > 1$).

2.2. **Nim on Graphs Essentials.** When a player, given a particular initialization, is on a known p-position, it is said that the player can win the graph, so the graph is ***winnable*** for a particular player and under the initialized state. If no initial vertex is assigned and the graph is said to be winnable, it is assumed that the graph is winnable for the winning player, regardless of the starting vertex.
It is straightforward to show that P_1 wins Nim on an odd path, and P_2 wins Nim on an even path given that Δ is initialized at a vertex of degree one. Supposing that G is a path, then given any other initialization of Δ, P_1 will win if there is at least one odd path option from the initialization, whereas P_2 will win if both paths are even.

2.3. **The Even Cycle Strategy for Nim on Graphs.** Given a graph, C_{2n}, the Even Cycle Strategy [8] finds the positional value for the C_{2n} under the given initialization and describes how to ensure that the player on a p-position continues to moves to 0-positions. First, notice that on any even cycle, if one player eliminates an edge before the other, the result is an odd path for the next player. Since an odd path is a first player win, the player to eliminate an edge on an even cycle will lose. Thus, all optimal moves from both players attempt to avoid eliminating an edge.
Let $G = C_{2n}$. Then P_1 under a given initialization is on a p-position if there is at least one odd path to a vertex incident with an edge of minimum weight. If all paths to a vertex incident with an edge of minimum weight are even, then P_1 is on a 0-position.
Supposing further that P_1 is on a p-position, the winning strategy is for P_1 to move along the odd path to the vertex incident with an edge of minimum (initial) weight and lower the weight of the edge traversed to equal the current game minimum weight. If and when P_2 lowers beyond the initialized minimum, P_1 then continues along the new odd path created, lowering each edge to the new minimum weight. If P_2 ever fails to match the

reduction to minimum weight, P_1 will "go back" over that edge, lowering to the minimum for an even faster victory. Continuing to lower to the current game minimum ensures that eventually P_2 will remove an edge.

3. A Study of Nim on the $K_{2,n}$

We now present seven distinct situations for initialization and weighting on the $K_{2,n}$ graph that collectively describe the complete solution to edge-Nim on this class of graphs.

Lemma 3.1. *Let $G = K_{2,n}$. Let Δ be initialized in the partite set of size 2. P_1 wins the arbitrarily-weighted $K_{2,3}$ if at least one C_4 is winnable for P_1 from either vertex in the partition of size 2.*

Proof. Enumerate the vertices as such: let v_1 and v_2 be the vertices in the partite set of size 2 and v_3, v_4, and v_5 be the vertices in the partite set of size 3 (see Figure 1).

Assume v_1, v_3, v_2, v_4 is a winnable 4-cycle for P_1 at v_1. Then this 4-cycle will terminate at v_3 or v_4 necessarily, as they are the only options for P_2. Thus P_1 retains the Even Cycle Strategy and $e_{15}, v_5,$ and e_{25} are not used.

Now assume Δ on v_1 but the induced 4-cycle subgraph is winnable only from v_2. P_1 moves to v_5 and removes all weight, forcing P_2 to make the same move on e_{25}. Here, a winnable C_4 exists for P_1 from v_2, so P_1 proceeds to win the graph with the Even Cycle Strategy. □

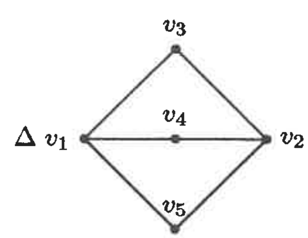

Figure 1. $K_{2,3}$

In most cases of randomly integrally weighted Nim on graphs, there exists a winnable 4-cycle for P_1. The following lemma addresses the cases where there is not a winnable C_4 for P_1 at v_1 or v_2. Specifically, there must be at least two uniform 2-paths and the weight of an edge from a non-uniform 2-path not incident with the starting position must be greater than the sum of an edge from each of the uniform 2-paths.

Lemma 3.2. *Let $G = K_{2,3}$ and let Δ be initialized in the partite set of size 2. Assume $\omega(e_{13}) > \omega(e_{23})$, $\omega(e_{14}) = \omega(e_{24})$, and $\omega(e_{15}) = \omega(e_{25})$. Then P_1 wins G if and only if $\omega(e_{23}) \leq \omega(e_{14}) + \omega(e_{15})$.*

Proof. Enumerate the vertices as shown in Figure 1. If $\omega(e_{14}) \geq \omega(e_{23})$ or $\omega(e_{15}) \geq \omega(e_{23})$, then there exists a winnable C_4 subgraph induced on v_1, v_2, v_3, v_4 or v_1, v_2, v_3, v_5, respectively, for P_1 from v_1. Thus P_1 wins the graph by the strategy outlined in Lemma 3.1.

In the case that both $\omega(e_{14})$ and $\omega(e_{15})$ are less than $\omega(e_{23})$, P_1 begins play by moving from v_1 to v_3 and reducing $\omega(e_{13})$ to $\omega(e_{23})$. Here, P_2 has 2 identical options, so we assume without loss of generality that P_2 moves to v_2 from v_3, reducing $\omega(e_{23})$ to m, where $m < \omega(e_{13})$.

Here, P_1 can move to either v_4 or v_5; without loss of generality, we assume that P_1 moves to v_4, reducing $\omega(e_{24})$ by 1. In this position, P_2 has two options: to move back to v_2 or on to v_1. In the case that P_2 moves back to v_2, a winnable C_4 subgraph induced on v_1, v_2, v_4, v_5 is created, which, by Lemma 3.1, is a win for P_1. In the case that P_2 moves on to v_1 and reduces $\omega(e_{14})$ to any weight except $\omega(e_{14}) - 1$, a winnable C_4 subgraph induced on v_1, v_2, v_4, v_5 is again created, resulting in a victory for P_1.

Thus, P_2 must move to v_1 with a best choice of reducing $\omega(e_{14})$ by 1. Now, P_1 moves to v_3 from v_1 and reduces $\omega(e_{13})$ to m matching $\omega(e_{23})$. From this point, P_1 forces P_2 to play exclusively on the 4-cycle induced on v_1, v_2, v_3, v_4 and mirrors P_2's weight reductions. Eventually, P_1 moves from v_1 to v_3, reducing $\omega(e_{13})$ with both edges incident on v_4 eliminated and both edges incident on v_3 having an equal weight of $\omega(e_{23}) - \omega(e_{14})$ or less. Because $\omega(e_{23}) - \omega(e_{14}) \leq \omega(e_{25})$, P_2 is on a 0-position. From here, P_1 uses the Even Cycle Strategy to win the graph.

If $\omega(e_{23}) > \omega(e_{14}) + \omega(e_{15})$, then P_1 will lose. P_1 moves from v_1 to v_3, reducing $\omega(e_{13})$ by some amount. P_1 would not move to either v_4 or v_5, as P_2 will match the weight of e_{24}. This puts P_1 back to a $K_{2,n}$ with two uniform 2-paths and one nonuniform 2-path from v_1 to v_2 through the partition of size n. If P_1 does not reduce $\omega(e_{13})$ to $\omega(e_{23})$, P_2 will reduce $\omega(e_{23})$ to the weight of $\omega(e_{13})$. This move creates a $K_{2,n}$ with all uniform 2-paths, which in unwinnable from the partition of size 2 for the first player. The graph is then played out under the same tit-for-tat strategy as above. P_1 will be force to take all weight from either e_{24} or e_{25}, and P_2 taking all weight from e_{14} or e_{15}, respectively, afterwards. This creates a situation in which P_1 is stuck on an unwinnable C_4 and loses the graph. Thus P_1 wins the graph only if $\omega(e_{23}) \leq \omega(e_{14}) + \omega(e_{15})$. □

Next, we generalize the proof from Lemma 3.1 to the $K_{2,n}$ graph.

Theorem 3.3. *For all $n \geq 3$, P_1 wins the $K_{2,n}$ with Δ initialized in the partition of size 2 if there exists a C_4 subgraph winnable for P_1 from either vertex in the partite set of size 2.*

Proof. We proceed by induction on n. Initialize Δ at v_1 and let $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4, \ldots, v_{n+2}\}$. Additionally, assume that Δ is initialized at v_1 (see Figure 3). For the base case of $n = 3$, by Lemma 3.1 P_1 wins under the given conditions of the graph.

Now assume that P_1 wins the $K_{2,k}$ for all $3 \leq k \leq n$, that Δ is initialized in the vertex set of size 2, and there is a C_4 winnable for P_1 from v_1 or v_2. Consider the $K_{2,n+1}$ with Δ initialized at v_1, $V_1 = \{v_1, v_2\}$, and $V_2 = \{v_3, v_4, \ldots, v_{n+3}\}$. If there exists a winnable C_4 for P_1 at v_1 on, without loss of generality, the vertices v_1, v_2, v_3, and v_4, then P_1 moves to, again without loss of generality, v_5 and removes all weight on e_{15}. P_2 must necessarily move to v_2 and remove all weight on e_{25} If P_2 does not remove all weight on e_{25}, P_1 moves back to v_5 on the next move and removes all weight for the win. P_1 moves to v_6, without loss of generality, and removes all weight on e_{26}, forcing P_2 to move back to v_1, removing all weight on e_{16}. We now have P_1 on v_1 with a winnable C_4 under the constraints of the assumption on the $K_{2,n-1}$, which, by the inductive hypothesis, is a P_1 win.

If the winnable C_4 starting on v_2 is induced on the vertices v_1, v_2, v_3, v_4 with Δ still initialized on v_1, then P_1 moves to v_5, without loss of generality, and removes all weight on e_{15}, forcing P_2 to move on to v_2, removing all weight from e_{25}. In this case, P_1 is on a $K_{2,n}$ with a winnable C_4 on the initialized vertex, which, by inductive hypothesis, is a P_1 win.

Therefore, P_1 wins the arbitrarily-weighted $K_{2,n}$ with Δ initialized in V_1 and a C_4 subgraph winnable from one of the vertices in the partite set of size 2. □

The following theorem demonstrates what occurs when the $K_{2,n}$ consists of uniform 2-paths.

Theorem 3.4. *Let $G = K_{2,n}$ and Δ initialized at v_1 with $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4, \ldots, v_{n+2}\}$ such that $\omega(e_{1i}) = \omega(e_{2i})$ for $3 \leq i \leq n+2$. If Δ is initialized in V_1, then P_2 wins G.*

Proof. We proceed by induction on n. Let Δ be initialized on v_1, and let v_2 be the other vertex in V_1. Enumerate the vertices in V_2 as $v_3, v_4, \ldots, v_{n+2}$.

For $n = 1$ we have an even path, which is a P_2 win. Similarly, for $n = 2$ we have a C_4 in which $\omega(e_{13}) = \omega(e_{23})$ and $\omega(e_{14}) = \omega(e_{24})$. Since Δ is initialized at v_1, the Even Cycle Strategy proves this results in a P_2 win.

Assume for all $1 \leq k \leq n$ that G is enumerated under these weighting conditions and that the given initialization is winnable for P_2.

Consider the $K_{2,n+1}$ under the same conditions. Without loss of generality, let Δ be initialized on v_1. Suppose P_1 moves from v_1 to $v_j \in V_2$, lowering $\omega(e_{1j})$ to l. P_2 responds by moving from v_j to v_2, lowering $\omega(e_{2j})$ to l, as well. Necessarily, P_1 moves from v_2 to $v_k \in V_2$, which allows P_2

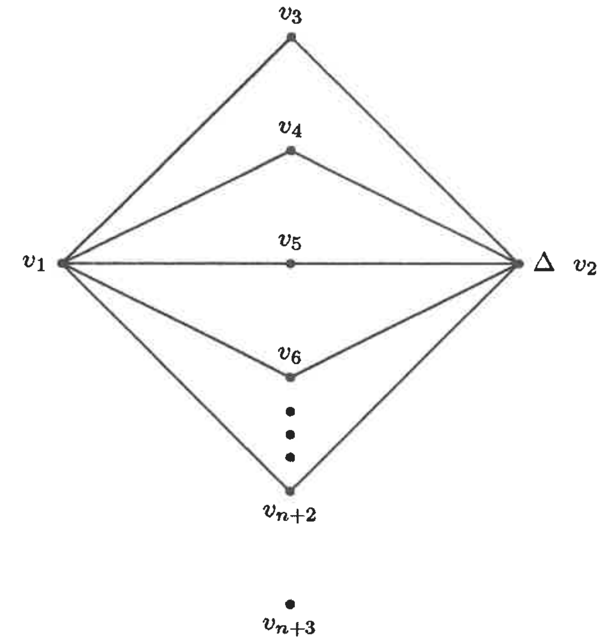

FIGURE 2. $G = K_{2,n+1}$ after both P_1 and P_2 eliminate an edge. This isolates a vertex, leaving a $K_{2,n}$.

to copy the weight reduction of P_1 to v_1. This strategy ensures that after every P_1-P_2 move combination, G is such that $\omega(e_{1j}) = \omega(e_{2j})$.

Eventually, P_1 is forced to lower an edge to weight 0. P_2 copies this weight reduction, and the resulting graph is a $K_{2,n}$ with Δ initialized in the partition of size 2, as shown in Figure 2. By the inductive assumption, then, this is a P_2 win.

Therefore, P_2 wins the $K_{2,n}$ where only uniform 2-paths traverse from v_1 to v_2 through the partition on size n. □

Next, we address the $K_{2,n}$ when P_1 is initialized in the partition of size 2 and the graph contains all but one uniform 2-paths from v_1 to v_2 through the partition of size n. Such graphs initially give P_1 no winnable C_4 subgraphs from either v_1 or v_2.

Theorem 3.5. *Let $G = K_{2,n}$ be initialized and enumerated as in Figure 3. Let $\omega(e_{13}) > \omega(e_{23})$ and $\omega(e_{1i}) = \omega(e_{2i})$ for $4 \leq i \leq n+2$. In this case, P_1 wins G if and only if $\omega(e_{23}) \leq \omega(e_{24}) + \omega(e_{25}) + \ldots + \omega(e_{2,n+2})$.*

Proof. Initialize Δ and enumerate the vertices as shown in Figure 3. Assume that $\omega(e_{13}) > \omega(e_{23})$ and $\omega(e_{1,i}) = \omega(e_{2,i})$ for $4 \leq i \leq n+2$. By

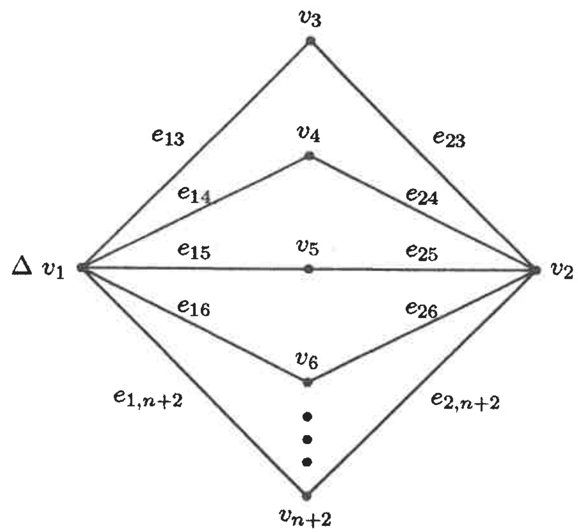

FIGURE 3. Enumeration of $K_{2,n}$

contrapositive, assume first that $\omega(e_{23}) > \omega(e_{24}) + \omega(e_{25}) + \ldots + \omega(e_{2,n+2})$. With Δ at v_1, all moves are isomorphic, but there are two distinct choices: (1) P_1 can move to v_3; (2) P_1 can move to v_k, $k > 3$.

In case (1), optimally P_1 reduces $\omega(e_{13})$ to equal $\omega(e_{23})$. After P_1's move, if $\omega(e_{13}) > \omega(e_{23})$ still, P_2 moves back to v_1, lowers the weight to match $\omega(e_{13})$, and the resulting graph is winnable for P_2 by Theorem 3.4. Likewise, if $\omega(e_{13}) < \omega(e_{23})$ after P_1's first move, P_2 moves on to v_2, lowers the weight to match $\omega(e_{13})$, and the resulting graph is again winnable for P_2 by Theorem 3.4.

Now with with Δ at v_3 and all 2-paths uniformly weighted, P_2 will move to, without loss of generality, v_2 and lower the weight of $\omega(e_{23})$ by 1. $(\star)$

Two nonisomorphic options result for P_1. First, if P_1 plays along e_{23}, P_2 will match the weight reduction on e_{13} so that once again, P_1 is left on a $K_{2,n}$ with all 2-paths uniformly weighted. Second, if P_1 plays along any of the e_{2k}, $k \geq 4$, then P_2 will again match the weight reduction of e_{2k} on e_{1k} coming to rest on v_1. If P_2 fails to match the weight reduction on e_{2k}, he loses the p-position as there is a winnable C_4 induced on v_1, v_3, v_2, v_4 for P_1 We are now back at the case of (1).

Since $\omega(e_{23}) > \sum_{k=4}^{n+2} \omega(e_{2k})$, either P_1 will exhaust the uniform 2-paths before the path $P = v_1, e_{13}, v_3, e_{23}, v_2$ is eliminated, in which case there is a final 2-path for a P_2 win, or P_1 will eliminate P leaving a $K_{2,n-1}$ with all 2-paths uniformly weighted, which again by Theorem 3.4 is a win for P_2.

For the case (2) where P_1 moves along any of the e_{1k}, $4 \leq k \leq n+2$, then P_2 matches P_1's weight on e_{1k} moving across e_{2k} and we are back at $(\star)$ of case (1).

For the other direction, assume that $\omega(e_{23}) \leq \sum_{k=4}^{n+2} \omega(e_{2k})$. In this case, P_2 will be forced to eliminate e_{23}, at which point there exists a 3-path for a P_1 win. The existence of the 3-path is guaranteed by the fact that $\omega(e_{23}) \leq \sum_{k=4}^{n+2} \omega(e_{2k})$. This inequality ensures that there are enough 2-paths to cover the possibility that P_2 removes weight 1 each time from e_{23}. □

The following corollary simply switches the weight assignment of the nonuniform 2-path from v_1 to v_2 through the partition of size n from Theorem 3.5.

Corollary 3.6. *Let Δ be initialized in the partite set of size 2. Consider a graph $G = K_{2,n}$ for $n \geq 3$ initialized and enumerated as in Figure 3. If $\omega(e_{13}) < \omega(e_{23})$ and $\omega(e_{1i}) = \omega(e_{2i})$ for $4 \leq i \leq n+2$, then P_1 wins G if and only if $\omega(e_{13}) \leq \omega(e_{14}) + \omega(e_{15}) + \ldots + \omega(e_{1,n+2}) - 1$.*

Proof. P_1 moves, without loss of generality, to v_2 across $e_{1,n+2}$, reducing the weight by 1. P_2 must do the same. We are now in the case of the graph of Theorem 3.5. □

Finally, we conclude with an analysis of the $K_{2,n}$ where Δ is initialized in the partition of size n. The winning strategies to these situations are largely informed by initialization in the partition of size 2.

Theorem 3.7. *Suppose $G = K_{2,n}$ is enumerated as shown in Figure 3 but initialized at v_3.*

(1) *If $\omega(e_{1i}) \neq \omega(e_{2i})$ and $\omega(e_{1j}) \neq \omega(e_{2j})$ for $i \neq j$, $4 \leq i,j \leq n+2$, then P_2 wins.*
(2) *If $\omega(e_{1i}) = \omega(e_{2i})$ for all $3 \leq i \leq n+2$, and if $\omega(e_{13}) - c > \omega(e_{14}) + \omega(e_{15}) + \ldots + \omega(e_{1,n+2}) - 1$, where c is the initial weight reduction of P_1, then P_1 wins. Otherwise, if $\omega(e_{13}) - c \leq \omega(e_{14}) + \omega(e_{15}) + \ldots + \omega(e_{1,n+2}) - 1$, then P_2 wins.*
(3) *If $\omega(e_{13}) \neq \omega(e_{23})$ and $\omega(e_{1i}) = \omega(e_{2i})$ for all $4 \leq i \leq n+2$, then P_1 wins.*
(4) *If $\omega(e_{14}) \neq \omega(e_{24})$ and $\omega(e_{1i}) = \omega(e_{2i})$ for all $3 \leq i \leq n+2$, where $i \neq 4$, then P_2 wins.*
(5) *Assume $\omega(e_{13}) \neq \omega(e_{23})$, $\omega(e_{14}) \neq \omega(e_{24})$, and $\omega(e_{1i}) = \omega(e_{2i})$ for all $5 \leq i \leq n+2$. Additionally, assume without loss of generality that $\omega(e_{14}) > \omega(e_{24})$. Then P_1 wins if and only if $\omega(e_{24}) > \omega(e_{23}) + \omega(e_{15}) + \omega(e_{16}) + \ldots + \omega(e_{1,n+2})$.*

Proof. Let $G = K_{2,n}$ with Δ initialized at v_3.

(1) Consider the C_4 subgraph induced on v_1, v_2, v_4, and v_5. This C_4 subgraph is winnable for P_2 from either vertex v_1 or v_2. If the C_4 is not winnable from v_1 for P_2, then $\omega(e_{14}) = \omega(e_{15}) < \omega(e_{24}), \omega(e_{25})$, which means that the C_4 subgraph is winnable for P_2 from v_2. Assume without loss of generality, then, that the C_4 subgraph is winnable for P_2 from v_1. If P_1 moves to v_1, then P_2 wins the graph by the strategy outlined in the second paragraph of Theorem 3.3. If P_1 moves to v_1, then P_2 wins the graph by the strategy outlined in the third paragraph of Theorem 3.3. As such, if more than two nonuniform 2-paths exist on the graph from v_1 to v_2 through the partition of size n, or exactly two nonuniform 2-paths exist on the graph but Δ is not initialized on one of them, then P_2 wins the graph.

(2) Let P_1 move to v_1 and lower $\omega(e_{13})$ by c. By Corollary 3.6, P_2 wins if and only if $\omega(e_{13}) - c \leq \omega(e_{14}) + \omega(e_{15}) + \ldots + \omega(e_{1,n+2}) - 1$.

(3) Assume without loss of generality that $\omega(e_{13}) < \omega(e_{23})$. P_1 moves to v_2 and lowers $\omega(e_{23})$ to $\omega(e_{13})$. Here, P_2 is left in the initial situation of Theorem 3.4, and hence P_2 loses.

(4) Let P_1 move to v_1 and lower $\omega(e_{13})$ by c. In this case, P_2 finds the same situation as described in Theorem 3.3, so P_2 wins the graph.

(5) Assume without loss of generality that $\omega(e_{13}) < \omega(e_{23})$. Let P_1 move to v_1 and lower $\omega(e_{13})$ to $\omega(e_{23})$. By Corollary 3.6, P_2 wins if and only if $\omega(e_{24}) \leq \omega(e_{23}) + \omega(e_{15}) + \omega(e_{16}) + \ldots + \omega(e_{1,n+2})$.

□

We conclude this study with a discussion of several questions that pertain to both alternate versions of play on the $K_{2,n}$ and strategies for edge-Nim on the wider class of multipartite graphs.

4. Open Questions

The analysis provided in this paper presented only a start to the study of Nim on complete bipartite graphs. Several questions remain unanswered, the largest of which is edge-Nim on the $K_{m,n}$ with arbirary weight. Altering the parameters of gameplay will create new research questions. For example, if we consider the misère form of Nim on the $K_{2,n}$ graph in which the first player unable to move wins, how does strategy for both players change? Alternatively, if we consider either of the vertex-Nim variations, how is the solution to edge-Nim on the $K_{2,n}$ affected?

Further, there remains much to be discovered regarding edge-Nim on the general class of complete multipartite graphs. For instance, can any parts of the solution to edge-Nim on the $K_{2,n}$ graph be generalized to all arbitrarily-weighted graphs of form $K_{m_1,m_2,\ldots,m_j}$? From our initial study of

the $K_{3,3}$ graph, such an easy generalization does not seem to exist, calling for a further study of Nim of this class of graphs.

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