# Claw-free Steinhaus Complements

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**Abstract:** Steinhaus graphs are a small (there are $2^{n-1}$ of them on $n$ vertices) but interesting family of graphs. They have been studied for over forty years and it has been shown that almost all graphs have certain properties if and only if almost all Steinhaus graphs have these properties. In this paper we find and count all the complements of Steinhaus graphs that are claw-free.

## Section 1: Introduction and History

Steinhaus in [21] posed as his first unsolved problem the following (which has been changed to the notation of this paper): given a string of 0s and 1s, create a new string by adding adjacent entries mod 2. In Figure 1 (ignoring the underlined 0s), the initial string is 0101, adding the adjacent entries we get the string 111, then 00 and finally 0. Steinhaus asked if there were strings so that the resulting triangle would have an equal number of 0s and 1s, as does the string 0101. Harborth [16] answered this question by giving strings that would give such balanced triangles. This problem has been generalized with still open questions, see for example [9].

Molluzzo [20] recognized that a triangle of 0s and 1s could easily be the adjacency matrix of a graph. In his paper, Molluzzo actually studied the complements of what are now known as Steinhaus graphs and some of his questions are also still open. For the basics on Steinhaus graphs see [13] and there is more on the compelements of Steinahus graphs in [12].

Why study these graphs? The main reason is that they are interesting, but it is also the case, as Brand and others in [3], [4], [5] showed, that almost all Steinhaus graphs have the first-order property $\mathcal{P}$ if and only if almost all graphs have property $\mathcal{P}$. This also applies to the complements of Steinhaus graphs. It was first shown that almost all Steinhaus graphs have diameter 2, see [1], [2], and [6]. Also, every graph with $n$ vertices is

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isomorphic to an induced subgraph of a Steinhaus graph with $\binom{n}{2}$ vertices, see [7] and [10]. Thus in some sense, the structure of Steinhaus graphs and their complements mirrors that of all graphs.

Since every line graph is claw-free, it is natural to think of claw-free graphs as a generalization of line graphs, but claw-free graphs are interesting without considering line graphs. For example, claw-free connected graphs of even order have perfect matchings, there are polynomial time algorithms for finding maximum independent sets in claw-free graphs, and the claw-free perfect graphs have been characterized. For a survey on claw-free graphs, see [15]. What is of interest to us in this paper is that the complement of any triangle-free graph is claw-free.

Our purpose is to classify the generating strings of claw-free Steinhaus complements. In Section 2, we give definitions and notation. In Section 3 we discuss bipartite Steinhaus graphs since their complements are claw-free and in the next section we state the two main theorems of this paper: one on the generating strings of the claw-free complements and the other on how many of the strings there are. Section 5 has the background theorems that we need and the next three sections are the classification for claw-free strings with exactly two 1s, three 1s, and four 1s. The last section is a discussion of the bounds on the number of claw-free complements.

## Section 2: Definitions and Notation

**Definition 1.** Let $T = a_{0,1} \cdots a_{0,n-1}$ be an $(n-1)$-long string of 0s and 1's. The *Steinhaus graph* generated by $T$ has as its adjacency matrix the *Steinhaus matrix* $A = [a_{i,j}]$ where

$$a_{i,j} = \begin{cases} 0, & \text{if } 0 \le i = j \le n-1; \\ a_{0,j}, & \text{if } 0 = i < j; \\ (a_{i-1,j-1} + a_{i-1,j}) \bmod 2, & \text{if } 0 < i < j \le n-1; \\ a_{j,i}, & \text{if } 0 \le j < i \le n-1. \end{cases}$$

The addition in the third part of the case statement is the *Steinhaus property* which is the same as the recursion for Pascal's Triangle mod 2. Given the entries $a_{i,i+1}$ for $0 \le i < n$, the same matrix can be generated using the Steinhaus property and this string is called the *diagonal generator* of the matrix and graph. In Figure 1 the diagonal generator is 0100. (Unless otherwise noted, all our generators are diagonal generators.)

$$\begin{array}{ccccc} \underline{0} & 0 & 1 & 0 & 1 \\ & \underline{0} & 1 & 1 & 1 \\ & & \underline{0} & 0 & 0 \\ & & & \underline{0} & 0 \\ & & & & \underline{0} \end{array}$$

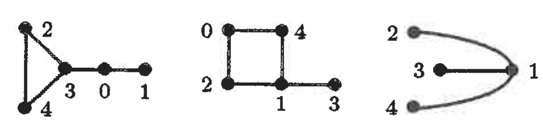



**Figure 1.** Complement generated by 0100, the Steinhaus graph, and the claw in the Steinhaus graph.



Note that the generating string is $(n-1)$-long and the corresponding graph has $n$ vertices. We always display the Steinhaus matrix but use 1s for adjacencies in the Steinhaus graph and 0s for adjacencies in the Steinhaus complement. The diagonal 0s of the matrices are always underlined. Also, the reverse of a diagonal generator generates a graph isomorphic to the graph generated by the original string. In Figure 1, the diagonal generator 0010 generates the same graph where the labels 0 and 4 and the labels 1 and 3 are switched.

**Notation 1.** *We let* $\lg(n) = \log_2(n)$ *and for a positive integer* $k$, $\lambda_k = \lfloor \lg k \rfloor$, *but we often use* $\lambda = \lfloor \lg(n-1) \rfloor$. *Also* $\nu_k$ *is the smallest index larger than 1 in the binary expansion of* $k = \tau_{\lambda_k} \cdots \tau_0$ *for which* $\tau_{\nu_k} = 1$ *or* $\nu_k$ *is 2 more than the largest power of 2 dividing* $\lfloor k/4 \rfloor$. *Hence* $k = 2^{\nu_k}(2q+1) + r$ *where* $q$ *is a non-negative integer and* $0 \le r < 4$.

**Notation 2.** *We always write the binary expansion of* $n-1$ *as* $b_{\lambda_{n-1}} \cdots b_0$ *and we use the following two particular indices in this expansion. If* $k = \min\{i : b_i = 1\}$, *then* $\omega = \max\{2, k+1\}$. *If* $j = \min\{\{\lambda_{n-1}\} \cup \{i : b_i = 0\}\}$, *then* $\zeta = \max\{2, j+1\}$.

**Notation 3.** *Two other notational shortcuts that we use are that instead of saying we are using the Steinhaus property, we denote it as* $a_{i+1,j+1} \overset{\text{S.P.}}{\equiv} a_{i,j} + a_{i,j+1}$. *Also, when we use* $\equiv$ *alone, it will mean equivalence modulo 2.*

**Notation 4.** *We use the Kronecker delta function in two ways. First,* $\delta_{a,b}$ *is 1 if* $a = b$ *and 0 otherwise. Second,* $\delta_n$ *is 1 if* $n$ *is a power of two and is 0 otherwise.*

**Definition 2.** The *claw* is the complete bipartite graph $K_{1,3}$, see Figure 1. A graph is *claw-free* if it has no induced claw. A *leaf* of the claw is one of the vertices of degree 1 and the *root* of the claw is the vertex of degree 3. The claw with root $u$ and leaves $x$, $y$, and $z$ is denoted $[u : x, y, z]$. In Figure 1, the complement is claw-free but the graph has the claw $[1 : 2, 3, 4]$.

**Notation 5.** *To show that* $[u : x, y, z]$ *is a claw in the complement of a Steinhaus graph with adjacency matrix* $A = [a_{i,j}]$, *we need to show that* $a_{u,x} = a_{u,y} = a_{u,z} = 0$ *and* $a_{x,y} = a_{x,z} = a_{y,z} = 1$. *In proofs, (i) will be to show* $a_{u,x} = 0$, *(ii)* $a_{u,y} = 0$, *(iii)* $a_{u,z} = 0$, *(iv)* $a_{x,y} = 1$, *(v)* $a_{x,z} = 1$, *and (vi)* $a_{y,z} = 1$.

**Definition 3.** A string that generates a complement that is claw-free is called claw-free and we also say that a string that generates a complement with a claw has a claw.

**Observation 1.** *Any substring of a claw-free string is a claw-free string. If a substring of a string has a claw, then the string has a claw.*

**Observation 2.** *A complete induced subgraph of a graph can have at most one leaf in a claw and possibly the center of the claw. If a graph has two induced subgraphs that are both complete and the vertices of these subgraphs partition the vertices of the graph into two subsets, then the graph is claw-free.*

Why are we not studying claw-free Steinhaus graphs? The answer is that there are only two for $n \geq 9$. Hence, unless stated otherwise, after our discussion of bipartite Steinhaus graphs, all graphs are the complements of Steinhaus graphs.

**Observation 3.** *For $n \geq 9$, all Steinhaus graphs have claws except for those generated by the diagonal generators $0^{n-1}$, generating the graph with no edges on $n$ vertices, and $1^{n-1}$, generating the path on $n$ vertices. There are six claw-free graphs on 4 vertices, seven on 4 vertices, eight on 6 vertices, five on 7 vertices, and three on 8 vertices.*

Let $\mathcal{P}$ be the property that a graph is claw-free or that it is bipartite. If a Steinhaus graph or complement does not have property $\mathcal{P}$, then adding a 0 or 1 to the end of its generating string results in a graph that also does not have property $\mathcal{P}$. Thinking of Steinhaus graphs as a tree with the child of a node as the generating string with a 0 or 1 appended, then searching for a graph with property $\mathcal{P}$ is a search through this tree. For example, the children of 0101 would be 01010 and 01011. Since there are not many of either type of graphs with property $\mathcal{P}$, the tree prunes quickly. One of the first author's students found all the claw-free complements with $n$ vertices for $1 < n \leq 288$. This paper is the result of using that data.

## Section 3: Bipartite Steinhaus graphs

Bipartite Steinhaus graphs have been much studied, see [8], [11], [14], [17], and [18]. While there are many results on bipartite Steinhaus graphs, we need just two. In [11] it was shown that

**Theorem 4.** *A Steinhaus graph is bipartite if and only if it has no triangles. Hence, the complement of a bipartite Steinhaus graph is claw-free.*

From [8] and [14] we have the following two theorems and since these are generators of the Steinhaus graph, adjacencies are indicated by 1s.

**Theorem 5.** *The diagonal generators of the bipartite Steinhaus graphs are*

*(a) Any substring of $(10^{2^m-1})^\infty$. These include the strings with no 1s, exactly one 1, and the string of all 1s.*

*(b) Any substring containing exactly two ones of $0^r10^{2^m(2q+1)-1}10^s$ where $1 < m$, $0 \leq r < 2^m$, $0 \leq s < 2^m$, and $0 < q$. Also, the following are bipartite: $10^{n-3}1$ for any $n > 3$, $010^{n-4}1$ and $10^{n-4}10$ for $n$ odd, and $01^{n-5}10$ for $n$ even.*



**Theorem 6.** *The number of strings that generate bipartite Steinhaus graphs is*

$$2n - 2 + b(n) \tag{1}$$

*where $b(n)$ is the number of generators in item (b) above. Also, $b(n)$ is a diatomic sequence satisfying the recursion in (2). The initial conditions for $b(n)$ are $b(2) = 0$, $b(3) = 0$ and for $n > 1$,*

$$b(2n) = b(n) + b(n+1) \quad \text{and} \quad b(2n+1) = 2b(n+1) + 1. \tag{2}$$

*There are exact formulas for $b(n)$ and tight upper and lower bounds, all given in [8]. The bounds are*

$$\frac{n-6}{8} \leq b(n) \leq \frac{n-3}{2} \tag{3}$$

*with equality on the left only for $n = \frac{1}{3}(4^m + 2)$ and on the right only for $n = 2^m + 1$.*

## Section 4: Generators of Claw-free Complements

In this section we state the two main theorems of this paper. In both of these we use Notation 1 and 2.

**Theorem 7.** *The claw-free strings are those that are bipartite and the following (and the reverse of each string):*

**Two 1s**

- *d.* $0^r110^s$, $\min\{r, s\} > 0$
- *2.* $10^{2^m-2}10^s$, $1 < m$, $0 < s$
- *22.* $10^{2^m(2q+1)-2}10^s$, $1 < m$, $0 < s < 2^m$, $0 < q$
- *8.* *0100100 and 0010010, $n = 8$*
- *2s.* *For $n \equiv 0 \pmod 4$, $10^{n-4}10$, $010^{n-4}1$, and for $n$ odd, $010^{n-5}10$*

**Three 1s**

- *3i.* $10^{2^m-2}10^{2^\mu(2q+1)-2}1$, $2 \leq m \leq \mu \leq \nu_n$, $0 \leq q$
- *3i.* $10^{2^m-2}10^{2^\mu(2q+1)-1}1$, $2 \leq m \leq \mu \leq \nu_n$, $0 \leq q$
- *3.* $10^{2^m-3}110^s$, $0 \leq s$, $2 \leq m$
- *31.* $10^{2^m(2q+1)-3}110^s$, $2 \leq m$, $0 \leq s < 2^m - 1$, $0 < q$
- *32.* $10^{2^m(2q+1)-2}110^s$, $2 \leq m$, $0 \leq s < 2^m - 1$, $0 < q$
- *3s.* *Not otherwise listed above:* $10^{n-4}11$, $110^{n-4}1$

**Four 1s**

- *4s.* $110^{n-5}11$
- *4.* $10^{2^m-2}110^{2^m-2}1$, $n = 2^{m+1} + 1$
- *4.* $10^{2^m-3}110^{2^\mu(2q+1)-3}1$, $2 \leq m \leq \mu \leq \nu_{n+1}$, $0 \leq q$
- *4.* $10^{2^m-2}110^{2^\mu(2q+1)-3}1$, $2 \leq m \leq \mu \leq \nu_n$, $0 \leq q$.

**Theorem 8.** *For $n \geq 2$ vertices, $n-1 = b_\lambda \dots b_0$ in binary, and $\lambda = \lfloor \lg(n-1) \rfloor$, the number of claw-free complements, $c(n)$, is given by*

$$c(n) = c_b(n) + c_d(n) + c_2(n) + c_{22}(n) + c_8(n) + c_{2s}(n) + c_{3i}(n) + c_3(n) + c_{31}(n) + c_{32}(n) + c_{3s}(n) + c_{4s}(n) + c_4(n)$$

*where $c_b(n)$ is the number of bipartite strings and the function is 0 below its indicated bounds:*

$c_b(n) = 2n - 2 + b(n),\ n \geq 1;$ $\qquad c_d(n) = n - 2,\ n \geq 3;$

$c_2(n) = 2\lambda_{n-2} - 2,\ n \geq 6;$ $\qquad c_{22}(n) = 2\sum_{k=\omega}^{\lambda-1} b_k,\ n \geq 14;$

$c_8(n) = 2 \cdot \delta_{n,8},$ $\qquad c_{2s}(n) = \begin{cases} 2, & n \equiv 0 \pmod 4,\ n \geq 8; \\ 1, & n \equiv 1,3 \pmod 4,\ n \geq 7; \\ 0, & \text{otherwise}; \end{cases}$

$c_3(n) = 2\lambda - 2,\ n \geq 5;$ $\qquad c_{3s}(n) = \begin{cases} 2, & n \equiv 0,3 \pmod 4,\ n \geq 7; \\ 0, & \text{otherwise}; \end{cases}$

$c_{31}(n) = 2\sum_{k=\zeta}^{\lambda-1} b_i,\ n \geq 13;$ $\qquad c_{32}(n) = 2\sum_{k=\omega}^{\lambda} b_i,\ n \geq 14;$

$c_{4s}(n) = 1,\ n \geq 6;$

$$c_{3i}(n) = \begin{cases} 2\nu_n - 2 - 3\delta_n, & n \equiv 0 \pmod 4, n \geq 8; \\ 2\nu_n - 2 - 2\delta_{n-1}, & n \equiv 1 \pmod 4, n \geq 9; \\ 0, & \text{otherwise}. \end{cases}$$

$$c_4(n) = \begin{cases} 2\nu_n - 2 - 2\delta_n, & n \equiv 0 \pmod 4, n \geq 8; \\ \delta_{n-1}, & n \equiv 1 \pmod 4, n \geq 9; \\ 2\nu_{n+1} - 2 - 3\delta_{n+1}, & n \equiv 3 \pmod 4, n \geq 7; \\ 0, & \text{otherwise}. \end{cases}$$

**Proof:** For $c_b(n)$, see (1). The formula for $c_d(n)$ is found in Proposition 16. Theorem 24 gives the proof for $c_2(n)$ and $c_{22}(n)$. For $c_8(n)$ and $c_{2s}(n)$), see Lemma 20 and Lemma 25 respectively. The proof of the formula for $c_{3i}(n)$ is in Theorem 27. For $c_{3s}(n)$ and $c_3(n)$, the proofs are in Theorem 28. Lemma 30 gives the proof for $c_{31}(n)$ and Lemma 29 has the proof for $c_{32}(n)$. In Lemma 31 we show that $110^{n-5}11$ is claw-free which is the only string with exactly four 1s that is not in a class and so $c_{4s}(n) = 1$. Finally, we verify $c_4(n)$ in Theorem 33. ▮

To simplify this, note that $\lambda_{n-2} + \lambda_{n-1} = 2\lambda_n - 2\delta_n - \delta_{n-1}$ and so

$$c_2(n) + c_3(n) = 4\lambda_n - 4 - 4\delta_n - 2\delta_{n-1}. \tag{4}$$

Since $b_\lambda$ is always 1, $\sum_{k=\omega}^{\lambda} b_k = 1 + \sum_{k=\omega}^{\lambda-1} b_k$ unless $n - 1 = 2^m$ in which case both sums are 0. Hence

$$\sum_{k=\omega}^{\lambda} b_k = 1 - \delta_{n-1} + \sum_{k=\omega}^{\lambda-1} b_k \quad \text{and so}$$
$$c_{22}(n) + c_{32}(n) = 2(1 - \delta_{n-1}) + 4\sum_{k=\omega}^{\lambda-1} b_k. \tag{5}$$



Using (1), (4), and (5), and combining all the case statements gives

**Corollary 9.** *For $n \geq 9$,*

$$c(n) = 3n - 5 + b(n) + 4\lambda_n + 4\sum_{k=\omega}^{\lambda_{n-1}-1} b_k + 2\sum_{k=\zeta}^{\lambda_{n-1}-1} b_k$$
$$+ \begin{cases} 4\nu_n - 9\delta_n, & n \equiv 0 \pmod 4; \\ 2\nu_n - 1 - 5\delta_{n-1}, & n \equiv 1 \pmod 4; \\ 0, & n \equiv 2 \pmod 4; \\ 2\nu_{n+1} + 1 - 3\delta_{n+1}, & n \equiv 3 \pmod 4. \end{cases} \tag{6}$$

To illustrate the above, here are the strings for $n = 8$ and the category to which each belongs. In Figure 2 we give the values needed to compute the result from Corollary 9 for $c(n)$, $b(n)$, the two sums, and the values in the case statement, called $\rho(n)$.

$n = 8$: Bipartite: 0000000, 0000001, 0000010, 0000100, 0001000, 0010000, 0100000, 1000000, 0101010, 1010101, 1000100, 0100010, 00100010, 1111111, counted by $b(8)$ 1000001. Counted by $c_d(8)$: 0000011, 0000110, 0001100, 0011000, 0110000, 1100000; $c_2(8)$: 1001000, 0001001; $c_8(8)$: 0100100, 0010010; $c_{2s}(8)$: 1000010, 0100001; $c_{3i}(8)$: 1001001; $c_3(8)$: 1011000, 0001011; $c_{32}(8)$: 1001100, 0011001; $c_{3s}(8)$: 1100001, 1000011; $c_4(8)$: 1001101, 1011001; $c_{4s}(8)$: 1100011.

For $1 \leq n \leq 4$, we have $c(n) = 2^{n-1}$ and $b(n) = 0$.

| $n$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c(n)$ | 14 | 21 | 29 | 37 | 37 | 38 | 46 | 52 | 54 | 57 | 67 | 74 | 71 | 69 | 76 | 81 | 82 | 85 | 96 | 103 | 100 |
| $b(n)$ | 1 | 0 | 1 | 1 | 3 | 1 | 1 | 1 | 3 | 2 | 3 | 4 | 7 | 4 | 3 | 2 | 3 | 2 | 3 | 4 | 7 |
| $\frac{1}{2}c_{22}(n)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| $\frac{1}{2}c_{31}(n)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| $\rho(n)$ | | | | 3 | 0 | 0 | 5 | 8 | 3 | 0 | 6 | 7 | 2 | 0 | 5 | 8 | 3 | 0 | 7 | 12 | 5 |

**Figure 2.** Table for values from Corollary 9

## Section 5: Proofs Techniques

The main result we need other than facts about Steinhaus graphs is the following (for a proof of this and many other interesting facts on binomial coefficients, see [19]).

**Theorem 10.** *Lucas' Theorem: If the binary expansion of $m = \tau_\ell \dots \tau_0$ ($\tau$ for top) and the binary expansion of $k = \beta_\ell \dots \beta_0$ ($\beta$ for bottom), then*

$$\binom{m}{k} \equiv \prod_{i=0}^{\ell} \binom{\tau_i}{\beta_i} \pmod 2.$$

**Corollary 11.** *Note that $\binom{m}{k}$ is even if and only if there is an $i$ such that $\tau_i = 0$ and $\beta_i = 1$. So if $m$ is even and $k$ is odd, then $\binom{m}{k} \equiv 0$. If $m \equiv 0 \pmod 4$ and $k \not\equiv 0 \pmod 4$, then $\binom{m}{k} \equiv 0$. Also, the binary expansion of $2^\ell - 2^m(2q+1)$ ends with $m$ 0s and the binary expansion of $2^\ell - 2^m(2q+1) - 1$ ends with $m$ 1s and with $\tau_m = 0$.*

In the matrices in Figure 3, note that there is a column of 1s and a row of 1s that meet at the diagonal. By the Steinhaus property, this means that the entries to the right of the column of 1s and above the row of 1s are elements of Pascal's Triangle mod 2. These are outlined with a box in Figure 3. In Theorem 13, we give the precise entries. If there is a string of 0s in the diagonal generator, then the corresponding vertices of the complement induce a complete graph. Finally, as can be seen in Figure 6, if $10^r1$ is a substring of a diagonal generator, then the corresponding vertices induce $K_2 \cup K_{r+1}$. We state this as

**Observation 12.** *If $a_{r+i,r+i+1} = 0$ for $0 \le i < s$, then in the complement the vertices $\{r, r+1, \dots r+s\}$ induce $K_{s+1}$. If $a_{r,r+1} = a_{s,s+1} = 1$ and $a_{i,i+1} = 0$ for $r < i < s$, then the vertices $r$ and $s$ induce $K_2$ and the vertices between $r$ and $s$ induce $K_{s-r}$ and all these vertices induce $K_2 \cup K_{s-r}$.*

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 2 | | | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| 3 | | | | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 4 | | | | | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 5 | | | | | | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| 6 | | | | | | | 0 | 1 | 0 | 1 | 0 | 0 |
| 7 | | | | | | | | 0 | 1 | 1 | 1 | 0 |
| 8 | | | | | | | | | 0 | 0 | 0 | 1 |
| 9 | | | | | | | | | | 0 | 0 | 1 |
| 10 | | | | | | | | | | | 0 | 1 |
| 11 | | | | | | | | | | | | 0 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| | | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| | | | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| | | | | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| | | | | | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| | | | | | | 0 | 1 | 1 | 1 | 1 | 1 |
| | | | | | | | 0 | 0 | 0 | 0 | 0 |
| | | | | | | | | 0 | 0 | 0 | 0 |
| | | | | | | | | | 0 | 0 | 0 |
| | | | | | | | | | | 0 | 0 |
| | | | | | | | | | | | 0 |

**Figure 3.** Steinhaus matrices generated by $10^5110^21$ (left) and $10^510^4$ (right).

The following is straightforward to prove, but is surprisingly useful in our proofs.

**Theorem 13.** *In the upper triangular part of a Steinhaus matrix, if there are $u$ and $v$ such that $a_{u-i,v} = 1$ for $1 \le i \le k$, $a_{u-(k+1),v} = 0$, and $a_{u,v+j} = 1$ for $1 \le j \le h$, then for these values of $i$ and $j$ and $\ell = 1 + \max\{\lfloor \lg(k) \rfloor, \lfloor \lg(h) \rfloor\}$, we have*

$$a_{u-i,v+j} \equiv \binom{2^\ell - 1 - i}{j} \quad \text{and} \quad a_{u-(k+1),v+j} \equiv 1 + \binom{2^\ell - 2 - k}{j}. \qquad (7)$$

**Proof:** First, $\ell$ is chosen to be large enough so that the entries of the matrix in row $u$ to the right of column $v$ are the entries of row $2^\ell - 1$ of Pascal's Triangle mod 2 and are all 1s. Column $v$ from rows $(u-k)$ to row $u$ are the coefficients $\binom{2^\ell - i}{0} = 1$ for $1 \le i \le k$. Since the Steinhaus property is the recursion for the binomial coefficients, we have the result on the left of (7).

If $a_{u-(k+1),v+j} = 1$, then $a_{u-(k+1),v} \equiv \binom{2^\ell-2-k}{j}$, but since $a_{u-(k+1),v+j} = 0$, in the matrix the 1s are switched to 0s and the 0s to 1s giving us the result on the right of (7). ∎



In Figure 3, the matrices are generated by $10^r110^21$ and $10^r10^4$ where $r = 5$ and so in the matrix on the left $u = v = 7$ and on the right, $u = 6$ and $v = 7$. The generic graph for $10^r10^s$ is shown in Figure 4 and for the matrix on the right in Figure 3, $K_{r+1} = \{1, \dots, 6\}$ and $K_{s+1} = \{7, \dots, 11\}$.

**Corollary 14.** *In particular, for the string $10^r10^s$ we have $u = r+1$ and $v = r+2$. (See the graph shown on the left in Figure 4.) Also, $a_{0,i} = 1$ for $0 < i < r+2$, $a_{0,r+2} = 0$, and $a_{0,r+2+j} \equiv 1 + \binom{2^\ell-r-2}{j}$ and $a_{r+1-i,n-1} \equiv \binom{2^\ell-1-i}{s}$. Note also that $a_{r+1+i,r+1+j} = 0$ for positive $i$ and $j$. The only possible claws have the form $[u : 0, d, v]$ where $u$ and $v$ are in $K_{s+1} = \{r+2, \dots, n-1\}$ and $d$ is in $K_{r+1} = \{1, \dots, r+1\}$.*

**Corollary 15.** *In particular, for the string $10^r110^s$ we have $u = v = r+2$. (See the graph on the left of Figure 10.) Also, $a_{0,i} = 1$ for $0 < i < r+2$, $a_{0,r+2} = 0$, $a_{0,r+2+j} \equiv 1 + \binom{2^\ell-r-3}{j}$, and $a_{r+2-i,n-1} \equiv \binom{2^\ell-1-i}{s+1}$. Note also that $a_{i,j} = 0$ for $0 < i < j < r+2$ and $r+2 < i < j < n$ and that vertex $r+2$ is only adjacent to vertex 0. The only possible claws have the form $[u : 0, d, v]$ where $u$ and $v$ are in $K_{s+1} = \{r+3, \dots, n-1\}$ and $d$ is in $K_{r+1} = \{1, \dots r+1\}$. (The generic graph is on the left in Figure 10.)*

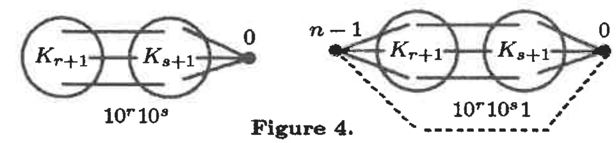



**Figure 4.**

## Section 6: Two 1s

All strings having no 1s or exactly one 1 are bipartite and hence claw-free. We carefully examine the claw-free strings with exactly two ones as we use these results in the subsequent sections.

**Proposition 16.** *The string $0^r110^s$ for $n = r+s+3$ with $r \ge 0$ and $s \ge 0$ generates a claw-free graph. There are $c_d(n) = n-2$ of these strings.*

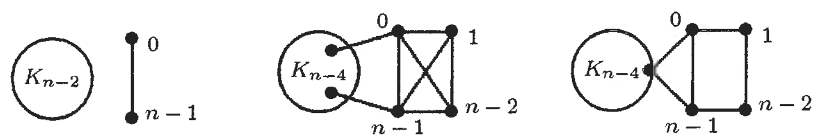
**Proof:** This graph is $K_{r+1} = \{0, \dots r\}$, $K_{s+1} = \{r+2, \dots n-1\}$ and the isolated vertex $\{r+1\}$. Hence the graph is claw-free. See the matrix on the left of Figure 3 but ignore the first row and last column. ∎

The only other claw-free strings containing exactly two 1s must be of the form $0^t10^r10^s$ with $r > 0$. We show if $r \equiv 3 \pmod 4$, then the string is bipartite and that $s$ and $t$ must be small.

**Lemma 17.** *The string $0^{2^m-1}10^{2^m(2q+1)-1}10^{2^m-1}$ is bipartite and hence claw-free. The string $10^{2^m(2q+1)-1}10^{2^m}$ has the claw $[r+3 : 0, 3, n-1]$ where $r = 2^m(2q+1)-1$ and $n = 2^{m+1}(q+1)+2$.*

**Proof:** If $r \equiv 3 \pmod 4$, then $r = 2^m(2q+1)-1$ for $m > 1$. By Theorem 5, if $s$ and $t$ are less than $2^m$, then the string is bipartite. If $s$ or $t$ is $2^m$ or larger, then the string has a claw. To prove this, we only need show that $10^{2^m(2q+1)-1}10^{2^m}$ has the claw $[r+3 : 0, 3, n-1]$ where $n = 2^{m+1}(q+1)+2$.

We use Notation 5 as a guide, Corollary 11, and Corollary 14. Here $u = r+1$ and $v = r+2$. **(i)** $a_{0,r+3} \equiv 1 + \binom{2^\ell-r-2}{1} \equiv 1+1 \equiv 0$ since the top is odd. **(ii)** $a_{3,r+3} \equiv \binom{2^\ell+1-r}{1} \equiv 0$ since the top is even and the bottom is odd. **(iii)** Since $r+3 > r+1$, $a_{r+3,n-1} = 0$. **(iv)** Since $r \geq 3$, $a_{0,3} = 1$. **(v)** $a_{0,n-1} \equiv 1 + \binom{2^\ell-r-2}{2^m} \equiv 1 + \binom{2^\ell-2^m(2q+1)-1}{2^m} \equiv 1+0 \equiv 1$ since $\tau_m = 0$ and $\beta_m = 1$. **(vi)** $a_{3,n-1} \equiv \binom{2^\ell-r+1}{2^m} \equiv \binom{2^\ell-2^m(2q+1)+2}{2^m} \equiv 1$ since $\tau_m = \beta_m = 1$. ∎




$10^{n-3}1$ $010^{n-5}10$, $n$ even $010^{n-5}10$, $n$ odd

**Figure 5.**

In the next three lemmas we take care of the special cases. In Lemma 20, we show that there are two sporadic strings for $n = 8$ that are not substrings of any claw-free strings.

**Lemma 18.** *The string $010^{n-5}10$ is claw-free. As substrings, both $10^{n-4}10$ and $010^{n-4}1$ are claw-free.*

**Proof:** The graph generated by $10^{n-3}1$ is shown on the left in Figure 5, but this is always bipartite. By Observation 2, the graph in the middle of Figure 5 is claw-free. In the graph on the right of Figure 5, we have for odd $n$, vertices 0 and $n-1$ are adjacent to the same vertices in $K_{s+1}$. (Note that this means that the Steinhaus graph has the triangles $\{0, k, n-2\}$ for any $1 < k < n-2$ and so is not bipartite.) The matrix for $010^4 10$ is on the left in Figure 6. ∎

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| 2 | | | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 3 | | | | 0 | 0 | 0 | 0 | 1 | 0 |
| 4 | | | | | 0 | 0 | 0 | 1 | 1 |
| 5 | | | | | | 0 | 0 | 1 | 0 |
| 6 | | | | | | | 0 | 1 | 1 |
| 7 | | | | | | | | 0 | 0 |
| 8 | | | | | | | | | 0 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| | | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| | | | 0 | 0 | 0 | 0 | 1 | 1 |
| | | | | 0 | 0 | 0 | 1 | 0 |
| | | | | | 0 | 0 | 1 | 1 |
| | | | | | | 0 | 1 | 0 |
| | | | | | | | 0 | 1 |
| | | | | | | | | 0 |

**Figure 6.** Matrices for $010^4 10$ and $110^4 11$



**Lemma 19.** *Any string containing 10100 or 00101 generates a complement with a claw. Note also that by Lemma 17, 01010 is claw-free, but bipartite.*

**Proof:** The graphs generated by these strings are given in Figure 7 on the right where the labelling is from 10100. A claw is $[4 : 0, 1, 5]$. ∎

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | | 0 | 1 | 0 | 0 |
| 2 | | | 0 | 1 | 0 |
| 3 | | | | 0 | 1 |
| 4 | | | | | 0 |


0111, claw is $[1 : 0, 3, 4]$ 10100, claw is $[4 : 0, 1, 5]$

**Figure 7.**

**Lemma 20.** *Both the strings 0100100 and 0010010 are claw-free. The string 01001000 has the claw $[8 : 0, 2, 5]$, the string 00100100 has the claw $[8 : 0, 3, 6]$, the string 01001001 has the claw $[1 : 0, 5, 8]$, and the string 00100101 has a claw by Lemma 19. Hence $c_8(n) = 2\delta_{n,8}$.*

**Proof:** The graphs generated by these strings are in Figure 8 where the labelling for 0010010 is from 0100100. ∎

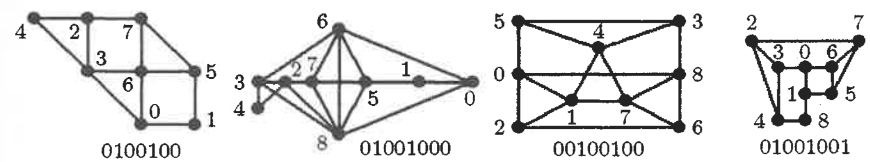



**Figure 8.** Graphs for Lemma 20

In the next theorem we show that $10^r100$ has a claw for $r \equiv 0, 1 \pmod 4$ and that $010^r100$ has a claw if $r \equiv 2 \pmod 4$.

**Theorem 21.** *The string $0^t10^r100$ with $n - 1 = r + 4 + t$ has the claw*

| | | | | |
|---|---|---|---|---|
| (a) | $[n-1 : 0, 3, n-2]$ | if | $r = 4q$, | $t = 0$, |
| (b) | $[n-2 : 0, 1, n-1]$ | if | $r = 4q+1$, | $t = 0$, |
| (c) | $[5 : 0, 4, n-1]$ | if | $r = 2^m(2q+1)-2$, | $t = 1$, $m > 1$, $q \geq 0$, |

*where (c) has the exception when $m = 2$ and $q = 0$ (see Lemma 20).*

**Proof:** For cases (a) and (b), we have $K_{r+1} = \{1, \ldots, r+1\}$, since $s = 2$, $K_{s+1} = \{n-3, n-2, n-1\}$, and $n = r+5$. Again we use Theorem 13 (in particular Corollary 14) and note that $u = r+1$ and $v = r+2$. For (a), $n - 1 \equiv 0 \pmod 4$. **(i)** $a_{0,n-1} \equiv 1 + \binom{2^\ell-1-(r+1)}{2} \equiv 1+1 \equiv 0$ since $2^\ell - r - 2 \equiv 2 \pmod 4$ and so $\tau_1 = \beta_1 = 1$. **(ii)** $a_{3,n-1} \equiv \binom{2^\ell-1-(r-2)}{2} \equiv 0$ since $2^\ell + 1 - r \equiv 1 \pmod 4$ and so has $\tau_1 = 0$ while $\beta_1 = 1$. **(iii)** Since $r+3 > r+1$, $a_{r+3,r+4} = a_{n-2,n-1} = 0$. **(iv)** Since $r \geq 4$, $a_{0,3} = 1$. **(v)** $a_{0,n-2} \equiv 1 + \binom{2^\ell-r-2}{1} \equiv 1+0 \equiv 1$ since the top is even and the bottom is odd. **(vi)** $a_{3,n-2} \equiv \binom{2^\ell+1-r}{1} \equiv 1$ since the top is odd.

Since the proof of (b) is similar to (a), we omit it and go to the proof of (c). For (c), $r = 2^m(2q+1) - 2 \geq 6$, $n = r + 6$, and from Corollary 14, $u = r + 2 = n - 4$ and $v = r + 3 = n - 3$. Thus **(i)**, **(ii)**, and **(iv)**: $a_{0,5} = a_{4,5} = 0$ and $a_{0,4} = 1$. **(iii)** Since $5 = u - (n - 9)$, we have $a_{5,n-1} = a_{(n-4)-(n-9),(n-3)+2} \equiv \binom{2^\ell - 1 - (n-9)}{2} \equiv \binom{2^\ell - 2^{m+1}q + 4}{2} \equiv 0$ since $\tau_1 = 0$ and $\beta_1 = 1$.

**(v)** From Figure 6 and noting that $n-3$ is the penultimate column for our current matrix, $a_{1,n-3} = 0$. Since $a_{0,\text{even}} = 1$ and $n-4$ is even, $a_{0,n-4} = 1$. Hence $a_{0,n-3} \overset{\text{S.P.}}{\equiv} a_{0,n-4} + a_{1,n-3} \equiv 1$. Next, $a_{1,n-2} \equiv 1 + \binom{2^\ell - n + 4}{1} \equiv 1 + 0 \equiv 1$ since the top is even and the bottom is odd. Thus $a_{0,n-2} \overset{\text{S.P.}}{\equiv} a_{0,n-3} + a_{1,n-2} \equiv 1 + 1 \equiv 0$. Finally, $a_{1,n-1} \equiv 1 + \binom{2^\ell - n + 4}{2} \equiv 1 + 0 \equiv 1$ since $\tau_1 = 0$ but $\beta_1 = 1$. Therefore $a_{0,n-1} \overset{\text{S.P.}}{\equiv} a_{0,n-2} + a_{1,n-1} \equiv 0 + 1 \equiv 1$.
**(vi)** $a_{4,n-1} \equiv \binom{2^\ell - 1 - (n-8)}{2} \equiv \binom{2^\ell - 2^{m+1}q + 3}{2} \equiv 1$ for $\tau_1 = \beta_1 = 1$. ∎

By Lemmas 17–20, we can now assume that $t = 0$. This is because if $t = s = 1$, then the string is claw-free, but if both $s$ and $t$ are positive and one of them is larger than 1, then either the string or its reverse has a claw (excepting the bipartite strings). We now consider our last case: $r \equiv 2 \pmod 4$.

**Theorem 22.** *For $m > 1$ and $q \geq 0$, both the strings $10^{2^m(2q+1)-2}10^s$ and $0^s10^{2^m(2q+1)-2}1$ are claw-free where if $q > 0$, then $0 \leq s < 2^m$ and if $q = 0$, then $s \geq 0$.*

**Proof:** Since $r = 2^m(2q+1) - 2$, by Corollary 14, the only possible claws have one leaf 0, the root $u$ and one leaf $v$ with $r + 2 \leq u, v \leq n - 1$ and the last leaf $1 \leq d \leq r + 1$. We need to know the adjacencies of 0 which are

$$a_{0,k} = \begin{cases} 1, & 0 < k < r+1 \\ 0, & k = r+2 \\ 1 + \binom{2^\ell - 2^m(2q+1)}{j} & k = r+2+j. \end{cases}$$

Note $\binom{2^\ell - 2^m(2q+1)}{j} \equiv 1$ only if $j \equiv 0 \pmod{2^m}$. So if $s < 2^m$, then there is no claw. If $q = 0$, then $\binom{2^\ell - 2^m}{j} \equiv 1$ if and only if $j = k \cdot 2^m$. Since $a_{r-i,k\cdot 2^m} \equiv \binom{2^\ell - 1 - i}{k \cdot 2^m - r - 2} \equiv \binom{2^\ell - 1 - i}{(k-1)2^m}$ and since $i < r = 2^m - 2$, the binary expansion of $i + 1$ has it left-most 1 before the $m^{\text{th}}$ position and hence the binary expansion of $2^\ell - 1 - i$ is all 1s from at least the $m^{\text{th}}$ position leftward. Therefore $a_{r-i,k\cdot 2^m} \equiv 1$ and this string is claw-free because whatever is adjacent to 0 cannot be adjacent to anything in the $K_{r+1}$. ∎

To conclude our classification of the strings with exactly two 1s, we need the following but since the proof is similar to that of Theorem 21 (a), we omit it.



**Lemma 23.** *If $r = 2^m(2q+1) - 2$ for $m > 1$, $q > 0$, and $s \geq 2^m$, then the string $10^r10^s$ has the claw $[n-1 : 0, 2^m + 1, r + 3]$.*

**Theorem 24.** *The number of strings of the form $10^{2^m(2q+1)-2}10^s$ and $0^s10^{2^m(2q+1)-2}1$ where $m > 1$ and $q \geq 0$ and where if $q > 0$, then $0 \leq s < 2^m$ $(c_{22}(n))$ and if $q = 0$, then $s \geq 0$ $(c_2(n))$ is*

$$\begin{cases} c_2(n) = 2(\lambda_{n-2} - 1), & q = 0; \\ c_{22}(n) = 2 \cdot \sum_{k=\omega}^{\lambda_{n-1}-1} b_k, & q > 0. \end{cases}$$

**Proof:** If $q = 0$, then $n - 1 = 2^m + s$ for $m > 1$ and $s \geq 1$. Thus there is a claw-free string if and only if $n - 1 \leq 2^m + 1$ for $m > 1$. Thus the number of strings of the form $10^{2^m-2}10^s$ for $s \geq 1$ is

$$\frac{1}{2} \cdot c_2(n) = \lfloor \lg(n-2) \rfloor - 1.$$

We must also count the reverse of these. If $q > 0$, then $2^m > s \geq 1$ and there is a string if and only if

$$n - 1 = 2 + \left(2^m(2q+1) - 2\right) + s = 2^m(2q+1) + s.$$

Since $2^m > s \geq 1$,

$$3 + \frac{1}{2^m} \leq 2q + 1 + \frac{1}{2^m} \leq \frac{n-1}{2^m} < 2q + 2. \tag{8}$$

Let $n - 1 = b_\lambda \ldots b_0$. If $b_m = 0$, then $\frac{n-1}{2^m}$ is an even number with remainder $b_{m-1} \ldots b_0$ (in binary) and hence this inequality cannot be satisfied. Thus the inequality can only be satisfied if $b_m = 1$ for $m > 1$. But if $b_i = 0$ for $i < m$, then $\frac{n-1}{2^m} = 2q + 1$ which contradicts the left side of (8). Using Notation 2, we have that the number of solutions (in $m$, $n$, and $q$) for (8) is the numbers of 1s in the binary expansion of $n - 1$ starting at the first 1 from the right (larger than 1) and not including $b_{\lambda_{n-1}}$ and hence

$$\frac{1}{2} \cdot c_{22}(n) = \begin{cases} \sum_{k=\omega}^{\lambda-1} b_k, & \omega < \lambda; \\ 0, & \text{otherwise.} \end{cases}$$ ∎

**Lemma 25.** *The non-bipartite claw-free strings with exactly two 1s that have not yet been counted are substrings of $010^r10$. There are $c_{2s}(n) = (2, 1, 0, 1)$ of these where the $0 \leq i < 4$ entry of the 4-tuple is for $n \equiv i \pmod 4$.*

**Proof:** For substrings containing two 1s of $010^r10$ that we have yet to count, if $r \equiv 3 \pmod 4$, then these are bipartite. If $r \equiv 1 \pmod 4$, then only $010^r10$ is not bipartite. If $r \equiv 2 \pmod 4$, then both $010^r1$ and $10^r10$ are counted by Theorem 24. If $r \equiv 0 \pmod 4$, then all have triangles but are claw-free.

If $n \equiv 0 \pmod 4$, then both $010^{n-4}1$ and $10^{n-4}10$ have not yet been counted. If $n \equiv 1 \pmod 4$, then only $010^{n-5}10$ has not yet been counted. If $n \equiv 2 \pmod 4$, then all three strings have been counted. If $n \equiv 3 \pmod 4$, then only $010^{n-5}10$ has not yet been counted. ∎

## Section 7: Three 1s

**Lemma 26.** *The string $10^r10^s1$ is claw-free if and only if both the strings $10^r10^s$ and $0^r10^s1$ are claw-free.*

**Proof:** If $10^r10^s1$ is claw-free, then since both $10^r10^s$ and $10^s10^r$ are substrings of $10^r10^s1$, then both $10^r10^s$ and $10^s10^r$ are claw-free.

Suppose that both $10^r10^s$ and $10^s10^r$ are claw-free. The graph generated by the string $10^r10^s1$ is shown on the right of Figure 4 (where the dotted line indicates that the vertices 0 and $n-1$ may or may not be adjacent) and its matrix is on the left of Figure 9 (ignoring the right-most column). This graph is obtained easily since it contains the graph generated by both $10^r10^s$ and $10^s10^r$. Thus the only possible claws are of the form $[d : e, u, n-1]$ or $[u : 0, d, v]$ where $d, e \in K_{r+1} = \{1, \ldots, r+1\}$ and $u, v \in K_{s+1} = \{r+2, \ldots, n-2\}$. The former claw is in the graph generated by $10^r10^s$, an impossibility, and the latter is in the graph generated by $10^s10^r$, another impossibility. Therefore $10^r10^s1$ is claw-free. ∎

**Theorem 27.** *A non-bipartite string of the form $0^r10^s10^t10^u$ with $s$ and $t$ positive is claw-free if and only if $r = u = 0$ and $s = 2^m - 2$ for some $m \geq 2$ and $t = 2^\mu(2\kappa+1) - \phi$ where $\phi$ is either 1 or 2 with $2 \leq m \leq \nu_n - \delta_n$ or the reverse of such a string. Also, the number of such strings is*

$$c_{3i}(n) = \begin{cases} 2\nu_n - 2 - 3\delta_n, & n \equiv 0 \pmod 4, \\ 2(\nu_n - 1 - \delta_{n-1}), & n \equiv 1 \pmod 4, \\ 0, & \textit{otherwise} \end{cases} \tag{9}$$

**Proof:** From Lemma 26, $10^s10^t1$ is claw-free only if both $10^s10^t$ and $10^t10^s$ are claw-free. We now proceed to eliminate values for $s$ and $t$.

By Theorem 21, part (a), if $s \equiv 0 \pmod 4$, then $10^s100$ has a claw and so $t = 1$. Since $s \geq 4$, by Lemma 19, $1010^s$ has a claw. Therefore neither $s$ nor $t$ is congruent to 0 mod 4.

By Theorem 21, part (b), if $s \equiv 1 \pmod 4$, then $10^s100$ has a claw and so $t = 1$. If $s > 1$, then by Lemma 19, $1010^s$ has a claw. Hence $s = t = 1$ and thus $r \leq 1$ and $u \leq 1$. Hence the string is bipartite. Therefore $s \not\equiv 1 \pmod 4$ and so $s$ and $t$ must be equivalent to 2 or 3 modulo 4.



We first consider $s \equiv 2 \pmod 4$ and we write $s = 2^m(2q+1) - 2$ where $m \geq 2$ and $q \geq 0$. By Theorem 21, part (3), $010^{2^m(2q+1)-2}100$ has a claw unless $q = 0$ and $m = 2$. By Lemma 20, 01001000, 00100100, and 01001001 have claws. Hence, if $s \equiv 2 \pmod 4$, then $r = 0$. Likewise, if $t \equiv 2 \pmod 4$, then $u = 0$.

If both $s$ and $t$ are congruent to 2 mod 4, then the string must have the form $10^{2^m(2q+1)-2}10^{2^\mu(2\kappa+1)-2}1$ where $m \geq 2$ and $\mu \geq 2$. If both $q > 0$ and $\kappa > 0$, then using Lemma 23 at the second and fourth less-than's,

$$2^m < 2^m(2q+1) - 2 < 2^\mu < 2^\mu(2\kappa+1) - 2 < 2^m, \tag{10}$$

which is a contradiction. If $q = 0$ and $\kappa > 0$, then $2^m - 2 < 2^\mu$ and so $m \leq \mu$. If $\kappa = 0$, then without loss of generality, we assume $m \leq \mu$. Thus if $s, t \equiv 2 \pmod 4$, we let $m \leq \mu$. Note also that since $2^m - 2 \geq 2$, Theorem 21, part (c) guarantees that the string has a claw if either $r$ or $u$ is positive.

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| 2 | | | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | | | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 3 | | | | 0 | 1 | 1 | 1 | 1 | 0 | 0 | | | | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 4 | | | | | 0 | 0 | 0 | 0 | 1 | 0 | | | | | 0 | 0 | 0 | 0 | 1 | 1 |
| 5 | | | | | | 0 | 0 | 0 | 1 | 1 | | | | | | 0 | 0 | 0 | 1 | 0 |
| 6 | | | | | | | 0 | 0 | 1 | 0 | | | | | | | 0 | 0 | 1 | 1 |
| 7 | | | | | | | | 0 | 1 | 1 | | | | | | | | 0 | 1 | 0 |
| 8 | | | | | | | | | 0 | 0 | | | | | | | | | 0 | 1 |
| 9 | | | | | | | | | | 0 | | | | | | | | | | 0 |

**Figure 9.** Steinhaus matrices generated by $10^210^310$ and $010^511$.

Suppose $t \equiv 3 \pmod 4$ and we let $t = 2^\mu(2\kappa+1) - 1$. We show that the only such strings are for $s = 2^m - 2$ and $2 \leq m \leq \mu$. If $q > 0$, then by Lemma 23, (10) holds if the last $-2$ is replaced by $-1$. Thus $q = 0$. Since $t \geq 3$, by the above, $r = 0$.

Is it possible for $u$ to be positive in such a claw-free string? The answer is no as the string $10^{2^m-2}10^{2^\mu(2\kappa+1)-1}10$ has the claw $[2^m : 0, 2^m+1, n-1]$ where $n - 1 = 1 + 2^m + 2^\mu(2\kappa+1)$. This is proven in Lemma 36.

Finally, suppose $s \equiv t \equiv 3 \pmod 4$. Let $s = 2^m(2q+1) - 1$ and $t = 2^\mu(2\kappa+1) - 1$. By Lemma 17, $s \leq 2^\mu - 1$ and $t \leq 2^m - 1$ and hence

$$2^m - 1 \leq 2^m(2q+1) - 1 \leq 2^\mu - 1 \leq 2^\mu(2\kappa+1) - 1 \leq 2^m - 1.$$

Thus $2^\mu = 2^m$ and $q = \kappa = 0$. Hence this is a bipartite string. Therefore the only non-bipartite claw-free strings of the form $0^r10^s10^t10^u$ are $10^s10^t1$ where one of $s$ or $t$ is $2^m - 2$ and the other is congruent to either 2 or 3 mod 4.

To count the non-bipartite, claw-free strings in this theorem, we first consider those of the form $10^{2^m-2}10^{2^\mu(2\kappa+1)-2}1$. Here $n = 2^m+2^\mu(2\kappa+1) \equiv 0 \pmod 4$ and $2 \le m \le \mu$ (if $\kappa = 0$, we choose $m \le \mu$). The string can be a palindrome if $\kappa = 0$ and $m = \mu$ in which case $n = 2^{m+1}$. If $m < \mu$, then $n = 2^m(2^{\mu-m}(2\kappa+1)+1)$ and so $m = \nu_n$. If $m = \mu$, then $n = 2^{m+1}(\kappa+1)$ and so $m < \nu_n$. If $n = 2^{\nu_n}$, then $m$ must be less than $\nu_n$ in order to have a string. Thus $2 \le m \le \nu_n - \delta_n$.

Conversely, suppose $n \equiv 0 \pmod 4$, $2 \le m \le \nu_n - \delta_n$, and $n \ge 8$. We need to show that $10^{2^m-2}10^{n-2-2^m}1$ is claw-free. Note $n - 2 - 2^m \equiv 2 \pmod 4$ and we write $n - 2 - 2^m = 2^\mu(2\kappa+1) - 2 > 0$ and so $\mu > 1$. If $\kappa = 0$, then we are done. If $\kappa > 0$ and $m > \mu$, then $n = 2^\mu(2^{m-\mu} + 2\kappa + 1)$ and so $\mu = \nu_n > m$, a contradiction. Thus we have a claw-free string and hence there are two strings for each $m$ with $2 \le m \le \nu_n$ unless $n = 2^{\nu_n}$ in which case the string is the palindrome, $10^{2^{\nu_n-1}-2}10^{2^{\nu_n-1}-2}1$. Therefore $c_{3i}(n) = 2(\nu_n - 1) - 3\delta_n$ when $n \equiv 0 \pmod 4$.

For strings of the form $10^{2^m-2}10^{2^\mu(2\kappa+1)-1}1$ we have $n \equiv 1 \pmod 4$ and the proof that there are $2(\nu_n - 1 - \delta_{n-1})$ of these is similar to what we have just done and we omit it. ∎

The only other strings with exactly three ones that are non-bipartite and claw-free have the form $0^t10^r110^s$ or the reverse.

**Theorem 28.** *All strings of the form $10^r11$ are claw-free (Lemma 31) and there are only two of these not counted below for $n \equiv 0,3 \pmod 4$. The only strings of the form $10^r110^s$ for $s > 0$ that are claw-free have $r \equiv 1,2 \pmod 4$. If $r \equiv 2 \pmod 4$, then writing $r = 2^m(2q+1) - 2$ with $m > 1$ we have $0 \le s < 2^m - 1$ ($c_{32}(n)$). If $r = 2^m - 3$, then $s$ can be any non-negative integer ($c_3(n)$). If $r = 2^m(2q+1) - 3$ for $m > 1$ and $q > 0$, then $0 \le s < 2^m - 1$ ($c_{31}(n)$). For all these strings,*

$$c_{32}(n) = 2\sum_{k=\omega}^{\lambda_{n-1}} b_k, \quad c_3(n) = 2\lfloor \lg(n-1) \rfloor - 2, \quad c_{31}(n) = 2\sum_{k=\zeta}^{\lambda_{n-1}} b_k.$$

**Proof:** From Lemmas 35 and 37, all strings containing 0111 or $010^r11$ have a claw. Thus we need only consider strings of the form $10^r110^s$ (or the reverse of such a string). From the matrix on the left in Figure 3 (ignoring the right-most column), it is easy to see that the graph of $10^r110^s$ must be as pictured on the left in Figure 10. Note that the only possible claws are of the form $[u : 0, d, v]$ where $1 \le d \le r+1$ and $r+3 \le u, v \le n-1$. Also, $n = r+s+4 \ge r+4$.

We consider the strings $10^r110^s$ where $r = 2^m(2q+1) - 2$ for $m \ge 2$, $q \ge 0$, and $0 \le s < 2^m - 1$. To have a claw we must have two vertices $u$ (the center of the claw) and $v$ such that $r+3 \le u, v \le r+s+3$ and $v$ is not adjacent to 0. So if $v = r+2+j$ where $j < 2^m$, $a_{0,r+2+j} \equiv$



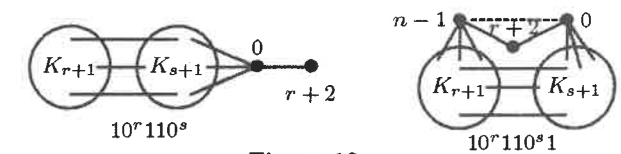



**Figure 10.**

$1 + \binom{2^\ell - 2^m(2q+1)-1}{j}$. Since $2^\ell - 1$ has a binary expansion of all 1s and subtracting $2^m(2q+1)$ from this leaves the right-most $m$ 1s as is and since the binary expansion of $j$ only has non-zero digits in the right-most $m$ positions, $a_{0,v} \equiv 1 + 1 \equiv 0$. Hence all the vertices larger than $r+2$ are adjacent to 0 and therefore there is no claw. To count these, see Lemma 29.

We next show the strings $10^r110^s$ where $r = 2^m(2q+1) - 3$ for $m \ge 2$, $q > 0$, and $0 \le s < 2^m - 1$ are claw-free. We need to compute $a_{0,r+2+j}$. Note that $2^\ell - r - 3 = 2^\ell - 2^m(2q+1)$ which has a binary expansion of $m$ right-most 0s. Also, $n - 1 = 2^m(2q+1) + s \ge r + 2 + j$ and so $j \le 2^m(2q+1) + s - r - 2 = s + 3 - 2 = s + 1 < 2^m - 1 + 1 = 2^m$. So the binary expansion of $j$ has a 1 in the right-most $m$ bits. Therefore $a_{0,r+2+j} \equiv 1 + \binom{2^\ell - 2^m(2q+1)}{j} \equiv 1 + 0 \equiv 1$ and the graph cannot have a claw. To count these, see Lemma 30.

All strings of the form $10^r110^s$ where $r = 2^m - 3$ for $m \ge 2$ and $0 \le s$ are claw-free. To have a claw in such a string, there must be two vertices $u$ (the center of the claw) and $v$ such that $r+3 \le u, v \le r+s+3$ and $v$ is not adjacent to 0. We now show that 0 is only adjacent to $r+2$ and hence there is no claw. We compute $a_{0,r+2+j} \equiv 1 + \binom{2^\ell - r - 3}{j} \equiv 1 + \binom{2^\ell - (2^m-3)-3}{j} \equiv 1 + \binom{2^\ell}{j} \equiv 1 + 0 \equiv 1$ since $0 < j < 2^\ell$.

To count these strings, note $n - 1 = 3 + (2^m - 3) + s$ where $s \ge 0$. Thus $n - 1 \ge 2^m$ where $m > 1$. Hence $1 \le m \le \lfloor \lg(n-1) \rfloor$. We double this to count the reverse of each string.

Finally, for $10^r11$, we have $n \equiv r \pmod 4$ and so these strings are not counted when $n \equiv 0,3 \pmod 4$. Hence

$$c_{3s}(n) = \begin{cases} 2, & n \equiv 0,3 \pmod 4; \\ 0, & n \equiv 1,2 \pmod 4. \end{cases}$$ ∎

**Lemma 29.** *The number of strings of length $n$ of the form $10^r110^s$ where $r = 2^m(2q+1) - 2$ for $m \ge 2$, $q \ge 0$, and $0 \le s < 2^m - 1$ is twice the following where $\lambda = \lfloor \lg(n-1) \rfloor$:*

$$\frac{1}{2} \cdot c_{32} = \begin{cases} \sum_{k=\omega}^{\lambda} b_k, & \omega \le \lambda; \\ 0, & \text{otherwise} \end{cases} = \frac{1}{2} \cdot c_{22}(n) + 1 - \delta_{n-1}.$$

**Proof:** Note that $n-1 = 2^m(2q+1) - 2 + s + 3$ and since $0 \le s < 2^m - 1$, we have $0 \le n - 2 - 2^m(2q+1) < 2^m - 1$. Adding $1 + 2^m(2q+1)$ to each part of the inequality and then dividing by $2^m$ gives

$$1 + \frac{1}{2^m} \le 2q + 1 + \frac{1}{2^m} \le \frac{n-1}{2^m} < 2q + 2. \tag{11}$$

Note that this inequality is almost (8) except that there is a 1 in the left-most part. Hence it has the same solution as (8) except that the sum extends to $\lambda$ and the only time the sum is empty is when $n-1$ is a power of two. ∎

**Lemma 30.** *For $n > 12$, the number of claw-free strings of the form $10^{2^m(2q+1)-3}110^s$ for $m > 1$, $q > 0$, and $0 \le s < 2^m - 1$ is*

$$\frac{1}{2} \cdot c_{31}(n) = \begin{cases} \sum_{k=\zeta}^{\lambda-1} b_k, & \zeta < \lambda; \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

**Proof:** Note that $n - 1 - 2^m(2q+1) = s$ and so

$$0 \le n - 1 - 2^m(2q+1) < 2^m - 1.$$

This inequality is equivalent to

$$3 \le 2q + 1 \le \frac{n-1}{2^m} < 2q + 2 - \frac{1}{2^m}. \tag{13}$$

As we did for (8), this has a solution if and only if $b_m = 1$, but for the right-hand inequality to be true, if $b_m = 1$, then there must be a 0 in the binary expansion of $n-1$ to the right of $m$. The 3 means that $m < \lambda$. Hence the number of solutions (in $m$, $n$, and $q$) for (13) is twice the expression in (12). ∎

## Section 8: Four 1s

**Lemma 31.** *The string $110^{n-5}11$ generates a claw-free graph. Hence both the strings $110^{n-4}1$ and $10^{n-4}11$ generate claw-free graphs. Also, $c_{4s}(n) = 1$.*

**Proof:** As can be seen from the matrix on the right in Figure 6, if $n$ is odd, then the graph is as the right-most graph in Figure 5 except the vertices 1 and $n-2$ are switched. If $n$ is even, then vertices 1 and $n-2$ are a component. In neither case does the graph have a claw. ∎

**Lemma 32.** *If $10^r10^s10^t1$ is claw-free for positive $r$, $s$, and $t$, then the string is bipartite.*



**Proof:** We have shown that any string with the following substrings have claws:
(a) 10100, (b) $010^{2 \bmod 4}100$, (c) 00100100, and (d) $10^{2^m-2}10^{2^\mu(2q+1)-1}10$. Note that if any of $r$, $s$, or $t$ is 1, then by (a), the others must be 1. Hence this is a bipartite string (see Lemma 19). So we assume that $r$, $s$, and $t$ are larger than 1. If $s \equiv 2 \pmod 4$, then by (b) or (c), the string has a claw. Thus by Theorem 27, $s \equiv 3 \pmod 4$. If either $r$ or $t$ is $2^m - 2$, then by (d), the string has a claw. Again by Theorem 27, all three of $r$, $s$, and $t$ must be of the form $2^m - 1$ and so the string is bipartite (from Theorem 5). ∎

**Theorem 33.** *The only claw-free strings with exactly four ones are the bipartite strings with exactly four 1s, $110^{n-5}11$, $10^{2^m-2}110^{2^m-2}1$ for $m > 1$, $10^{2^m-3}110^{2^\mu(2q+1)-3}1$ for $2 \le m \le \mu \le \nu_{n+1}$ and $q \ge 0$, and for $2 \le m \le \mu \le \nu_n$ and $q \ge 0$, $10^{2^m-2}110^{2^\mu(2q+1)-3}1$. For these strings*

$$c_4(n) = \begin{cases} 2\nu_n - 2 - 2\delta_n, & n \equiv 0 \pmod 4; \\ 2\nu_{n+1} - 2 - 3\delta_{n+1}, & n \equiv 3 \pmod 4; \\ \delta_{n-1}, & n \equiv 1 \pmod 4; \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

**Proof:** By Lemma 35, there can be three consecutive 1s in a claw-free string with exactly four 1s if and only if the string is 1111. If there are no two adjacent 1s in the string, then by Lemma 32, the string is bipartite. Note that $010^r11$ and $110^r110$ have claws, Lemmas 37 and 39. The only possible claw-free strings with four 1s with two 1s adjacent are of the form $10^r110^s1$. By Theorem 28 the string $10^r110^s$ for $s > 0$ is claw-free only if $r, s \equiv 1, 2 \pmod 4$. We write $r = 2^m(2q+1) - \pi$ and $s = 2^\mu(2\kappa+1) - \epsilon$ where $\pi$ and $\epsilon$ are 2 or 3 and both $m$ and $\mu$ are greater than 1. But not all strings with $r$ and $s$ of this type are claw-free. In what follows we assume the string $10^r110^s1$ is claw-free and note that $n - 1 = r + s + 4$.

Assume first that $\pi = \epsilon = 2$. By Theorem 28, $10^r110^s$ must have $s < 2^m - 1$ and likewise $10^s110^r$ must have $r < 2^\mu - 1$. Thus

$$2^m - 2 \le 2^m(2q+1) - 2 = r \le 2^\mu - 2 \le 2^\mu(2\kappa+1) - 2 = s \le 2^m - 2.$$

Hence $r = s$ and $q = \kappa = 0$ and so $r = s = 2^m - 2$. Thus $n - 1 = 2^{m+1}$. So if $n = 2^{m+1} + 1$ with $m > 1$, then $10^{2^m-2}110^{2^m-2}1$ is a possible claw-free string.

Next suppose $\pi = 2$ and $\epsilon = 3$. Again, $s < 2^m - 1$. If $\kappa > 1$, then by Theorem 28, $r < 2^\mu - 1$. Hence

$$2^\mu < 2^\mu(2\kappa+1) - 3 = s \le 2^m - 2 \le 2^m(2q+1) - 2 = r \le 2^\mu - 2,$$

a contradiction. Thus $\kappa = 0$ and $r = 2^m(2q+1) - 2$ and $s = 2^\mu - 3$.

The final case is $\pi = \epsilon = 3$. If $q$ and $k$ are both positive, then by Theorem 28, $r < 2^\mu - 1$ and $s < 2^m - 1$. Thus

$$2^\mu < 2^\mu(2\kappa+1) - 3 = s < 2^m - 1 < 2^m(2q+1) - 3 = r < 2^\mu - 1,$$

a contradiction. Therefore at least one of $q$ or $\kappa$ is 0. The claw-free strings containing exactly four 1s with adjacent 1s must be of the following types:

$$10^{2^m-2}110^{2^m-2}1 \quad \text{or} \quad 10^{2^m-3}110^{2^\mu(2k+1)-\pi}1 \tag{15}$$

where $\pi = 2$ or $\pi = 3$ and $m > 1$ and $\mu > 1$ or the reverse of the second string.

We now show that the strings in (15) are claw-free. The matrix for the string $10^r110^s1$ is shown in Figure 3 on the left and the resulting graph is on the right in Figure 10 where vertices 0 and $n-1$ may or may not be adjacent. Since neither $0, r+2$, or $n-1$ can be the root of a claw, the only possible claw in the graph is either $[d : n-1, e, u]$ or $[u : 0, d, v]$ where $d$ and $e$ are in $K_{r+1} = \{1, \ldots, r+1\}$ and $u$ and $v$ are in $K_{s+1} = \{r+3, \ldots, n-1\}$. In the first case, the claw would be in the graph generated by $0^r110^s1$ but for the parameters in (15), these are all claw-free. In the second case, the claw would be in the graph generated by $10^r110^s$ which is also claw-free. Therefore these graphs are claw-free.

Since $n-1 = r+s+4$ and $r$ and $s$ are only 1 or 2 modulo 4, $n$ cannot be congruent to 2 modulo 4. As shown above, if $r \equiv s \equiv 2 \pmod 4$, then $n-1$ must be a power of two which gives us the third line in (14).

Let $r = 2^m - 3$ and $s = 2^\mu(2k+1) - 2$. Since $2^m - 3 = r < 2^\mu - 1$, we have $m \leq \mu$. Now

$$n = r + s + 5 = (2^m - 3) + \big(2^\mu(2\kappa+1) - 2\big) + 5 = 2^m\big(2^{\mu-m}(2\kappa+1) + 1\big).$$

If $m < \mu$, then $m = \nu_n$. If $m = \mu$, then $m < \nu_n$. If $n = 2^{\nu_n}$, then $m < \nu_n$. So there are at most $2(\nu_n - 1 - \delta_n)$ of these strings where $n \equiv 0 \pmod 4$.

Conversely, assume $2 \leq m \leq \nu_n - \delta_n$ where $n \equiv 0 \pmod 4$. Consider the string $10^{2^m-3}110^{n-2-2^m}1$. Note that $n - 2 - 2^m \equiv 2 \pmod 4$. If $n = 2^{\nu_n}(2q+1)$, then $n-2-2^m = 2^{\nu_n}(2q+1)-2-2^m > 0$ if $q > 0$ and if $q = 0$, then $m < \nu_n$. So $n-2-2^m > 0$. Let $n-2-2^m = 2^\mu(2\kappa+1)-2$ for $\mu > 1$ and $\kappa \geq 0$. Suppose $\mu < m$. Hence $n = 2^m + 2^\mu(2\kappa+1) = 2^\mu(2^{m-\mu} + (2\kappa+1))$ and so $\mu = \nu_n \geq m > \mu$, a contradiction. Thus $2 \leq \mu \leq m$. This gives the first line of in (14).

If $r = 2^m - 3$ and $s = 2^\mu(2\kappa+1) - 3$, then the argument proceeds similarly to the previous. In this case, the string $10^{2^m-3}110^{2^m-3}1$ is a palindrome which only occurs for $n = 2^{m+1} - 1$ and so $\delta_{n+1}$ must be subtracted giving us the second line of (14). ▮



**Theorem 34.** *The only claw-free strings with more than four 1s are bipartite.*

**Proof:** There are no non-bipartite claw-free strings with exactly four 1s for which a 0 or a 1 can be appended and still be claw-free. Therefore, the only claw-free strings with more than four 1s are bipartite. ▮

## Section 9: With claws

In this section we give proofs that certain strings have claws.

**Lemma 35.** *Any string containing 0111 or 1110 generates a complement with a claw.*

**Proof:** As seen in Figure 7, the claw in the complement is $[1 : 0, 3, 4]$. ▮

**Lemma 36.** *For $2 \leq m \leq \mu$ and $\kappa \geq 1$, the string $10^{2^m-2}10^{2^\mu(2\kappa+1)-1}10$ has the claw $[2^m : 0, 2^m+1, n-1]$ where $n - 1 = 1 + 2^m + 2^\mu(2\kappa+1)$.*

**Proof:** **(i)** and **(ii)** $a_{0,2^m} = 0$ and $a_{2^m,2^m+1} = 0$ by Corollary 14 with $u = 2^m - 1$ and $v = 2^m$ (see the matrix on the left of Figure 9 where $m = \mu = 2$). **(iv)** $a_{0,2^m+1} \equiv 1 + \binom{2^\ell - 2^m}{1} \equiv 1 + 0 \equiv 1$ since the top is even and the bottom is odd.

Using Figure 9, we now let $u = n-3$ and $v = n-2$. **(iii)** $a_{2^m,n-1} = a_{n-3-(n-3-2^m),n-2+1} \equiv \binom{2^\ell-1-(n-3-2^m)}{1} \equiv 0$ since the top is even and the bottom is odd. **(v)** $a_{2^m+1,n-1} = a_{n-3-(n-4-2^m),n-2+1} \equiv \binom{2^\ell-1-(n-4-2^m)}{1} \equiv 1$ since the top is odd and the bottom is 1.

**(vi)** We need $a_{0,n-1} = 1$. Using the notation from the previous paragraph, we know $a_{2^m-1,n-2} = 0$ and $a_{2^m-1,n-1} = a_{n-3-(n-2-2^m),n-2+1} \equiv 1 + \binom{2^\ell-1-(n-2-2^m)}{1} \equiv 1 + 1 \equiv 0$ since the top is odd and the bottom is 1. Next we consider the matrix with entries $a'_{i,j}$ where the only change is that instead of $a_{2^m-1,n-2} = a_{2^m-1,n-1} = 0$, we let $a'_{2^m-1,n-2} = a'_{2^m-1,n-1} = 1$. Note that this just changes the last two columns of the matrix in Figure 9. In this situation, we again let $u = 2^m - 1$ and $v = 2^m$. Thus $a'_{0,n-1} \equiv a'_{2^m-1-(2^m-1),2^m+(n-1-2^m)} \equiv 1 + \binom{2^\ell-1-(2^m-1)}{n-1-2^m} \equiv 1 + 1 \equiv 0$ since the top is even and the bottom is odd. By the Steinhaus property, $a_{2^m-1-i,n-2} \equiv 1 + a'_{2^m-1-i,n-2}$ for $0 \leq i < 2^m$.

Note that $a_{2^m-1,n-1} \equiv 1 + a'_{2^m-1,n-1}$. So $a_{2^m-2,n-1} \overset{\text{S.P.}}{\equiv} a_{2^m-2,n-2} + a_{2^m-1,n-1} \equiv (1+a'_{2^m-2,n-2}) + (1+a'_{2^m-1,n-1}) \equiv a'_{2^m-2,n-2} + a'_{2^m-1,n-1} \overset{\text{S.P.}}{\equiv} a'_{2^m-2,n-1}$. Once again, $a_{2^m-3,n-1} \overset{\text{S.P.}}{\equiv} a_{2^m-3,n-2} + a_{2^m-2,n-1} \equiv (1 + a'_{2^m-3,n-2}) + a'_{2^m-2,n-1} \overset{\text{S.P.}}{\equiv} 1 + a'_{2^m-3,n-1}$. Inductively,

$$a_{2^m-1-i,n-1} \equiv \begin{cases} a'_{2^m-1-i,n-1}, & i \text{ odd;} \\ 1 + a'_{2^m-1-i,n-1}, & i \text{ even.} \end{cases}$$

Thus $a_{0,n-1} \equiv a'_{0,n-1} \equiv 1$. Therefore there is a claw. ▮

**Lemma 37.** *For non-negative* $r$*, the string* $010^r11$ *has a claw.*

**Proof:** By Lemma 35, if $r=0$, then the string has a claw. If $r>0$, then by using the matrix on the right of Figure 9, the vertices $\{1,2,\ldots,r+3\}$ induce $K_2 \cup K_{r+1}$ with $K_2$ induced by $\{1,r+3\}$ and hence $a_{1,r+3}=0$ and $a_{2,3}=0$. It is easy to see that the equations in (16) are true if $1 \le 2q+\rho \le r+2$ for the first equation and if $2 \le 2q+\rho \le r+3$ for the second.

$$a_{0,2q+\rho} \equiv \rho+1 \quad \text{and} \quad a_{2q+\rho,r+4} \equiv r+\rho. \tag{16}$$

Thus $a_{0,r+3} \overset{\text{S.P.}}{\equiv} a_{0,r+2}+a_{1,r+3} \equiv a_{0,r+2} \equiv r+1$ and $a_{1,r+4} \overset{\text{S.P.}}{\equiv} a_{1,r+3}+a_{2,r+4} \equiv a_{2,r+4} \equiv r$. Finally, $a_{0,r+4} \overset{\text{S.P.}}{\equiv} a_{0,r+3}+a_{1,r+4} \equiv (r+1)+r \equiv 1$. To summarize we have

$$a_{0,r+3} \equiv r+1, \quad a_{0,r+4}=1, \quad \text{and} \quad a_{1,r+4} \equiv r. \tag{17}$$

If $r$ is even, we show that the string has the claw $[1:0,r+3,r+4]$. **(i)** and **(vi)** From (16), $a_{0,1}=0$ and $a_{r+3,r+4} \equiv (r+1)+r \equiv 1$. **(ii)** From the discussion at the beginning of the proof, $a_{1,r+3}=0$. **(iii)**, **(iv)**, and **(v)** From (17), $a_{1,r+4} \equiv r \equiv 0$, $a_{0,r+3} \equiv r+1 \equiv 1$, and $a_{0,r+4}=1$.

If $r$ is odd, we show that the string has the claw $[3:0,2,r+4]$. **(i)**, **(iii)**, **(iv)**, and **(vi)** From (16) we know that $a_{0,3}=0$, $a_{3,r+4} \equiv 1+r \equiv 0$, $a_{0,2}=1$, $a_{2,r+4} \equiv r+2 \equiv 1$. **(ii)** From the discussion at the beginning of the proof, $a_{2,3}=0$. **(v)** From (17) we have $a_{0,r+4}=1$ and hence $[3:0,2,r+4]$ is a claw. ∎

**Theorem 38.** *If* $p=2^m-1$*, then the string* $10^r110^s$ *has the claw*

| | | | |
|---|---|---|---|
| (1) $[r+3:0,1,r+4]$ | if | $r \equiv 0 \pmod 4$, | $s>0$; |
| (2) $[r+4:0,3,r+3]$ | if | $r \equiv 3 \pmod 4$, | $s>0$; |
| (3) $[r+3:0,1,r+2+2^m]$ | if | $r=2^m(2q+1)-2$, $m>1$, $q \ge 0$, | $s \ge p$; |
| (4) $[r+2+2^m:0,2^m+1,r+3]$ | if | $r=2^m(2q+1)-3$, $m>1$, $q>0$, | $s \ge p$. |

**Proof:** From Corollary 15 note that $K_{r+1}=\{1,\ldots,r+1\}$ and $K_{s+1}=\{r+3,\ldots,n-1\}$ where $n=r+s+4$. The only possible claws have the form $[u:0,d,v]$ where $1 \le d \le r+1$ and $r+3 \le u,v \le n-1$. Also,

(a) $a_{0,1}=a_{0,3}=1$,
(b) $a_{0,r+2}=a_{r+3,r+4}=a_{r+3,r+2+2^m}=0$,
(c) $a_{0,r+2+j} \equiv 1+\binom{2^\ell-r-3}{j}$
(d) $a_{r+2-i,n-1} \equiv \binom{2^\ell-1-i}{s+1}$
(e) $a_{3,r+2+j} \equiv \binom{2^\ell-r}{j}$
(f) $a_{1,r+2+j} \equiv \binom{2^\ell-r-2}{j}$

(1) In this case, $r=4q$, $s=1$, and $n=r+5$. **(i)** $a_{0,r+3} \equiv 1+\binom{2^\ell-r-3}{1} \equiv 1+1 \equiv 0$ since the top is odd and the bottom is 1. **(ii)** $a_{1,r+3} \equiv \binom{2^\ell-r-2}{1} \equiv 0$ since the top is even and the bottom is odd. **(iii)** and **(iv)** $a_{r+3,r+4}=0$ and $a_{0,1}=1$ from the displayed equations. **(v)** $a_{0,r+4} \equiv 1+\binom{2^\ell-r-3}{2} \equiv 1+0 \equiv 1$ since $\tau_1=0$ and $\beta_1=1$. **(vi)** $a_{1,r+4} \equiv \binom{2^\ell-r-2}{2} \equiv 1$ since $\tau_1=\beta_1=1$.

(2) This case is similar to (1) and so we omit it.

(3) In this case, $r=2^m(2q+1)-2$ where $m \ge 2$, $q \ge 0$, and $s=2^m-1$. **(i)** $a_{0,r+3} \equiv 1+\binom{2^\ell-2^m(2q+1)-1}{1} \equiv 1+1 \equiv 0$ since the top is odd and the bottom is 1. **(ii)** $a_{1,r+3} \overset{\text{S.P.}}{\equiv} a_{0,r+2}+a_{0,r+3} \equiv 0+0=0$. **(iii)** and **(iv)** $a_{r+3,r+2+2^m}=0$ and $a_{0,1}=1$ from the displayed equations. **(v)** $a_{0,r+2+2^m} \equiv 1+\binom{2^\ell-2^m(2q+1)-1}{2^m} \equiv 1+0 \equiv 1$ since $\tau_m=0$ and $\beta_m=1$. **(vi)** $a_{1,r+2+2^m} \equiv \binom{2^\ell-2^m(2q+1)}{2^m} \equiv 1$ since $\tau_m=1$ and the only 1 in the expansion of $2^m$ is $\beta_m=1$.

(4) This case is similar to (3) and so we omit it. ∎

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| | | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 2 |
| | | | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | · |
| | | | | 0 | 0 | 0 | 0 | 1 | 1 | 1 | · |
| | | | | | 0 | 0 | 0 | 1 | 0 | 0 | $r$ |
| | | | | | | 0 | 0 | 1 | 1 | 0 | |
| | | | | | | | 0 | 1 | 0 | 1 | |
| | | | | | | | | 0 | 1 | 1 | |
| | | | | | | | | | 0 | 0 | |
| | | | | | | | | | | 0 | |

**Figure 11.** $110^r110$

**Lemma 39.** *For* $r$ *positive, the string* $110^r110$ *always has a claw.*

**Proof:** For $r \equiv 0,3 \pmod 4$, $10^r110$ has a claw by Theorem 38. For $r \equiv 1,2 \pmod 4$ we use Corollary 15 but since our string starts 11, $u=v=r+3$ and not $r+2$ and row 1 is now what row 0 is in Corollary 15. See Figure 11. Here is what we know

(a) $a_{1,r+3}=0$,
(b) $a_{0,j} \equiv j$ for $0<j<r+3$,
(c) $a_{i,j}=0$ for $2 \le i < j \le r+2$
(d) $a_{r+2-i,n-1} \equiv \binom{2^\ell-1-i}{s+1}$
(e) $a_{r+3-i,r+4} \equiv 1+i$, for $0 \le i < r+1$,
(f) $a_{r+3-i,r+3+j} \equiv \binom{2^\ell-1-i}{j}$

From (e), $a_{i,r+4}=a_{r+3-(r+3-i),r+4} \equiv r+3-i+1 \equiv r+i$.

If $r \equiv 1 \pmod 4$, then a claw is $[2:0,3,r+3]$. **(i)**, **(ii)**, **(iv)** $a_{0,2} \equiv 2 \equiv 0$, $a_{0,3} \equiv 3 \equiv 1$, and $a_{2,3}=0$ from (b) and (c). **(iii)** $a_{2,r+5} \equiv \binom{2^\ell-1-(r+2)}{2} \equiv 0$ since $2^\ell-2-r \equiv 1 \pmod 4$ ($\tau_1=0$ and $\beta_1=1$). **(v)** Since $r \equiv 1 \pmod 4$, $2^\ell-1-(r+2) \equiv 0 \pmod 4$. Thus $\binom{2^\ell-1-(r+2)}{j} \equiv 0$ for both $j=1,2$. Hence $a_{1,r+3+j} \equiv 1+0 \equiv 1$ for $j=1$ and $j=2$. Since $a_{0,r+2} \equiv r+2 \equiv 1$, the top corner of the matrix can be completed using the Steinhaus property (see the matrix below on the left). Thus $a_{0,5}=1$.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| row 0 | … | 1 | 1 | 0 | 1 | row 0 | … | 0 | 0 | 1 | 0 |
| row 1 | … | | 0 | 1 | 1 | row 1 | … | | 0 | 1 | 1 |

**(vi)** $a_{3,r+5} \equiv \binom{2^\ell-1-r}{2} \equiv 1$ since $2^\ell-1-r \equiv 2 \pmod 4$.

The case $r \equiv 2 \pmod 4$ is similar to the previous and is omitted. ∎

## Section 10: Bounds

**Theorem 40.** *For $n > 16$ and with equality for $n = 2^m$,*

$$c(n) \leq \left\lfloor \frac{7n}{2} + 11\lfloor \lg(n+1) \rfloor - 26 \right\rfloor . \quad (18)$$

**Proof:** Using (2), (3), and induction it is easy to see that $b(2^m) = 2^{m-1} - m$ and using Corollary 9, $c(2^m) = 7 \cdot 2^{m-1} + 11m - 26$ which is equal to the upper bound. To show that no other value of $c(n)$ exceeds or equal this bound requires a proof that $b(n) \leq \frac{1}{2}n - \lambda_n$ and $b(4q+3) \leq 2q + 3 - 2\lambda_{4q+3}$. While these are not difficult to prove, we do not do so here. ∎

The following conjecture has much evidence to support it. First, the right-hand inequality is the same as (18) for $n = 2^m$ and is true for $n > 2^{12}$ (the only values where it is not true are $n = 24$ and $n = 2^m - 1$ for $m < 13$). The bound on the left in the following is achieved for $n = \frac{1}{3}(4^m - 10)$. From Corollary 9, and the bounds on $b(n)$, it is easily seen that a lower bound for $c(n)$ is $\frac{1}{8}(25n + 6) + 4\lfloor \lg(n) \rfloor$. To prove that (19) is the best lower bound requires a careful analysis of the interplay between $b(n)$ and the two sums and case statement in (6), which has not completely been done.

**Conjecture 1.** *For $n > 2^{12}$,*

$$\left\lceil \frac{25n+6}{8} + 7\lfloor \lg(n) \rfloor - 16 \right\rceil \leq c(n) \leq \left\lfloor \frac{7n}{2} + 11\lfloor \lg(n) \rfloor - 26 \right\rfloor . \quad (19)$$

We also have formulas for the number of claw-free strings with exactly $k$ 1s, but with the results in this paper, these are mainly proofs about the number of bipartite strings with exactly $k$ 1s and so we did not include these results.

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