Cycle Extendability in Tournaments

LeRoy B. Beasley*
Sarah K. Merz† David E. Brown‡ and Brent J. Thomas§

Abstract

Let D be a digraph on n vertices. A cycle, C, in D is said to be 1-extendable if there is a cycle, C', in D such that the vertex set of C' contains the vertex set of C and C' contains exactly one additional vertex. A digraph is 1-cycle-extendable if every non Hamiltonian cycle is 1-extendable. A cycle, C, in D is said to be 2-extendable if there is a cycle, C', in D such that the vertex set of C' contains the vertex set of C and C' contains exactly two additional vertices. A digraph is 2-cycle-extendable if every cycle on at most $n-2$ vertices is 2-extendable. A digraph is 1,2-cycle-extendable if every non Hamiltonian cycle is either 1-extendable or 2-extendable. It has been previously shown that not all strong tournaments (orientations of a complete undirected graph) are 1-extendable, but are 2-extendable. The structure of all non 1-extendable tournaments is shown as a type of block Kroneker product of 1-extendable subtournaments.

1 Introduction.

Let $\mathcal{Q}$ be a set and "+" be a binary operation (addition) on $\mathcal{Q}$. Then $(\mathcal{Q},+)$ is a *monoid* if $(\mathcal{Q},+)$ is an algebraic system such that $\mathcal{Q}$ is closed under $+$, $+$ is associative ($(a+b)+c = a+(b+c)$) and $+$ has an identity, O ($O+a = a+O = a$). If $(\mathcal{Q},+)$ is commutative, that is $a+b=b+a$, then we say that $(\mathcal{Q},+)$ is a *commutative* or an *Abelian* monoid.

If $\mathbb{S}$ is a set and "+" and "$\star$" are binary operations such that:

*550 N. Main, Ste 317, Logan, Utah 84321, U.S.A,, leroy_beas@aol.com
†Department of Mathematics, University of the Pacific, 3601 Pacific Ave., Stockton, CA 95211, U.S.A., smerz@pacific.edu
‡Department of Mathematics and Statistics, Utah State University, Logan, Utah 84322-3900, U.S.A., dbrown@usu.edu
§Department of Mathematics and Statistics, Utah State University, Logan, Utah 84322-3900, U.S.A., brent.thomas@usu.edu

1. $(\mathbb{S}.+)$ is an Abelian monoid with identity 0,
2. $(\mathbb{S}, \star)$ is a monoid with identity 1,
3. for $a \in \mathbb{S}$, $0 \star a = a \star 0 = 0$ and
4. multiplication distributes over addition, that is for $a, b, c \in \mathbb{S}$, $a \star (b + c) = a \star b + a \star c$ and $(b + c) \star a = b \star a + c \star a$

then $(\mathbb{S}, +, \star)$ is a *semiring*. Note that in this article, all semirings have a multiplicative identity, which is not a usual requirement.

A semiring, $\mathbb{S}$, is said to be *antinegative* if for $a, b \in \mathbb{S}$, $a + b = 0$ implies that $a = b = 0$. A semiring we are particularly interested in is the binary Boolean semiring, $\{0, 1\}$ with addition and multiplication the same as for reals except that $1 + 1 = 1$

Let $\mathcal{M}_{m,n}(\mathbb{S})$ denote the set of all $m \times n$ matrices with entries from the semiring $\mathbb{S}$. We let $J_{m,n}$ denote the $m \times n$ matrix of all ones, $O_{m,n}$ denote the $m \times n$ matrix of all zeros, and I_n the identity matrix of order n. If $m = n$ we shorten the notation to $\mathcal{M}_n(\mathbb{S}), J_n$ and O_n. If the order of a matrix is obvious from the context we write J, O, I. If X is an $m \times n$ matrix then we let X^t denote the transpose of X.

Definition 1.1 *Let $M_1, M_2, \cdots, M_k$ and N be matrices of order $n_1, n_2, \cdots, n_k$ and k respectively with entries in the semiring $\mathbb{S}$. Define the* block Kroneker product $(M_1, M_2, \cdots, M_k) \boxtimes N = \hat{M}$ *to be the $k \times k$ block matrix whose (i, i) block entry is M_i and for $i \neq j$ whose (i, j) block entry is $n_{i,j} J_{n_i, n_j}$.*

By considering the adjacency matrix of a graph we can state the above definition for directed graphs:

Definition 1.2 *Let G_1, G_2, $\cdots$, G_k and H be directed graphs of orders $n_1, n_2, \cdots, n_k$ and k respectively. . Define the* block Kroneker product $(G_1, G_2, \cdots, G_k) \boxtimes H = \hat{H}$ *to be the directed graph whose adjacency matrix is $(A(G_1), A(G_2), \cdots, A(G_k)) \boxtimes A(H)$.*

Example 1.1 *Let $\mathbb{S} = \Re$ be the real numbers and consider the matrices M_1 of order 3, M_2 of order 2, M_3 of order 1, and M_4 of order 2. Let*

$$N = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ Then}$$

$$(M_1, M_2, M_3, M_4) \boxtimes N = \begin{bmatrix} M_1 & J_{3,2} & O_{3,1} & 2J_{3,2} \\ O_{2,3} & M_2 & O_{2,1} & 4J_{2,2} \\ J_{1.3} & O_{1,2} & M_3 & O_{1,2} \\ O_{2,1} & O_{2,2} & O_{2,1} & M_4 \end{bmatrix}.$$

Some routinely verified facts about the block Kroneker product are:

1. $(K_1, K_2, \cdots, K_k) \boxtimes N_1 + (L_1, L_2, \cdots, L_k) \boxtimes N_2 = ((K_1 + L_1), (K_2 + L_2), \cdots, (K_k + L_k)) \boxtimes (N_1 + N_2)$;
2. $[(K_1, K_2, \cdots, K_k) \boxtimes N]^t = (K_1^t, K_2^t, \cdots, K_k^t) \boxtimes N^t$;

If N is a $k \times k$ matrix and M is $\ell \times \ell$ then $M \boxtimes N$ is the $(k \cdot \ell) \times (k \cdot \ell)$ matrix $(M, M, \cdots, M) \boxtimes N$ where there are k matrices M in $(M, M, \cdots, M)$.

2 Matrix Results

Recall that M is a tournament matrix if and only if M is a $(0, 1)$ matrix such thah $M + M^t = J \setminus I$.

Theorem 2.1 *If $M_i, i = 1, \cdots, k$ and N are tournament matrices then $(M_1, M_2, \cdots, M_k) \boxtimes N$ is a tournament matrix.*

Proof. Suppose that $M_i, i = 1, \cdots, k$ and N are tournament matrices. then M_i is a $(0, 1)$ matrix such that $M_i + M_i^t = J \setminus I$, and $N + N^t = J \setminus I$. Now,

$$(M_1, M_2, \cdots, M_k) \boxtimes N + [(M_1, M_2, \cdots, M_k) \boxtimes N]^t$$
$$= (M_1, M_2, \cdots, M_k) \boxtimes N + (M_1^t, M_2^t, \cdots, M_k^t) \boxtimes N^t$$
$$= ((M_1 + M_1^t), (M_2 + M_2^t), \cdots, (M_k + M_k^t)) \boxtimes (N + N^t)$$
$$= ((J \setminus I), (J \setminus I), \cdots, (J \setminus I)) \boxtimes (J \setminus I) = J \setminus I.$$

That is, since each of the matrices are $(0, 1)$-matrices, $(M_1, M_2, \cdots, M_k) \boxtimes N$ is a tournament matrix. ■

Theorem 2.2 *Let $\mathbb{S}$ be an antinegative semiring, $n_1, n_2, \cdots n_k$ be arbitrary positive integers, and $M_1, M_2, \cdots, M_k$ be arbitrary square matrices* *of order $n_1, n_2, \cdots n_k$ with entries in $\mathbb{S}$. If N is an irreducible matrix then $(M_1, M_2, \cdots, M_k) \boxtimes N$ is irreducible.*

Proof. It is a routine check to see that if N is irreducible then the block matrix whose (i, j) entry is $n_{i,j} J_{n_i,n_j}$ for $i \neq j$ is irreducible. Further if $\mathbb{S}$ is antinegative and N is irreducible, then $N + X$ is irreducible for any matrix of the same order as N. ■

Theorem 2.3 *Let m and q be positive integers, $M_i, i = 1, \cdots, k$ be $(0,1)$-matrices all of order m and $0 \leq \ell \leq m$. If all row sums of every $M_i, i = 1, \cdots, k$ equal ℓ and N is a $(0,1)$-matrix such that all the row sums of N are equal to q, then all row sums of $(M_1, M_2, \cdots, M_k) \boxtimes N = \hat{M}$ are equal to $\ell + qm$.*

Proof. Every row sum of each row in the matrix M corresponding to a row in the i^{th} block is the sum of the row sum of a row in M_i plus the number of nonzero columns of N times the order of M_i that is every row of M has row sum $(r_1(M_1) + (n_1 \cdot r_1(N))) = \ell + qm$. . ■

3 Graph results

The above theorems can be restated for directed graphs by considering the adjacency matrix of the graph:

Theorem 3.1 *If $G_i, i = 1, \cdots, k$ and H are tournaments then $(G_1, G_2, \cdots, G_k) \boxtimes H = T$ is a tournament.*

Theorem 3.2 *If $G_i, i = 1, \cdots, k$ are any digraphs and H is strongly connected, then $(G_1, G_2, \cdots, G_k) \boxtimes H$ is strongly connected.*

Theorem 3.3 *If the digraphs $G_i, i = 1, \cdots, k$ are all regular digraphs of the same order and H is a regular digraph then $(G_1, G_2, \cdots, G_k) \boxtimes H =$ is a regular digraph.*

Definition 3.1 *Let G be a graph on n vertices and let D be a directed graph on n vertices. Let $S \subset \{1, 2, \cdots, n\}$. A cycle, C, of length k in G (D) is* S-extendable *if there is an $i \in S$ such that the k vertices on C together with i vertices not on C induce a graph (digraph) which contains a cycle of length $k + i$, i.e., which induces a Hamiltonian graph (digraph). A graph, G (digraph, D), is* S-cycle-extendable *if every cycle in G (D) is S-extendable.*

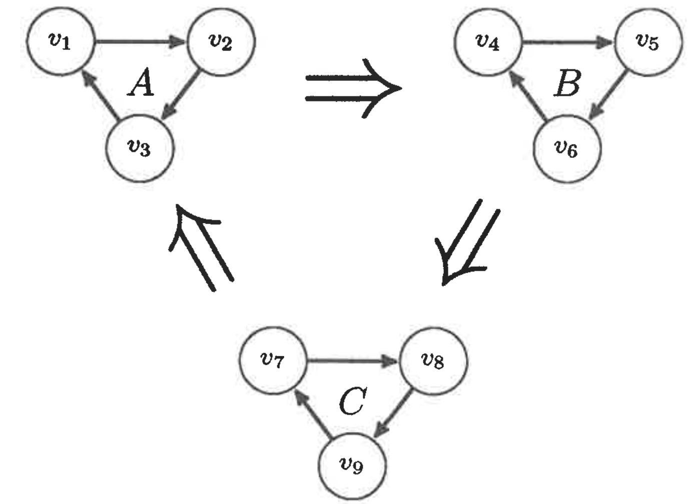

Figure 1: 9 vertex example..

The usual concept of "cycle extendability" is just the case $S = \{1\}$. The standing conjecture of Hendry is that every chordal Hamiltonian graph is $\{1\}$-cycle-extendable. The case for directed graphs is quite different.

Example 3.1 *(See Figure 1.) Let D be the digraph on 9 vertices, $v_1, v_2, \cdots, v_9$ such that v_1, v_2, v_3 (indicated by "A") induces a 3-cycle, v_4, v_5, v_6 (indicated by "B") induces a 3-cycle, v_7, v_8, v_9 (indicated by "C") induces a 3-cycle, and there is an arc from each vertex of A to each vertex of B, an arc from each vertex of B to each vertex of C, and an arc from each vertex of C to each vertex of A. Then the 3-cycle on v_1, v_2, v_3 cannot be extended to a 4-cycle containing v_1, v_2, v_3, but every cycle in D is $\{1, 2\}$-extendable.*

Let G and H be disjoint sets of vertices of a digraph K. We shall use the notation $G \Rightarrow H$ to indicate that the arc set of K contains every arc of the form (g, h) where g is any vertex in G and h is any vertex in H. That is, $\{(g, h) \mid g \in G, h \in H\} \subseteq A(K)$.

Note a digraph consisting of an isolated vertex may be considered a strong (regular) tournament.

Lemma 3.1 *Let T be a strong tournament. If H_1 and H_2 are 1-extendable strong subtournaments whose longest cycles are not 1-extendable in T, then either $H_1 \Rightarrow H_2$ or $H_2 \Rightarrow H_1$.*

Proof. Let T be a strong tournament, and let H_1 and H_2 be strong subtournaments. If H_1 and H_2 are both graphs on a single vertex, then since T is a tournament, either $H_1 \Rightarrow H_2$ or $H_2 \Rightarrow H_1$. Thus, assume that H_2 is not a single vertex. Since H_2 is a strong tournament, there is a cycle containing every vertex in H_2.

If H_1 and H_2 are 1-extendable subtournaments whose longest cycles are not 1-extendable in T, then by Moon, [4], the vertex set of T can be partitioned into three sets, $V(H_1)$, X and Y such that $H_1 \Rightarrow X$, and $Y \Rightarrow H_1$. Now suppose that $V(H_2) \cap X \neq \emptyset$ and $V(H_2) \cap Y \neq \emptyset$. Then there are vertices u and v on a longest cycle in H_2 that are adjacent on that cycle with $u \rightarrow v$ with $u \in Y$ and $v \in X$. But then if that longest cycle is $a \rightarrow b \rightarrow \cdots \rightarrow u \rightarrow v \rightarrow \cdots \rightarrow z$ then for any vertex $h \in V(H_1)$, $a \rightarrow b \rightarrow \cdots \rightarrow u \rightarrow h \rightarrow v \rightarrow \cdots \rightarrow z$ is a 1-extension of that long cycle in T, a contradiction. Thus, either $V(H_2) \subseteq X$ or $V(H_2) \subseteq Y$. We now have that either $H_1 \Rightarrow H_2$ or $H_2 \Rightarrow H_1$. ■

Theorem 3.4 *T is a strong tournament if and only if there exist strong 1-extendable subtournaments $S_1, S_2, \cdots, S_k$ and $\hat{T}$ such that*

$$T = (S_1, S_2, \cdots, S_k) \boxtimes \hat{T}.$$

Proof. If T is 1-extendable, then for $k = 1, S_1 = T$ and $\hat{T}$ a digraph consisting of an isolated vertex, $T = S_1 \boxtimes \hat{T}$.

Now, suppose that T is not 1-extendable. Let $\mathcal{C} = \{C_1, C_2, \cdots, C_\ell\}$ be a set of vertex disjoint cycles (of length at least 3) in T such that

1. each C_i is not 1-extendable in T; and
2. $< (V(T) \setminus \{V(C_1) \cup \cdots \cup V(C_\ell)\}) >$ has no 1-extendable cycle.

Let $V(T) \setminus \{V(C_1) \cup \cdots \cup V(C_\ell)\} = \{v_1, \cdots, v_j\}$. Let $S_i =< C_i >, i = 1, \cdots, \ell$ and $S_{\ell+i} =< v_i >$, $i = 1, \cdots, j$. Let $k = \ell + j$. Note that S_i, $i = 1, \cdots, k$, is a strong subtournament since S_i is induced by the set of vertices on a cycle or is an isolated vertex.

By Lemma 3.1, given any $1 \leq i, j \leq k$, $i \neq j$ either $S_i \Rightarrow S_j$ or $S_j \Rightarrow S_i$. Define $\hat{T} = (V, A)$ where $V = \{u_1, \cdots, u_k\}$ and $(u_i, u_j) \in A$ if and only if $S_i \Rightarrow S_j$. Then $\hat{T}$ is a tournament. Now, $(S_1, S_2, \cdots, S_k) \boxtimes \hat{T}$ is a tournament, and all arcs in $(S_1, S_2, \cdots, S_k) \boxtimes \hat{T}$ are arcs in T. Thus $T = (S_1, S_2, \cdots, S_k) \boxtimes \hat{T}$.

The converse follows from Theorems 3.1 and 3.2. ■

References

[1] Beasley, L. B. and Brown, D. E., Cycle extendability in graphs and digraphs, *Linear Algebra and its Applications*, **435**(2011) 1513–1519.

[2] Bondy, J. A. and Murty, U. S. R., *Graph theory* (Graduate Texts in Mathematics, 244. Springer, New York, 2008).

[3] Marcus, M. and Minc, H. *A survey of matrix theory and matrix inequalities* (Reprint of the 1969 edition. Dover Publications, Inc., New York, 1992).

[4] Moon, J.W., On cycles in tournaments, Matematický Časopis, **19**(1969), No. 2, 121–125.