

# DELTA MATROIDS FROM MATROIDS

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ABSTRACT. We give necessary and sufficient conditions for two matroids on the same ground set to be the upper and lower matroid of a  $\Delta$ -matroid.

## 1. MATROIDS AND $\Delta$ -MATROIDS

A matroid  $M$  is a finite set  $E$  and a collection  $\mathcal{B}$  of subsets of  $E$  satisfying the condition that

(MB) If  $B_1$  and  $B_2$  are in  $\mathcal{B}$  and  $x \in B_1 \setminus B_2$  then there exists a  $y \in B_2 \setminus B_1$  such that  $(B_1 \cup \{y\}) \setminus \{x\} = B_1 \Delta \{x, y\} \in \mathcal{B}$

Axiom (MB) is called the *basis exchange axiom*. Sets in  $\mathcal{B}$  are called *bases* of  $M$ . Subsets of bases are called *independent sets*, sets which are not independent are called *dependent*, minimal dependent sets are called *cycles*, and sets containing a basis are called *spanning*.

Matroids were introduced by Whitney [16] in 1935. Since then many texts on matroid theory have appeared; see [14], or the standard text for the graph theorist [15], the standard resource for the geometer/algebraist [11], an applied approach [13], or a new text [12].

Replacing the set difference in Axiom (MB) by the symmetric difference we obtain the symmetric exchange axiom ( $\Delta$ F) used by Bouchet [2] to define  $\Delta$ -matroids.

A  $\Delta$ -matroid  $D$  is a finite set  $E$  and a collection  $\mathcal{F}$  of subsets of  $E$  satisfying the condition that

( $\Delta$ F) If  $F_1$  and  $F_2$  are in  $\mathcal{F}$  and  $x \in F_1 \Delta F_2$  then there exists a  $y \in F_2 \Delta F_1$  such that  $F_1 \Delta \{x, y\} \in \mathcal{F}$ .

Axiom ( $\Delta$ F) is called the *symmetric exchange axiom* and the sets in  $\mathcal{F}$  are called the *feasible sets* of  $D$ . It is important to note that  $y$  may equal  $x$ , so  $|F_1 \Delta \{x, y\}| - |F_1| \in \{0, \pm 1, \pm 2\}$ .

$\Delta$ -matroids were independently introduced by Dress and Havel [9] as *matroids*, and by Chandasekaran and Kabadi [8] as *pseudomatroids*. Bouchet developed the properties of  $\Delta$ -matroids and related structures in a series of papers on multimatroids [4, 5, 7, 6].

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Note that every matroid  $M$  is a  $\Delta$ -matroid whose feasible sets are the bases of  $M$ , and, conversely, every  $\Delta$ -matroid with equicardinal feasible sets forms a matroid with those feasible sets as bases. More generally, Bouchet pointed out that there are two obvious matroids associated with every  $\Delta$ -matroid;  $M_u$ , the *upper matroid*, whose bases are the feasible sets with largest cardinality, and  $M_l$ , the *lower matroid*, whose bases are the feasible sets with least cardinality, [3]. It is the point of view of this paper to study the  $\Delta$ -matroid specifically with regard to the relation it bears to its upper and lower matroids.

Considering the bases of a matroid as feasible sets yields a  $\Delta$ -matroid whose upper and lower matroids are identical. For an example where they are distinct, let  $E$  be a set on  $n$  elements,  $0 < k < n$ , and let  $\mathcal{F}$  consist of all subsets of  $E$  that have either  $k - 1$  or  $k + 1$  elements. Then it is easy to check that  $\mathcal{F}$  is the collection of feasible sets of a  $\Delta$ -matroid on  $E$ . The upper matroid  $M_u$  is the uniform matroid  $U_{k+1,n}$ , and the lower matroid is  $U_{k-1,n}$ . However, we also have the following extreme examples.

**Theorem 1.1.** *Let  $M$  be a matroid and let  $\mathcal{I}$  be the collection of independent sets of  $M$ . Then  $\mathcal{I}$  satisfies the symmetric exchange axiom.*

*Proof.* Let  $I_1, I_2 \in \mathcal{I}$  and let  $x \in I_1 \Delta I_2$ .

If  $x \in I_1 \setminus I_2$ , then  $I_1 \Delta \{x, x\} = I_1 \setminus \{x\}$  is independent as required, so assume that  $x \in I_2 \setminus I_1$ . If  $I_1 \cup \{x\}$  is independent, then set  $y = x$ , so that  $I_1 \Delta \{x, y\} = I_1 \cup \{x\}$  is independent. If, on the other hand,  $I_1 \cup \{x\}$  is dependent, then  $I_1 \cup \{x\}$  contains a unique cycle containing  $x$ , but which is not contained in  $I_2$ , so there exists  $y \in I_1 \setminus I_2$ , so that  $I_1 \Delta \{x, y\} = (I_1 \setminus \{x\}) \cup \{y\}$  is independent.  $\square$

So  $\mathcal{I}$  is the collection of feasible sets of a  $\Delta$ -matroid with upper-matroid  $M$  and lower matroid with the empty set as only basis. Not surprisingly, the dual result also holds.

**Theorem 1.2.** *Let  $M$  be a matroid and let  $\mathcal{S}$  be the collection of spanning sets of  $M$ . Then  $\mathcal{S}$  satisfies the symmetric exchange axiom.*

*Proof.* Let  $S_1, S_2 \in \mathcal{S}$  and let  $x \in S_1 \Delta S_2$ .

If  $x \in S_2 \setminus S_1$ , then  $S_1 \Delta \{x, x\} = S_1 \cup \{x\}$  is spanning as required, so assume that  $x \in S_1 \setminus S_2$ . If  $S_1 \setminus \{x\}$  is spanning, then set  $y = x$ , so that  $S_1 \Delta \{x, y\} = S_1 \setminus \{x\}$  is spanning. If, on the other hand,  $S_1 \setminus \{x\}$  is not spanning, then since  $S_2$  is spanning it contains an element  $y \notin S_1$  so that  $(S_1 \setminus \{x\}) \cup \{y\} = S_1 \Delta \{x, y\}$  spans.  $\square$

So any matroid  $M$  may be naturally viewed as a  $\Delta$ -matroid in three different ways, namely by considering  $\mathcal{F} = \mathcal{B}$  where  $M_u = M_l = M$ , or  $\mathcal{F} = \mathcal{I}$ , where  $M_u = M$  and the lower matroid has rank zero, or  $\mathcal{F}$  equals the collection of spanning sets of  $M$  where  $M_l = M$  and the upper matroid has rank  $|E|$ .

The feasible sets of cardinality between the ranks of  $M_l$  and  $M_u$  can also be related to the upper and lower matroids.

**Theorem 1.3.** *If  $D$  is a  $\Delta$ -matroid and  $F$  is a feasible set of  $D$ , then  $F$  is spanning in the lower matroid and independent in the upper matroid.*

*Proof.* Let  $F$  be a feasible set and let  $L$  be the basis of  $M_l$  with the largest possible intersection with  $F$ . If  $L \not\subseteq F$ , there exists  $x \in L \setminus F$ . By the symmetric exchange axiom, there must be a  $y \in L \Delta F$  such that  $L \Delta \{x, y\}$  is feasible. If  $y \in L \setminus F$ , then  $|L \Delta \{x, y\}| < |L|$ , which is impossible, so  $y \in F \setminus L$  and  $L \Delta \{x, y\}$  is a basis of  $M_l$  which intersects  $F$  in more elements than  $L$  does, a contradiction. Therefore  $L \subseteq F$ , that is,  $F$  is spanning in  $M_l$ .

For independence, let  $F$  be a feasible set and let  $U$  be a basis of  $M_u$  with largest possible intersection with  $F$ . If  $|F| \not\subseteq |U|$ , there exists  $x \in F \setminus U$ . By the symmetric exchange axiom, there must be a  $y \in F \Delta U$  such that  $U \Delta \{x, y\}$  is feasible. If  $y \in F \setminus U$ , then  $|U \Delta \{x, y\}| > |U|$ , which is impossible, so  $y \in U \setminus F$  and  $U \Delta \{x, y\}$  is a basis of  $M_u$  which intersects  $F$  in more elements than  $U$  does, a contradiction. Therefore  $F$  is contained in some basis of  $M_u$  and is independent in  $M_u$ .  $\square$

In particular, Theorem 1.3 leads to a necessary condition on two matroids of different ranks on a set  $E$  to be the upper and lower matroids of a  $\Delta$ -matroid.

**Corollary 1.4.** *If  $D$  is a  $\Delta$ -matroid on a set  $E$  with upper matroid  $M_u$  and lower matroid  $M_l$ , then every basis of  $M_u$  is a spanning set of  $M_l$  and every basis of  $M_l$  is independent in  $M_u$ .*

The converse of Corollary 1.4 is the topic of the next section.

## 2. CHARACTERIZING UPPER AND LOWER MATROIDS

If we wanted to study a  $\Delta$ -matroid  $D$ , and were given its upper and lower matroids,  $M_u$  and  $M_l$ , it would be convenient if we could simply compute the rest of the feasible sets from these extreme classes, but that is impossible, as the simple feasible collections  $\{\{a, b\}, \{a\}, \{b\}, \emptyset\}$  and  $\{\{a, b\}, \emptyset\}$  demonstrate, since they have the same upper and lower matroids.

In general, if the rank difference between upper and lower matroid is larger than 2, it is impossible for there to be no feasible sets of intermediate cardinality, since symmetric exchange will force the existence of feasible sets both of cardinality one or two greater than minimum, and of cardinality one or two less than maximum. It is more likely to find a general construction by declaring as many of the sets of intermediate cardinality to be feasible as possible, within the restrictions of Theorem 1.3. This construction works in the examples of Theorems 1.1 and 1.2 as well as in the following theorem relating the cycle matroid of a simple graph  $G = (V, E)$  on the ground set

$E$  and its (2-dimensional generic)rigidity matroid, where a set  $F \subseteq E$  of edges is independent if

$$|F'| \leq 2|V(F')| - 3$$

holds for all non-empty subsets  $F' \subseteq F$ .  $V(F)$  here denotes set of endpoints of edges in  $F$ . Edge sets violating this inequality are called overbraced. For an introduction to combinatorial rigidity see [10].

**Theorem 2.1.** Let  $G = (V, E)$  be a connected simple graph and consider the connectivity matroid (cycle matroid)  $M_c$  and the 2-dimensional generic rigidity matroid  $M_r$  on the edge set  $E$  of  $G$ . Let  $\mathcal{F}$  denote the collection of edge sets which induce graphs containing a spanning tree, but are not over-braced. Then  $\mathcal{F}$  satisfies the symmetric exchange property.

*Proof.* Let  $F, F' \in \mathcal{F}$  and let  $x \in F \setminus F'$ . If  $(V, F \setminus \{x\})$  is connected, then choose  $y = x$ , and  $F \Delta \{x, x\} = F \setminus \{x\}$  contains a spanning tree and is not over-braced. Otherwise,  $F \setminus \{x\}$  is disconnected, and  $x$  is a bridge of  $(V, F)$ . Since  $(V, F')$  contains a spanning tree, and does not contain  $x$ , it contains an edge  $y$  so that every spanning tree of  $(V, (F \setminus \{x\}) \cup \{y\})$  contains  $y$ . Moreover,  $G(V, (F \setminus \{x\}) \cup \{y\})$  cannot be over-braced, since  $G(V, F \setminus \{x\})$  is not over-braced and no rigidity cycle can contain a bridge.

Now we must consider the case when  $x \in F' \setminus F$ . If  $(V, F \cup \{x\})$  is not over-braced, we can choose  $y = x$ , and  $F \Delta \{x, y\} = F \cup \{x\}$  has the desired property. Otherwise,  $(V, F \cup \{x\})$  contains a unique rigidity cycle, which is necessarily edge 2-connected. Since the set of edges of that rigidity cycle is not contained in  $F'$ , there is an edge  $y \in F \setminus F'$  whose deletion does not disconnect it, and so does not increase the number of connected components of  $(V, F \cup \{y\})$ , but whose deletion does leave  $(V, F \cup \{y\})$  no longer over-braced. So  $(F \setminus \{y\}) \cup \{x\}$  has the desired property.  $\square$

This proof depended on the fact that the graph of a rigidity cycle, i.e. a minimally overbraced graph, must be a 2-connected graph, or, equivalently, that the cycle matroid on those edges is connected. This is not true in general, even if  $M_u$  and  $M_l$  are the upper and lower matroids of a  $\Delta$ -matroid, as the following example shows. Consider  $E = \{1, 2, 3, a, b, c\}$ ,  $M_u = U_{5,6}(E)$ ,  $M_l = U_{2,3}(\{1, 2, 3\}) \oplus U_{2,3}(\{a, b, c\})$ . Then it is easy to check that  $M_u, M_l$  are the upper and lower matroids of a delta-matroid  $D$ , with feasible sets  $\{B_1 \cup B_2 \mid B_1 \in \mathcal{B}(M_u), B_2 \in \mathcal{B}(M_l)\}$ . Here the upper matroid is a cycle, and the lower matroid is disconnected.

We would like to have that every cycle in  $M_u$  is a union of cycles in  $M_l$ , which is not necessarily the case even if  $M_u$  and  $M_l$  are matroids such that every basis of  $M_l$  is independent in  $M_u$ , and every basis of  $M_u$  is spanning in  $M_l$ , as the following example demonstrates. Consider the cycle matroids of multigraphs of Figure 1. Every basis of  $M_u$ , the matroid for the graph on the left, contains the element  $a$  and so spans the matroid  $M_l$  for the graph on the right; and each of the bases of  $M_l$  have one element and so

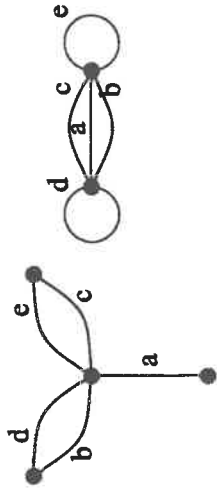


FIGURE 1. Two connectivity matroids not yielding a  $\Delta$ -matroid.

are independent in  $M_u$ , which has no loops. Note that the cycle  $\{d, b\}$  of  $M_u$  is not a union of cycles of  $M_l$ . However,  $M_u$  and  $M_l$  cannot be the upper and lower matroids of any  $\Delta$ -matroid. Consider  $\{a, d, e\}$  and  $\{b\}$ . Then  $a \in \{a, d, e\} \Delta \{b\}$ , and, if there was a  $\Delta$ -matroid, there would exist a  $y \in \{a, d, e\} \Delta \{b\} = \{a, b, d, e\}$  so that  $\{a, d, e\} \Delta \{a, y\}$  is feasible. But  $y \notin \{a, d, e\}$  since  $\{d, e\}$  is not lower spanning, and  $y \neq b$  since  $\{b, d, e\}$  is not upper independent. So no choice of  $y$  could give a feasible set.

But we do have the following result, which gives us a new necessary condition and will allow us to characterize upper and lower matroids of  $\Delta$ -matroids.

**Theorem 2.2.** Let  $D$  be a delta matroid with ground set  $E$ , with upper matroid  $M_u$  and lower matroid  $M_l$ . Then every cycle in  $M_u$  is a union of cycles in  $M_l$ .

*Proof.* Let  $C$  be a cycle in  $M_u$ , and consider the restriction of  $D$  to  $C$ , see [1]. In this restriction, the upper matroid is a single cycle  $C$ . If the lower matroid is not a union of lower cycles, then there is an edge  $e$  which is contained in no lower cycle, so  $C \setminus \{e\}$  which is an upper basis, hence feasible, is not lower spanning, a contradiction.  $\square$

It is easy to show that this necessary condition is stronger than the earlier necessary conditions, since if every cycle in  $M_u$  is a union of cycles in  $M_l$  then every basis of  $M_l$  is independent in  $M_u$ , and every basis of  $M_u$  is spanning in  $M_l$ . In fact, it is a necessary and sufficient condition for two matroids to be the upper and lower matroids of a  $\Delta$ -matroid.

**Theorem 2.3.** Let  $M_u$  and  $M_l$  be matroids on the same ground set  $E$ . Then  $M_u$  and  $M_l$  are the upper and lower matroids of a  $\Delta$ -matroid if and only if every cycle in  $M_u$  is a union of cycles in  $M_l$ .

*Proof.* We only have to show sufficiency. Suppose that every cycle in  $M_u$  is a union of cycles in  $M_l$ , so that every basis of  $M_l$  is independent in  $M_u$ , and every basis of  $M_u$  is spanning in  $M_l$ . We will construct a  $\Delta$  matroid realizing  $M_u$  and  $M_l$  as the upper and lower matroids. Set  $\mathcal{F}$  to be the collection of all subsets of  $E$  which are both upper independent and lower spanning. So, in particular, the bases of  $M_l$  and  $M_u$  are contained in  $\mathcal{F}$ .

Let  $F, F' \in \mathcal{F}$  and let  $x \in F \setminus F'$ . If  $x$  is an element of a lower cycle in  $F$ , then chose  $y = x$ , and  $F \setminus \{x\}$  is both lower spanning and upper independent, so feasible. So suppose that  $x$  belongs to no lower cycle in  $F$ . Then  $F \setminus \{x\}$  is not lower spanning, and, since  $F'$  is lower spanning, there exists an element  $y \in F' \setminus F$  so that  $(F \setminus \{x\}) \cup \{y\}$  is lower spanning. Moreover,  $y$  is contained in no lower cycle in  $(F \setminus \{x\}) \cup \{y\}$ . We know that  $F$  is upper independent, and so  $(F \setminus \{x\}) \cup \{y\}$  is either upper independent, or contains a unique upper cycle, which contains  $y$ . That upper cycle is a union of lower cycles, contradicting the fact that  $y$  is contained in no lower cycle in  $(F \setminus \{x\}) \cup \{y\}$ . So  $(F \setminus \{x\}) \cup \{y\}$  is lower spanning and upper independent.

Now we must consider the case when  $x \in F' \setminus F$ . If  $F \cup \{x\}$  is upper independent, we can choose  $y = x$ , and  $F \Delta \{x, y\} = F \cup \{x\}$  is upper independent and lower spanning, as required. So suppose that  $F \cup \{x\}$  is not upper independent, and so has a unique upper cycle  $C$  containing  $x$ . Since  $C \not\subseteq F'$ , there exists an element  $y \in F' \setminus F$  so that  $(F \setminus \{x\}) \cup \{y\}$  is upper independent. Moreover, since  $y$  is contained in a lower cycle  $C'$  contained in the lower spanning set  $F \cup \{x\}$ ,  $(F \setminus \{x\}) \cup \{y\}$  is also lower spanning.  $\square$

So  $\mathcal{F}$  is the collection of feasible sets of a  $\Delta$ -matroid on  $E$ . Consider a  $\Delta$ -matroid  $D$  on a ground set  $E$ . From the symmetric exchange axiom it is clear that replacing the feasible sets  $\mathcal{F}$  with their complements  $E \setminus F$ , yields another delta matroid  $D^*$  with feasible sets  $\mathcal{F}^* = \{E \setminus F \mid F \in \mathcal{F}\}$ . Note that  $(M_l)^* = (M^*)_u$  and  $(M_u)^* = (M^*)_l$ . For a subset  $X$  of the ground set which is contained in some feasible set  $F$ , we define a  $\Delta$  matroid  $D \setminus X$ , on the ground set  $E \setminus X$ , whose feasible sets are  $\{F \setminus X \mid F \in \mathcal{F}\}$  and a  $\Delta$  matroid  $D/X = (D^* \setminus X)^*$ .

Oxley [11] defines a matroid  $Q$  to be a *quotient* of a matroid  $M$  if there is a matroid  $N$  and a subset  $X$  of the ground set of  $N$  such that  $M = N \setminus X$  and  $Q = N/X$  and proves that  $Q$  is a quotient of  $M$  if and only if every circuit of  $M$  is a union of circuits of  $Q$ , so we have the following.

**Corollary 2.4.** *Let  $M_u$  and  $M_l$  be matroids on the same ground set  $E$ . Then  $M_u$  and  $M_l$  are the upper and lower matroids of a  $\Delta$ -matroid if and only if  $M_l$  is a quotient of  $M_u$ .*

There may be many matroids  $N$  on  $E \cup X$  with the property that  $M_l = N \setminus X$  and  $M_u = N/X$ . For our  $\Delta$ -matroid example on the edge set of a graph  $G = (V, E)$  with upper matroid the 2-dimensional generic rigidity matroid of  $G$  and lower matroid the cycle matroid of  $G$ , Corollary 2.4 implies that the connectivity matroid of a graph must be a quotient of its 2-dimensional generic rigidity matroid. Geometrically, this does not seem obvious, however the following is a realization of this relationship. Given  $G = (V, E)$ , the cone of  $G$ ,  $G_c = (V \cup \{x_0\}, E \cup X)$  is obtained by

adding a new vertex  $x_0$  and  $|V|$  new edges connecting the  $x_0$  to each vertex in  $V$ , so  $X = \{(x, v) \mid v \in V\}$ . See Figure 2.

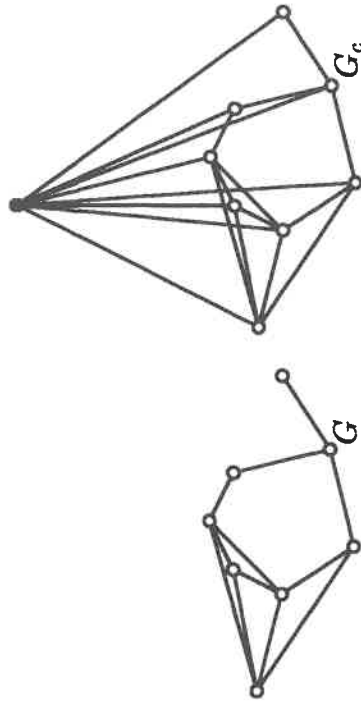


FIGURE 2. A graph  $G$  and its cone  $G_c$ .

**Theorem 2.5.** *Given a graph  $G$ , then its 2-dimensional generic rigidity matroid is  $M_r(G) = M_r(G_c) \setminus X$  and its connectivity matroid is  $M_c(G) = M_r(G_c)/X$ .*

*Proof.* Clearly  $M_r(G) = M_r(G_c) \setminus X$ , so we need only show  $M_c(G) = M_r(G_c)/X$ .

The circuits of  $M_r(G_c)/X$  are the minimal sets in  $D = \{C \setminus N \mid C \in \mathcal{C}(M_r(G_c))\}$ .

Since the cone of every connectivity cycle is a circuit in the 2-dimensional generic rigidity matroid, the minimal sets in  $D$  can contain at most the edges of a single connectivity cycle.

On the other hand, since the cone of an acyclic graph is independent in the 2-dimensional generic rigidity matroid, every edge set in  $D$  contains at least one connectivity cycle.

So the minimal elements of  $D$  are exactly the connectivity cycles of the graph  $G = (V, E)$ , and  $M_c(G) = M_r(G_c)/X$ .  $\square$

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