C_4-FACE-MAGIC TORUS LABELINGS ON $C_4 \times C_4$

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ABSTRACT. For a toroidal graph $G = (V, E)$ embedded in the torus, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a C_n-face-magic torus graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labelings along C_n is a constant S. Let $x_v = f(v)$ for all $v \in V(G)$. We call $\{x_v : v \in V(G)\}$ a C_n-face magic torus labeling on G. We say that a C_4-face-magic torus labeling $\{x_{i,j}\}$ on $C_{2n} \times C_{2n}$ is antipodal balanced if $x_{i,j} + x_{i+n,j+n} = \frac{1}{2}S$, for all $(i, j) \in V(C_{2n} \times C_{2n})$. We determine all antipodal balanced C_4-face-magic torus labelings on $C_4 \times C_4$ up to symmetries on a torus.

1. INTRODUCTION

Graph labelings form an important part of graph theory. First formally introduced in the 1970s by Kotzig and Rosa [8], this area of research has been the subject matter for many papers in the mathematical literature. Certain types of graph labelings have applications to graph decomposition problems, radar pulse code designs, X-ray crystallography and communication network models. The interested reader is directed to J.A. Gallian's comprehensive dynamic survey on graph labelings [5].

For concepts and notation not explicitly defined in this paper, the reader can refer to [3]. All graphs in this paper are simple and connected.

The focus of this paper is motivated by a *magic labeling of type* (a, b, c), introduced by Lih [9]. Subsequent papers on (a, b, c)-type magic labelings are found in [1, 2, 6, 7]. For a planar graph $G = (V, E)$ embedded in $\mathbb{R}^2$, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a *C_n-face-magic* graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labelings along C_n is a constant S. Here, the constant S is called a *C_4-face-magic value* of G. For a toroidal graph $G = (V, E)$ embedded in the torus, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a *C_n-face-magic torus* graph if there exists

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a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labelings along C_n is a constant S.

In this paper, we study C_4-face-magic torus labelings on $C_{2n} \times C_{2n}$. In the manuscript [4], the authors investigate further results on C_4-face-magic torus labelings on $C_{2m} \times C_{2n}$.

2. Preliminaries

We begin by showing that any C_4-face-magic labeling on $P_{2n} \times P_{2n}$ yields a C_4-face-magic torus labeling on $C_{2n} \times C_{2n}$.

Lemma 1. *Let n be a positive integer. A C_4-face-magic labeling on $P_{2n} \times P_{2n}$ always yields a C_4-face-magic labeling on $C_{2n} \times C_{2n}$ with its natural embedding on the torus.*

Proof. Let $x_{i,j}$ be the C_4-face-magic labeling on vertex (i,j), for $i = 1, 2, \ldots, 2n$ and $j = 1, 2, \ldots, 2n$. Let S be the C_4-face-magic value on $P_{2n} \times P_{2n}$. Then we have $x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} = S$, for all $i = 1, 2, \ldots, 2n-1$ and $j = 1, 2, \ldots, 2n-1$. We observe that

$$n^2 S = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_{2i-1,2j-1} + x_{2i,2j-1} + x_{2i-1,2j} + x_{2i,2j})$$
$$= \sum_{i=1}^{4n^2} i = \tfrac{1}{2}(4n^2)(4n^2+1).$$

Thus, $S = 2(4n^2+1)$. Since $x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = S = x_{i+1,j} + x_{i+2,j} + x_{i+1,j+1} + x_{i+2,j+1}$, we have $x_{i,j} + x_{i,j+1} = x_{i+2,j} + x_{i+2,j+1}$. An induction argument shows that $x_{i,j} + x_{i,j+1} = x_{i+2k,j} + x_{i+2k,j+1}$. Since $x_{i+2k,j} + x_{i+2k,j+1} + x_{i+2k+1,j} + x_{i+2k+1,j+1} = S$, we have $x_{i,j} + x_{i,j+1} + x_{i+2k+1,j} + x_{i+2k+1,j+1} = S$. A similar argument shows that $x_{i,j} + x_{i+1,j} + x_{i,j+2\ell+1} + x_{i+1,j+2\ell+1} = S$. This, in turn, yields $x_{i,j} + x_{i+2k+1,j} + x_{i,j+2\ell+1} + x_{i+2k+1,j+2\ell+1} = S$. Hence, we have $x_{1,j} + x_{1,j+1} + x_{2n,j} + x_{2n,j+1} = S$, for all $j = 1, 2, \ldots, 2n-1$. Similarly, we have $x_{i,1} + x_{i+1,1} + x_{i,2n} + x_{i+1,2n} = S$, for all $i = 1, 2, \ldots, 2n-1$. Lastly, we have $x_{1,1} + x_{2n,1} + x_{1,2n} + x_{2n,2n} = S$. Therefore, the C_4-face-magic labeling on $P_{2n} \times P_{2n}$ yields a C_4-face-magic labeling on $C_{2n} \times C_{2n}$ with its natural embedding on the torus. □

Definition 2. We say that the C_4-face-magic torus labeling $\{x_{i,j} : i = 1, 2, \ldots, 2n$ and $j = 1, 2, \ldots, 2n\}$ on $C_{2n} \times C_{2n}$ is *antipodal balanced* if $x_{i,j} + x_{i+n,j+n} = \frac{1}{2}S = 4n^2 + 1$, for all integers i and j such that $1 \le i \le 2n$ and $1 \le j \le 2n$.

Remark 3. We give a brief explanation for the term *antipodal balanced.* On the n-sphere $S^n \subsetneq \mathbb{R}^{n+1}$, the antipodal map $p : S^n \to S^n$ is given by $p(\mathbf{x}) = -\mathbf{x}$. Similarly, on the torus $T^2 = S^1 \times S^1 \subsetneq \mathbb{C}^2 \cong \mathbb{R}^4$, we define the antipodal map $p : T^2 \to T^2$ by $p(e^{i\theta_1}, e^{i\theta_2}) = -(e^{i\theta_1}, e^{i\theta_2}) = (e^{i(\theta_1+\pi)}, e^{i(\theta_2+\pi)})$. Thus an antipodal balanced C_4-face-magic torus label on $C_{2n} \times C_{2n}$ is one in which the sum of the labels on a vertex and its antipodal vertex is constant for all vertices in $C_{2n} \times C_{2n}$.

Lemma 4. *Let n be a positive integer. Let $\{x_{i,j} : i,j = 1, 2, \ldots, 2n\}$ be an antipodal balanced C_4-face-magic torus labeling on $C_{2n} \times C_{2n}$. For all integers i such that $1 \le i \le n$, we define*

$$d_i = x_{i(n-1)+1,in+1} - x_{(i-1)(n-1)+1,(i-1)n+1}.$$

Then for all integers i and j such that $1 \le i \le n$ and $1 \le j \le 2n$, we have

$$x_{i(n-1)+1,in+j} = x_{(i-1)(n-1)+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

Proof. By the definition of d_i, we have

$$x_{i(n-1)+1,in+1} = x_{(i-1)(n-1)+1,(i-1)n+1} + d_i.$$

We apply an induction argument involving the value j. Thus we assume that

$$x_{i(n-1)+1,in+j-1} = x_{(i-1)(n-1)+1,(i-1)n+j-1} + (-1)^j d_i. \tag{1}$$

Since the labeling is antipodal balanced, we have

$$x_{i(n-1)+2,in+j-1} = \tfrac{1}{2}S - x_{(i-1)(n-1)+1,(i-1)n+j-1}, \text{ and} \tag{2}$$

$$x_{i(n-1)+2,in+j} = \tfrac{1}{2}S - x_{(i-1)(n-1)+1,(i-1)n+j}, \tag{3}$$

where $S = 2(4n^2+1)$ is the C_4-face-magic value of $\{x_{i,j}\}$. Since $\{x_{i,j}\}$ is a C_4-face-magic torus labeling on $C_{2n} \times C_{2n}$, we have

$$x_{i(n-1)+1,in+j-1} + x_{i(n-1)+1,in+j} + x_{i(n-1)+2,in+j-1} + x_{i(n-1)+2,in+j} = S. \tag{4}$$

When we substitute the expressions from equations (1), (2) and (3) into equation (4), we obtain

$$x_{i(n-1)+1,in+j} = x_{(i-1)(n-1)+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

This completes the proof. □

We investigate how the conditions of Lemma 4 apply to the case $C_{2n} \times C_{2n}$, where n is even.

Lemma 5. *Let n be a positive even integer. Let $\{x_{i,j} : i,j = 1, 2, \ldots, 2n\}$ be an antipodal balanced C_4-face-magic torus labeling on $C_{2n} \times C_{2n}$. For all integers i such that $1 \le i \le n$, we define*

$$d_i = x_{i(n-1)+1,in+1} - x_{(i-1)(n-1)+1,(i-1)n+1}.$$

Then, by Lemma 4, for all integers i and j such that $1 \le i \le n$ and $1 \le j \le 2n$, we have

$$x_{i(n-1)+1,in+j} = x_{(i-1)(n-1)+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

Also, for all integers i and j such that $n+1 \le i \le 2n$ and $1 \le j \le 2n$, we have

$$x_{i(n-1)+1,in+j} = x_{(i-1)(n-1)+1,(i-1)n+j} + (-1)^j d_{i-n}.$$

For all integers j such that $1 \le j \le n$, we define

$$d'_j = x_{jn+1,j(n-1)+1} - x_{(j-1)n+1,(j-1)(n-1)+1}.$$

Then for all integers i and j such that $1 \le i \le 2n$ and $1 \le j \le n$, we have

$$x_{jn+i,j(n-1)+1} = x_{(j-1)n+i,(j-1)(n-1)+1} + (-1)^{i+1} d'_j.$$

Also, for all integers i and j such that $1 \le i \le 2n$ and $n+1 \le j \le 2n$, we have

$$x_{jn+i,j(n-1)+1} = x_{(j-1)n+i,(j-1)(n-1)+1} + (-1)^i d'_{j-n}.$$

Proof. By Lemma 4, for all integers i and j such that $1 \le i \le n$ and $1 \le j \le 2n$, we have

(5) $$x_{i(n-1)+1,in+j} = x_{(i-1)(n-1)+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

Since the indices $in+j$ and $(i-1)n+j$ are reduced modulo $2n$, equation (5) holds for all integers j. Let i be an integer such that $n+1 \le i \le 2n$. We replace i with $i-n$ and j with $n+j$ in equation (5) to obtain

$$x_{(i-n)(n-1)+1,(i-n)n+n+j} = x_{(i-n-1)(n-1)+1,(i-n-1)n+n+j} + (-1)^{n+j+1} d_{i-n}.$$

This reduces to

(6) $$x_{i(n-1)+n+1,(i+1)n+j} = x_{(i-1)(n-1)+n+1,in+j} + (-1)^{j+1} d_{i-n}.$$

Since $\{x_{i,j}\}$ is antipodal balanced, we have

(7) $$x_{i(n-1)+n+1,(i+1)n+j} = \tfrac{1}{2}S - x_{i(n-1)+1,in+j}, \text{ and}$$

(8) $$x_{(i-1)(n-1)+n+1,in+j} = \tfrac{1}{2}S - x_{(i-1)(n-1)+1,(i-1)n+j},$$

When we substitute the expressions in equations (7) and (8) into equation (6), we have, for all integers i and j such that $n+1 \le i \le 2n$ and $1 \le j \le 2n$,

(9) $$x_{i(n-1)+1,in+j} = x_{(i-1)(n-1)+1,(i-1)n+j} + (-1)^j d_{i-n}.$$

When we interchange the roles of i and j in the previous argument, when $1 \le i \le 2n$ and $1 \le j \le n$, we have

$$x_{jn+i,j(n-1)+1} = x_{(j-1)n+i,(j-1)(n-1)+1} + (-1)^{i+1} d'_j;$$

and when $1 \le i \le 2n$ and $n+1 \le j \le 2n$, we have

$$x_{jn+i,j(n-1)+1} = x_{(j-1)n+i,(j-1)(n-1)+1} + (-1)^i d'_{j-n}.$$

□

3. Results on $C_4 \times C_4$

We first investigate the C_4-face-magic torus labelings on $C_4 \times C_4$. We begin with the following definition.

Definition 6. Let n be a positive integer Let $\{x_{i,j} : i,j = 1,2,\ldots,2n\}$ be a C_4-face-magic torus labeling on $C_{2n} \times C_{2n}$. We say that the C_4-face-magic labeling $\{x_{i,j}\}$ on $C_{2n} \times C_{2n}$ is *torus symmetric* if all row sums, column sums, and diagonal sums have a constant value S'. I.e., the sums

$$R_i = \sum_{j=1}^{2n} x_{i,j} = S' \qquad \text{for all } i = 1,2,\ldots,2n$$

$$C_j = \sum_{i=1}^{2n} x_{i,j} = S' \qquad \text{for all } j = 1,2,\ldots,2n$$

$$D_j = \sum_{i=1}^{2n} x_{i,i+j} = S' \qquad \text{for all } j = 1,2,\ldots,2n, \text{ and}$$

$$D'_j = \sum_{i=1}^{2n} x_{i,j-i} = S' \qquad \text{for all } j = 1,2,\ldots,2n.$$

are constant.

Notation 7. Let $\{x_{i,j} : i,j = 1,2,\ldots,2n\}$ be a C_4-face-magic torus labeling on $C_{2n} \times C_{2n}$. For all integers i and j, we define

$$B_{i,j} = x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2}.$$

In Lemma 8, we show that a C_4-face-magic torus labeling on $C_4 \times C_4$, in which each 2 by 2 block sum value $B_{i,j}$ has the C_4-face-magic value S, is equivalent to a torus symmetric C_4-face-magic torus labeling on $C_4 \times C_4$.

Lemma 8. *Consider the system of linear equations $x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} = 34$ (S1), for all $i = 1,2,3$ and $j = 1,2,3$ for a C_4-face-magic labeling on $P_4 \times P_4$. Let (S2) be the system (S1) together with the equations $B_{1,1} = x_{1,1} + x_{1,3} + x_{3,1} + x_{3,3} = 34$, $B_{1,2} = x_{1,2} + x_{1,4} + x_{3,2} + x_{3,4} = 34$, and $B_{2,1} = x_{2,1} + x_{2,3} + x_{4,1} + x_{4,3} = 34$. If the labeling $\{x_{i,j}\}$ satisfies system (S2), then $\{x_{i,j}\}$ is torus symmetric. Also, let (S3) be the system (S1) together with the equations $R_1 = x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 34$, $C_1 = x_{1,1} + x_{2,1} + x_{3,1} + x_{4,1} = 34$, and $D_4 = x_{1,1} + x_{2,2} + x_{3,3} + x_{4,4} = 34$. Then, (S2) is equivalent to (S3).*

Proof. First, we show that system S3 yields a torus symmetric labeling on $C_4 \times C_4$. For convenience, let $S = 34$ be the C_4-face-magic value of $\{x_{i,j}\}$. Observe that the equations $x_{i,1} + x_{i,2} + x_{i+1,1} + x_{i+1,2} = S$ and $x_{i,3} + x_{i,4} + x_{i+1,3} + x_{i+1,4} = S$ produces the equation

$$(x_{i,1}+x_{i,2}+x_{i,3}+x_{i,4}) + (x_{i+1,1}+x_{i+1,2}+x_{i+1,3}+x_{i+1,4}) = 2S.$$

Thus if

$$R_i = x_{i,1}+x_{i,2}+x_{i,3}+x_{i,4} = S,$$

then

$$R_{i+1} = x_{i+1,1}+x_{i+1,2}+x_{i+1,3}+x_{i+1,4} = S.$$

Since

$$R_1 = x_{1,1}+x_{1,2}+x_{1,3}+x_{1,4} = S,$$

we have

(10) $\quad R_i = x_{i,1}+x_{i,2}+x_{i,3}+x_{i,4} = S,$ for $i = 1,2,3$ and 4.

Similarly, we have

(11) $\quad C_j = x_{1,j}+x_{2,j}+x_{3,j}+x_{4,j} = S,$ for $j = 1,2,3$ and 4.

Consider the diagonal sums

$$D_j = x_{1,j+1}+x_{2,j+2}+x_{3,j+3}+x_{4,j+4}, \text{ for } j = 1,2,3 \text{ and } 4.$$

Then,

$$\begin{aligned} D_j + 2D_{j+1} + D_{j+2} = &\,(x_{1,j+2}+x_{1,j+3}+x_{2,j}+x_{2,j+1}) \\ &+ (x_{2,j+3}+x_{2,j}+x_{3,j+3}+x_{3,j}) \\ &+ (x_{3,j}+x_{3,j+1}+x_{4,j}+x_{4,j+1}) \\ &+ (x_{4,j+1}+x_{4,j+2}+x_{1,j+1}+x_{1,j+2}) = 4S. \end{aligned}$$

Thus

$$D_j + 2D_{j+1} + D_{j+2} = 4S = D_{j+1} + 2D_{j+2} + D_{j+3}$$

implies that

$$D_j + D_{j+1} = D_{j+2} + D_{j+3}.$$

Since

$$D_j + D_{j+1} + D_{j+2} + D_{j+3} = 4S,$$

we have

$$D_j + D_{j+1} = 2S.$$

Since

$$D_4 = x_{1,1}+x_{2,2}+x_{3,3}+x_{4,4} = S,$$

we have

(12) $\quad D_j = x_{1,j+1}+x_{2,j+2}+x_{3,j+3}+x_{4,j+4} = S,$ for $j = 1,2,3$ and 4.

Let

$$D_1' = x_{1,4}+x_{2,3}+x_{3,2}+x_{4,1}.$$

Observe that

$$\begin{aligned} D_1' + D_2 = &\,(x_{1,3}+x_{1,4}+x_{2,3}+x_{2,4}) \\ &+ (x_{3,1}+x_{3,2}+x_{4,1}+x_{4,2}) = 2S. \end{aligned}$$

Since $D_2 = S$, we have

$$D_1' = x_{1,4}+x_{2,3}+x_{3,2}+x_{4,1} = S.$$

An argument similar to the one for equation (12), shows that

(13) $\quad D_j' = x_{4,j}+x_{3,j+1}+x_{2,j+2}+x_{1,j+3} = S,$ for $j = 1,2,3,$ and 4.

Second, we show that we can obtain system (S2) from system (S3). We observe that

$$\begin{aligned} 6S &= R_i + R_{i+2} + C_j + C_{j+2} + D_{j-i} + D_{j-i+2} \\ &= 2(x_{i,j}+x_{i,j+2}+x_{i+2,j}+x_{i+2,j+2}) + \left(\sum_{i=1}^{4}\sum_{j=1}^{4} x_{i,j}\right) \\ &= 2B_{i,j} + 4S. \end{aligned}$$

Hence,

(14) $\quad B_{i,j} = x_{i,j}+x_{i,j+2}+x_{i+2,j}+x_{i+2,j+2} = S,$ for $i,j = 1$ and 2.

Third, we show that we can obtain system (S3) from system (S2). We have the equations

$B_{i,j} = x_{i,j}+x_{i,j+2}+x_{i+2,j}+x_{i+2,j+2} = S,$ for $(i,j) = (1,1),\ (1,2)$ and $(2,1)$.

Since

$$3S + B_{2,2} = B_{1,1} + B_{1,2} + B_{2,1} + B_{2,2} = \left(\sum_{i=1}^{4}\sum_{j=1}^{4} x_{i,j}\right) = 4S,$$

we have

$$B_{2,2} = x_{2,2}+x_{2,4}+x_{4,2}+x_{4,4} = S.$$

Since

$$\begin{aligned} 4S &= \sum_{i=1,4}\sum_{j=1,3}\left(x_{i,j}+x_{i,j+1}+x_{i+1,j}+x_{i+1,j+1}\right) \\ &= R_4 + 2R_1 + R_2 = 2R_1 + B_{2,1} + B_{2,2} = 2R_1 + 2S, \end{aligned}$$

we have

$$R_1 = \sum_{j=1}^{4} x_{1,j} = S.$$

A similar argument using the expression $C_4 + 2C_1 + C_2$ yields the equation

$$C_1 = \sum_{i=1}^{4} x_{i,1} = S.$$

Since

$$4S = \sum_{i=1}^{4}\Big(x_{i,i} + x_{i,i+1} + x_{i+1,i} + x_{i+1,i+1}\Big)$$
$$= D_3 + 2D_4 + D_1 = 2D_4 + B_{1,2} + B_{2,1} = 2D_4 + 2S,$$

we have

$$D_4 = \sum_{i=1}^{4} x_{i,i} = S.$$

□

Definition 9. Consider the natural embedding of $C_{2n} \times C_{2n}$ on the torus. We say that two torus symmetric C_4-face-magic torus labelings $\{x_{(i,j)}\}$ and $\{x'_{(i,j)}\}$ on $C_{2n} \times C_{2n}$ are *torus equivalent* if there is a homeomorphism of the torus $\phi : T^2 \longrightarrow T^2$ that induces a graph isomorphism on $C_{2n} \times C_{2n}$ such that the first C_4-face-magic torus labeling $\{x_{(i,j)}\}$ on $C_{2n} \times C_{2n}$ is mapped to the second C_4-face-magic torus labeling $\{x'_{(i,j)}\}$ on $C_{2n} \times C_{2n}$. That is, we have $x'_{\phi(i,j)} = x_{(i,j)}$, for all $(i,j) \in V(C_{2n} \times C_{2n})$.

Theorem 10. *There are three distinct torus inequivalent torus symmetric C_4-face-magic torus labelings on $C_4 \times C_4$. These three distinct torus inequivalent torus symmetric C_4-face-magic torus labelings on $C_4 \times C_4$ are given below:*

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

TABLE 1. Torus symmetric C_4-face-magic torus labeling A on $C_4 \times C_4$, with C_4-face-magic value 34.

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

TABLE 2. Torus symmetric C_4-face-magic torus labeling B on $C_4 \times C_4$, with C_4-face-magic value 34.

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

TABLE 3. Torus symmetric C_4-face-magic torus labeling C on $C_4 \times C_4$, with C_4-face-magic value 34.

Remark 11. We observe that the vertex labeled 1 in each of the three C_4-face-magic torus labelings on $C_4 \times C_4$ in the previous theorem are adjacent to the vertices with the labelings 8, 12, 14 and 15. What distinguishes the three torus inequivalent labelings on $C_4 \times C_4$ is determined by which vertex labeling occurs opposite from the vertex labeled 8 with respect to the vertex labeled 1. In labeling A, the vertex labeled 12 is opposite the vertex labeled 8 with respect to the vertex labeled 1. In labeling B, the vertex labeled 14 is opposite the vertex labeled 8 with respect to the vertex labeled 1. In labeling C, the vertex labeled 15 is opposite the vertex labeled 8 with respect to the vertex labeled 1.

Remark 12. The 16 C_4 face sums of one of the C_4-face-magic torus labelings on $C_4 \times C_4$ are the 4 row sums and 4 column sums of the other two C_4-face-magic torus labelings on $C_4 \times C_4$. For example, the 16 C_4 face sums of labeling A are the 4 row sums and 4 column sums of labelings B and C.

In Lemma 13, we show that every torus symmetric C_4-face-magic torus labeling on $C_4 \times C_4$ is antipodal balanced.

Lemma 13. *Let $\{x_{i,j}\}$ be a torus symmetric C_4-face-magic torus labeling on $C_4 \times C_4$. Then, for all i and j, we have $x_{i,j} + x_{i+2,j+2} = 17$ where the indices are taken modulo 4. That is, the labeling $\{x_{i,j}\}$ is antipodal balanced.*

Proof. Consider the sum

$$\begin{aligned}2S &= D_{j-i} + D'_{i+j}\\ &= \big(x_{i,j} + x_{i+1,j+1} + x_{i+2,j+2} + x_{i+3,j+3}\big)\\ &\quad + \big(x_{i,j} + x_{i+1,j+3} + x_{i+2,j+2} + x_{i+3,j+1}\big)\\ &= 2(x_{i,j} + x_{i+2,j+2}) + B_{i+1,j+1} = 2(x_{i,j} + x_{i+2,j+2}) + S.\end{aligned}$$

Thus, $x_{i,j} + x_{i+2,j+2} = \frac{1}{2}S = 17$. □

When we specialize Lemma 5 to the case when $n = 2$, we have the following corollary.

Corollary 14. *Let $\{x_{i,j}\}$ be a torus symmetric C_4-face-magic torus labeling on $C_4 \times C_4$. Let $d_i = x_{i+1,2i+1} - x_{i,2i-1}$, for $i = 1$ and 2. Also, let $d_{j+2} = d'_j = x_{2j+1,j+1} - x_{2j-1,j}$, for $j = 1$ and 2. Then for all integers i and j, when $1 \le i \le 2$ and $1 \le j \le 4$, we have*

$$x_{i+1,2i+j} = x_{i,2i-2+j} + (-1)^{j+1} d_i, \tag{15}$$

and when $3 \le i \le 4$ and $1 \le j \le 4$, we have

$$x_{i+1,2i+j} = x_{i,2i-2+j} + (-1)^{j} d_{i-2}. \tag{16}$$

Furthermore, for all integers i and j, when $1 \le i \le 4$ and $1 \le j \le 2$, we have

$$x_{2j+i,j+1} = x_{2j-2+i,j} + (-1)^{i+1} d_{j+2} \tag{17}$$

and when $1 \le i \le 4$ and $3 \le j \le 4$, we have

$$x_{2j+i,j+1} = x_{2j-2+i,j} + (-1)^{i} d_{j}. \tag{18}$$

By relabeling the indices in each of the equations in Corollary 14, we obtain the following corollary.

Corollary 15. *Let $\{x_{i,j}\}$ be a torus symmetric C_4-face-magic torus labeling on $C_4 \times C_4$. Let $d_i = x_{i+1,2i+1} - x_{i,2i-1}$, for $i = 1$ and 2. Also, let $d_{j+2} = d'_j = x_{2j+1,j+1} - x_{2j-1,j}$, for $j = 1$ and 2. Then for all integers i and j, when $1 \le i \le 2$ and $1 \le j \le 4$, we have*

$$x_{i+1,j+2} = x_{i,j} + (-1)^{j+1} d_i, \tag{19}$$

and when $3 \le i \le 4$ and $1 \le j \le 4$, we have

$$x_{i+1,j+2} = x_{i,j} + (-1)^{j} d_{i-2}. \tag{20}$$

Furthermore, for all integers i and j, when $1 \le i \le 4$ and $1 \le j \le 2$, we have

$$x_{i+2,j+1} = x_{i,j} + (-1)^{i+1} d_{j+2} \tag{21}$$

and when $1 \le i \le 4$ and $3 \le j \le 4$, we have

$$x_{i+2,j+1} = x_{i,j} + (-1)^{i} d_{j}. \tag{22}$$

Proof. By replacing j with $j - 2i + 2$ in equations (15) and (16), we obtain equations (19) and (20). Also, by replacing i with $i - 2j + 2$ in equations (17) and (18), we obtain equations (21) and (22). □

Lemma 16. *Let $\{x_{i,j}\}$ be a torus symmetric C_4-face-magic torus labeling on $C_4 \times C_4$. Let $d_i = x_{i+1,2i+1} - x_{i,2i-1}$, for $i = 1$ and 2, and let $d_{j+2} = d'_j = x_{2j+1,j+1} - x_{2j-1,j}$, for $j = 1$ and 2. Then each value $x_{i,j}$ can be expressed in terms of $x_{1,1}$, and d_i, for $i = 1, 2, 3, 4$, as rendered in Table 4.*

$x_{1,1}$	$x_{1,2} = x_{1,1} + d_1 + d_2 + d_3$	$x_{1,3} = x_{1,1} + d_3 + d_4$	$x_{1,4} = x_{1,1} + d_1 + d_2 + d_4$
$x_{2,1} = x_{1,1} + d_1 + d_3 + d_4$	$x_{2,2} = x_{1,1} + d_2 + d_4$	$x_{2,3} = x_{1,1} + d_1$	$x_{2,4} = x_{1,1} + d_2 + d_3$
$x_{3,1} = x_{1,1} + d_1 + d_2$	$x_{3,2} = x_{1,1} + d_3$	$x_{3,3} = x_{1,1} + d_1 + d_2 + d_3 + d_4$	$x_{3,4} = x_{1,1} + d_4$
$x_{4,1} = x_{1,1} + d_2 + d_3 + d_4$	$x_{4,2} = x_{1,1} + d_1 + d_4$	$x_{4,3} = x_{1,1} + d_2$	$x_{4,4} = x_{1,1} + d_1 + d_3$

TABLE 4. C_4-face-magic torus labeling involving the differences d_1, d_2, d_3 and d_4 on $C_4 \times C_4$.

Proof. We first verify the formulas in Table 4 for the values $x_{1,2}$, $x_{1,3}$, and $x_{1,4}$. By equation (19), we have $x_{2,3} = x_{1,1} + d_1$ and $x_{3,1} = x_{2,3} + d_2$. Thus, $x_{3,1} = x_{1,1} + d_1 + d_2$. By equation (21), we have $x_{1,2} = x_{3,1} + d_3$, $x_{3,2} = x_{1,1} + d_3$, and $x_{1,3} = x_{3,2} + d_4$. By equation (22), we have $x_{3,1} = x_{1,4} - d_4$. Since $x_{1,2} = x_{3,1} + d_3$ and $x_{3,1} = x_{1,1} + d_1 + d_2$, we have $x_{1,2} = x_{1,1} + d_1 + d_2 + d_3$. Since $x_{1,3} = x_{3,2} + d_4$ and $x_{3,2} = x_{1,1} + d_3$, we have $x_{1,3} = x_{1,1} + d_3 + d_4$. Since $x_{1,4} = x_{3,1} + d_4$ and $x_{3,1} = x_{1,1} + d_1 + d_2$, we have $x_{1,4} = x_{1,1} + d_1 + d_2 + d_4$. From equations (19) and (20), we can generate the other formulas for $x_{i,j}$ in Table 4. This completes the proof. □

Notation 17. Let $V' = \{\{1,16\},\{2,15\},\{3,14\},\{4,13\},\{5,12\},\{6,11\},\{7,10\},\{8,9\}\}$. Let d be an integer such that $-8 \le d \le 8$. We define a symmetric relation $\sim_d$ on V' by $\{y_1, y_2\} \sim_d \{z_1, z_2\}$ if and only if either $y_1 = z_1 + d$ and $y_2 = z_2 - d$, or $y_1 = z_1 - d$ and $y_2 = z_2 + d$. By Lemma 13, we have $V' = \{\{x_{i,j}, x_{i+2,j+2}\} : i = 1, 2, 3, 4 \text{ and } j = 1, 2\}$.

Lemma 18. *Let $\{x_{i,j}\}$ be a torus symmetric C_4-face-magic torus labeling on $C_4 \times C_4$. Let $d_i = x_{i+1,2i+1} - x_{i,2i-1}$, for $i = 1$ and 2, and let $d_{j+2} = d'_j = x_{2j+1,j+1} - x_{2j-1,j}$, for $j = 1$ and 2. Then for all integers i and j such that $1 \le i \le 2$ and $1 \le j \le 4$, we have*

$$\{x_{i,j}, x_{i+2,j+2}\} \sim_{d_i} \{x_{i+1,j+2}, x_{i+3,j}\}.$$

Also, for all integers i and j such that $1 \le i \le 4$ and $1 \le j \le 2$, we have

$$\{x_{i,j}, x_{i+2,j+2}\} \sim_{d_{j+2}} \{x_{i+2,j+1}, x_{i,j+3}\}.$$

Proof. By equation (19), we have $x_{i+1,j+2} = x_{i,j} + (-1)^{j+1} d_i$. When we replace i with $i+2$ and j with $j+2$ in equation (20), we have $x_{i+3,j} = x_{i+2,j+2} + (-1)^{j} d_i$. Thus, $\{x_{i,j}, x_{i+2,j+2}\} \sim_{d_i} \{x_{i+1,j+2}, x_{i+3,j}\}$.

By equation (21), we have $x_{i+2,j+1} = x_{i,j} + (-1)^{i+1} d_{j+2}$. When we replace i with $i+2$ and j with $j+2$ in equation (22), we have $x_{i,j+3} = x_{i+2,j+2} + (-1)^{i} d_{j+2}$. Thus, $\{x_{i,j}, x_{i+2,j+2}\} \sim_{d_{j+2}} \{x_{i+2,j+1}, x_{i,j+3}\}$. □

Notation 19. Let $\{x_{i,j}\}$ be a torus symmetric C_4-face-magic torus labeling on $C_4 \times C_4$. Let $d_i = x_{i+1,2i+1} - x_{i,2i-1}$, for $i = 1$ and 2, and let $d_{j+2} = d'_j = x_{2j+1,j+1} - x_{2j-1,j}$, for $j = 1$ and 2. By Lemma 18, we have $\{x_{i,j}, x_{i+2,j+2}\} \sim_{d_i} \{x_{i+1,j+2}, x_{i+3,j}\}$. For $i = 1, 2$ and $j = 1, 2, 3, 4$, let

$$A_{i,j} = \{x_{i,j}, x_{i+2,j+2}, x_{i+1,j+2}, x_{i+3,j}\} \quad \text{and}$$

$$V_i = \bigcup_{j=1}^{4} A_{i,j}.$$

Also, by Lemma 18, we have $\{x_{i,j}, x_{i+2,j+2}\} \sim_{d_{j+2}} \{x_{i+2,j+1}, x_{i,j+3}\}$. For $i = 1, 2, 3, 4$ and $j = 1, 2$, let

$$A'_{j+2,i} = \{x_{i,j}, x_{i+2,j+2}, x_{i+2,j+1}, x_{i,j+3}\} \quad \text{and}$$

$$V_{j+2} = \bigcup_{i=1}^{4} A'_{j+2,i}.$$

By Lemma 13, $V_i = V'$, for $i = 1, 2, 3, 4$.

Notation 20. We observe that $\{1,16\} \sim_1 \{2,15\}$, $\{3,14\} \sim_1 \{4,13\}$, $\{5,12\} \sim_1 \{6,11\}$ and $\{7,10\} \sim_1 \{8,9\}$. Let $B_{1,1} = \{1,16,2,15\}$, $B_{1,2} = \{3,14,4,13\}$, $B_{1,3} = \{5,12,6,11\}$ and $B_{1,4} = \{7,10,8,9\}$. Also, we observe that $\{1,16\} \sim_2 \{3,14\}$, $\{2,15\} \sim_2 \{4,13\}$, $\{5,12\} \sim_2 \{7,10\}$ and $\{6,11\} \sim_2 \{8,9\}$. Let $B_{2,1} = \{1,16,3,14\}$, $B_{2,2} = \{2,15,4,13\}$, $B_{2,3} = \{5,12,7,10\}$ and $B_{1,4} = \{6,11,8,9\}$. Next, we observe that $\{1,16\} \sim_4 \{5,12\}$, $\{2,15\} \sim_4 \{6,11\}$, $\{3,14\} \sim_4 \{7,10\}$ and $\{4,13\} \sim_4 \{8,9\}$. Let $B_{4,1} = \{1,16,5,12\}$, $B_{4,2} = \{2,15,6,11\}$, $B_{4,3} = \{3,14,7,10\}$ and $B_{4,4} = \{4,13,8,9\}$. Lastly, we observe that $\{1,16\} \sim_8 \{8,9\}$, $\{2,15\} \sim_8 \{7,10\}$, $\{3,14\} \sim_8 \{6,11\}$ and $\{4,13\} \sim_8 \{5,12\}$. Let $B_{8,1} = \{1,16,8,9\}$, $B_{8,2} = \{2,15,7,10\}$, $B_{8,3} = \{3,14,6,11\}$ and $B_{8,4} = \{4,13,5,12\}$.

Lemma 21. *Let $\{x_{i,j}\}$ be a torus symmetric C_4-face-magic torus labeling on $C_4 \times C_4$. Let $d_i = x_{i+1,2i+1} - x_{i,2i-1}$, for $i = 1$ and 2, and let $d_{j+2} = d'_j = x_{2j+1,j+1} - x_{2j-1,j}$, for $j = 1$ and 2. Let $A_{i,j}$ and $A'_{j+2,i}$ be defined as in Notation 19. Then, for $i = 1$, 2, 3 and 4, we have $d_i \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$. Furthermore, we have $\{A_{i,j} : j = 1,2,3,4\} = \{B_{|d_i|,j} : j = 1,2,3,4\}$, for $i = 1, 2$, and $\{A'_{j,i} : i = 1,2,3,4\} = \{B_{|d_j|,i} : i = 1,2,3,4\}$, for $j = 3, 4$.*

Proof. By Lemma 13, the set $\{\{x_{i,j}, x_{i+2,j+2}\} : i = 1,2,3,4 \text{ and } j = 1,2\}$ is the set $V' = \{\{1,16\},\{2,15\},\{3,14\},\{4,13\},\{5,12\},\{6,11\},\{7,10\},\{8,9\}\}$. Let d be an integer. We define a symmetric relation $\sim_d$ on the set V' by $\{y_1, y_2\} \sim_d \{z_1, z_2\}$ if and only if either $y_1 = z_1 + d$ and $y_2 = z_2 - d$, or $y_1 = z_1 - d$ and $y_2 = z_2 + d$. Let G_d be the graph with vertex set V' and edge set with an edge from $\{y_1, y_2\}$ to $\{z_1, z_2\}$ if and only if $\{y_1, y_2\} \sim_d \{z_1, z_2\}$.

By Lemma 18, for $i = 1$ or 2, we have $\{x_{i,j}, x_{i+2,j+2}\} \sim_{d_i} \{x_{i+1,j+2}, x_{i+3,j}\}$, for $j = 1, 2, 3, 4$. Thus, these four relations yield a perfect matching

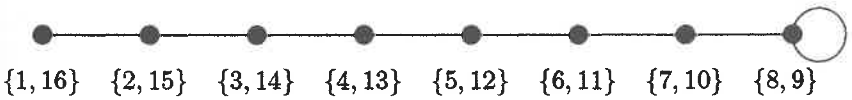

FIGURE 1. Graph G_1 has a perfect matching.

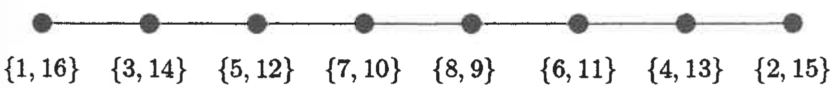

FIGURE 2. Graph G_2 has a perfect matching.

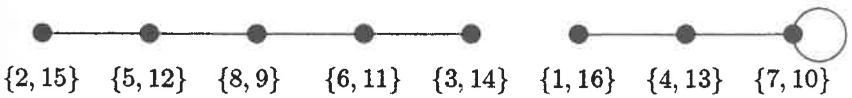

FIGURE 3. Graph G_3 has no perfect matching.

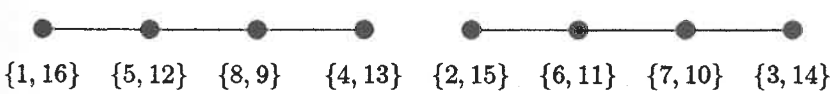

FIGURE 4. Graph G_4 has a perfect matching.

on G_{d_i} with vertex set $V_i = V'$. However, by Lemma 18, for $j = 3$ or 4, we have $\{x_{i,j}, x_{i+2,j+2}\} \sim_{d_j} \{x_{i+2,j+1}, x_{i,j+3}\}$, for $i = 1, 2, 3, 4$. Thus, these four relations yield a perfect matching on G_{d_j} with vertex set $V_j = V'$. In all cases, there exists $\{y_1, y_2\} \in V'$ such that $\{8, 9\} \sim_{d_i} \{y_1, y_2\}$. Thus, we have either $1 \leq 8 + d_i \leq 16$ or $1 \leq 8 - d_i \leq 16$. Hence, $-8 \leq d_i \leq 8$. Since $G_{-d_i} = G_{d_i}$, we need only consider the values where $1 \leq d_i \leq 8$. In Figures 1 through 8, we observe that the only graphs with a perfect matching are the graphs G_1, G_2, G_4 and G_8.

Therefore, either $d_i = \pm 1$, ± 2, ± 4 or ± 8. We prove the lemma for the value d_i when $i = 1$ or 2. The proof of the lemma for the value d_i, when $i = 3$ or 4, is made by replacing the sets $A_{i,j}$, for $j = 1, 2, 3, 4$, with the sets $A'_{i,j}$, for $j = 1, 2, 3, 4$, (see Notation 19) in the remainder of the proof of this lemma.

Let $d_i = \pm 1$. Since the only perfect matching in the graph $G_1 = G_{-1}$ is given by the relations $\{1,16\} \sim_1 \{2,15\}$, $\{3,14\} \sim_1 \{4,13\}$, $\{5,12\} \sim_1 \{6,11\}$ and $\{7,10\} \sim_1 \{8,9\}$ (see Figure 1), the sets $A_{i,j}$, for $j = 1, 2, 3, 4$, are a permutation on the sets $B_{1,j}$, for $j = 1, 2, 3, 4$.

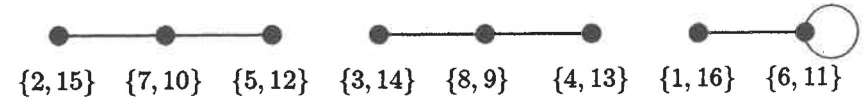

FIGURE 5. Graph G_5 has no perfect matching.

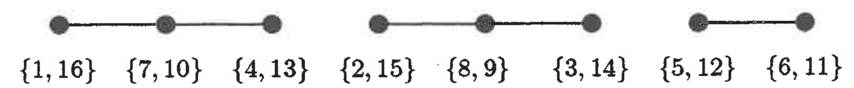

FIGURE 6. Graph G_6 has no perfect matching.

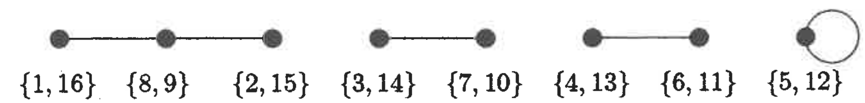

FIGURE 7. Graph G_7 has no perfect matching.

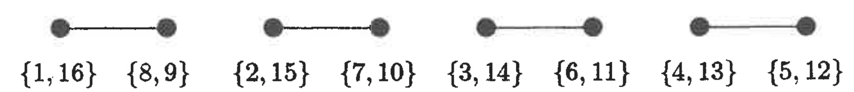

FIGURE 8. Graph G_8 has a perfect matching.

Let $d_i = \pm 2$. Since the only perfect matching in the graph $G_2 = G_{-2}$ is given by the relations $\{1,16\} \sim_2 \{3,14\}$, $\{2,15\} \sim_2 \{4,13\}$, $\{5,12\} \sim_2 \{7,10\}$ and $\{6,11\} \sim_2 \{8,9\}$ (see Figure 2), the sets $A_{i,j}$, for $j = 1,2,3,4$, are a permutation on the sets $B_{2,j}$, for $j = 1,2,3,4$.

Let $d_i = \pm 4$. Since the only perfect matching in the graph $G_4 = G_{-4}$ is given by the relations $\{1,16\} \sim_4 \{5,12\}$, $\{2,15\} \sim_4 \{6,11\}$, $\{3,14\} \sim_4 \{7,10\}$ and $\{4,13\} \sim_4 \{8,9\}$ (see Figure 4), the sets $A_{i,j}$, for $j = 1,2,3,4$, are a permutation on the sets $B_{4,j}$, for $j = 1,2,3,4$.

Let $d_i = \pm 8$. Since the only perfect matching in the graph $G_8 = G_{-8}$ is given by the relations $\{1,16\} \sim_8 \{8,9\}$, $\{2,15\} \sim_8 \{7,10\}$, $\{3,14\} \sim_8 \{6,11\}$ and $\{4,13\} \sim_8 \{5,12\}$ (see Figure 8), the sets $A_{i,j}$, for $j = 1,2,3,4$, are a permutation on the sets $B_{8,j}$, for $j = 1,2,3,4$. □

Notation 22. We consider several symmetries of the torus that result in graph automorphisms on $C_4 \times C_4$. Let (i,j) be a vertex on $C_4 \times C_4$ where we take the values of i and j, modulo 4. Let $T_{(a,b)}$ be the translation given by $T_{(a,b)}(i,j) = (i+a, j+b)$. Let R_1 be the graph automorphism given by $R_1(i,j) = (j,i)$. This graph automorphism is induced by the homeomorphism on the torus that interchanges the longitudinal cycles with the meridional cycles. Let R_2 be the graph automorphism given by $R_2(i,j) = (i,j)$ if j is odd and $R_2(i,j) = (i, j+2)$ if j is even. This graph automorphism is induced by the homeomorphism on the torus that reflects the torus across the plane intersecting the cycles with vertices (i,j) where j is odd. Let R_3 be the graph automorphism given by $R_3(i,j) = (i,j)$ if i is odd and $R_3(i,j) = (i, j+2)$ if i is even. This graph automorphism is induced by the homeomorphism on the torus that reflects the torus across the plane intersecting the cycles with vertices (i,j) where i is odd.

Lemma 23. *Let $\{x'_{i,j}\}$ be a torus symmetric C_4-face-magic torus labeling on $C_4 \times C_4$. Then it is torus equivalent to a torus symmetric C_4-face-magic torus labeling $\{x_{i,j}\}$ on $C_4 \times C_4$ such that $x_{1,1} = 1$ and either*

(1) $(d_1, d_2, d_3, d_4) = (1,2,4,8)$,
(2) $(d_1, d_2, d_3, d_4) = (1,4,2,8)$, *or*
(3) $(d_1, d_2, d_3, d_4) = (1,8,2,4)$.

Proof. Suppose $x'_{a,b} = 1$. Then apply the translation $T_{(1-a,1-b)}$ so that $x_{1,1} = 1$. See Notation 22. By Lemma 21, $d_i \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$ for $i = 1$, 2, 3 and 4. By Lemma 16, we have $x_{2,3} = x_{1,1} + d_1$, $x_{4,3} = x_{1,1} + d_2$, $x_{3,2} = x_{1,1} + d_3$, and $x_{3,4} = x_{1,1} + d_4$. Thus $d_i \in \{1,2,4,8\}$ for $i = 1$, 2, 3 and 4. Also, $d_i \neq d_j$, for all $i \neq j$. Thus (d_1, d_2, d_3, d_4) is a permutation on the set $\{1,2,4,8\}$. If $1 \in \{d_3, d_4\}$, we apply the graph automorphism R_1 so that $1 \in \{d_1, d_2\}$. If $d_2 = 1$, we apply the graph automorphism R_2 so that $d_1 = 1$. If $d_3 > d_4$, we apply the graph automorphism R_3 so that $d_3 < d_4$. The result follows. □

We complete the proof of Theorem 10 below.

Proof. By Lemma 23, we may assume that $x_{1,1} = 1$ and either $(d_1, d_2, d_3, d_4) = (1,2,4,8)$, $(d_1, d_2, d_3, d_4) = (1,4,2,8)$, or $(d_1, d_2, d_3, d_4) = (1,8,2,4)$, The choice $x_{1,1} = 1$ and $(d_1, d_2, d_3, d_4) = (1,2,4,8)$ produces Table 1, the choice $x_{1,1} = 1$ and $(d_1, d_2, d_3, d_4) = (1,4,2,8)$ produces Table 2, and the choice $x_{1,1} = 1$ and $(d_1, d_2, d_3, d_4) = (1,8,2,4)$ produces Table 3. □

Remark 24. We can view the three C_4-face-magic torus labelings on $C_4 \times C_4$ in Theorem 10 as one common phenomenon. Label the vertices of $C_4 \times C_4$ with the elements from the set $\{0,1\}^4$ so that the labelings on each C_4 face adds to $(2,2,2,2)$. This labeling is given in Table 5. Then the corresponding C_4-face-magic torus labeling on $C_4 \times C_4$ in Theorem 10 is given by $x_{i,j} = x_{1,1} + a_1 d_1 + a_2 d_2 + a_3 d_3 + a_4 d_4$ where $x_{1,1} = 1$, (a_1, a_2, a_3, a_4) is the labeling on vertex (i,j) in $C_4 \times C_4$ given by Table 5, and (d_1, d_2, d_3, d_4) is one of the three choices of either $(1,2,4,8)$, $(1,4,2,8)$ or $(1,8,2,4)$.

$(0,0,0,0)$	$(1,1,1,0)$	$(0,0,1,1)$	$(1,1,0,1)$
$(1,0,1,1)$	$(0,1,0,1)$	$(1,0,0,0)$	$(0,1,1,0)$
$(1,1,0,0)$	$(0,0,1,0)$	$(1,1,1,1)$	$(0,0,0,1)$
$(0,1,1,1)$	$(1,0,0,1)$	$(0,1,0,0)$	$(1,0,1,0)$

TABLE 5. C_4-face-magic torus labeling with elements from $\{0,1\}^4$ on $C_4 \times C_4$.

REFERENCES

[1] Baca, M., On magic labelings of Mobius ladders. *J. Franklin Inst.* **326** (1989), 885-888.

[2] Baca, M., On magic and consecutive labelings for the special classes of plane graphs. *Utilitas Math.* **32** (1987), 59-65.

[3] Bondy, J.A. and Murty, U.S.R., *Graph Theory with Applications*, MacMillan, New York, (1976).

[4] Curran, S.J. and Low, R.M., On C_4-face-magic torus labelings on $C_{2m} \times C_{2n}$, manuscript.

[5] Gallian, J.A., A dynamic survey of graph labeling. *The Electronic Journal of Combinatorics.* **20** (2017), #DS6.

[6] Hsieh, J., Lee, S-M., Liang, P-J. and Low, R.M., On C_4-face-$(1,0,0)$ magic polyomino graphs and their related graphs., manuscript.

[7] Kasif, A., Muhammad, H., Ali, A. and Miller, M., Magic labeling of type (a,b,c) of families of wheels. *Mathematics in Computer Science.* **7** (2013), 315-319.

[8] Kotzig, A. and Rosa, A., Magic valuations of finite graphs. *Canad. Math. Bull.* **13** (1970), 451-461.

[9] Lih, K-W., On magic and consecutive labelings of plane graphs. *Utilitas Math.* **24** (1983), 165-197.

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