Making the Most of Your Decycling Sets

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1 Introduction

The decycling number of a graph G (denoted $\nabla(G)$) is the smallest size of a subset S of the vertex set $V(G)$ such that $G - S$ is acyclic [1]. Clearly a decycling set of order $\nabla(G)$ is minimal with respect to the decycling property. A natural question to ask is, "Are there larger subsets of the vertex set which are also decycling sets yet minimal with respect to that property?"

We define a ∇-critical set S of a graph G to be a subset of the vertex set which is a decycling set, but for every vertex v in S, $G - \{S - v\}$ contains a cycle. The *maximum decycling number* of a graph G (denoted $\nabla_m(G)$) is the maximum order of a ∇-critical set of G.

2 Preliminaries

Obervation 2.1: If S is a ∇-critical set of a graph G and v is any vertex in S, then v has at most deg(v) - 2 neighbors in the set S.
When G is a cycle or a complete graph, any ∇-critical set is of order $\nabla(G)$ ($\nabla(C_n) = \nabla_m(C_n) = 1$ and $\nabla(K_n) = \nabla_m(K_n) = n - 2$). Are there graphs G for which $\nabla(G) < \nabla_m(G)$? An answer to that question can be found in the Petersen Graph. From [1] we know that $\nabla(P) = 3$ (for example $\{a, d, w\}$ is a minimal decycling set).

Proposition 2.2: For the Petersen Graph P, $\nabla_m(P) = 4$.

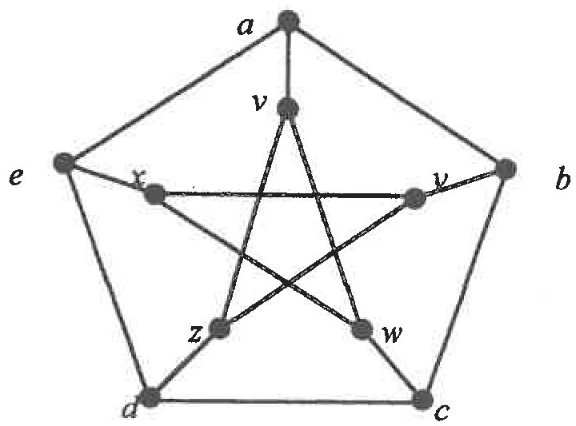

Figure 2.1

Proof: Clearly the set $S = \{a, b, w, x\}$ is a ∇-critical set. Can we find a ∇-critical set of order 5? In any set of 5 vertices at least 3 must be in either the pentagon or the pentagram. Assume (WOLOG) they are in the pentagon. By Observation 2.1 they cannot induce a P_3, so one of these vertices is not adjacent to either of the others. In this case, the only remaining cycle is the pentagram, and the decycling set can be completed with a single vertex.

Proposition 2.3: The difference between $\nabla(G)$ and $\nabla_m(G)$ can be arbitrarily large.
Proof: Recall from [1] that, for $n \geq 3$, $\nabla(W_n) = 2$.

Lemma 2.3: For $n \geq 3$, $\nabla_m(W_n) = \left\lfloor \frac{2n}{3} \right\rfloor$. If we label the central vertex v_0 and the peripheral vertices $v_1, v_2, v_3, \ldots v_n$, going clockwise around the cycle, then if $n \equiv 0$ or $2 \; mod \; 3$ the set $\{ v_i : i \equiv 1 \text{ or } 2 \; mod \; 3 \}$ or if $n \equiv 1 \; mod \; 3$ the set $\{ v_i : i \equiv 1 \text{ or } 2 \; mod \; 3 \text{ and } i < n-2 \} \cup \{v_{n-1}\}$ are ∇-critical sets. If a decycling set S has more than $\left\lfloor \frac{2n}{3} \right\rfloor$ vertices, then some vertex in S which lies on the cycle must have 2 neighbors in the ∇-critical set, which contradicts Observation 2.1.

3 Products of Paths and Products of Cycles

For the grid graphs we will use the standard labeling [2] for the vertices of a product of two graphs.

Lemma 3.1: If S is a ∇-critical set of $P_m \times P_n$ then any copy of P_m (respectively P_n) can contain at most $m - 1$ $(n - 1)$ vertices of S.

Proof: If $|S \cap P_m| = m$, then S is not ∇-critical since any cycle containing one vertex of the P_m must contain another vertex from the same P_m, hence if you remove any vertex in $S \cap P_m$ from S it will still be a decycling set.

Theorem 3.2: For $n \geq 2$, $\nabla_m(P_2 \times P_n) = \left\lfloor \frac{2n}{3} \right\rfloor$.

Proof: The sets $\{ v_{1,i} : i \equiv 0 \text{ or } 1 \; mod \; 3 \}$ when $n \equiv 0$ or $2 \; mod \; 3$ or $\{ v_{1,i} : i \equiv 0 \text{ or } 1 \; mod \; 3 \text{ and } i < n-2 \} \cup \{ v_{1,n-1} \}$ when $n \equiv 1 \; mod \; 3$ are ∇-critical sets of order $\left\lfloor \frac{2n}{3} \right\rfloor$. Are there larger ∇-critical sets?
Note that no ∇-critical set S may contain more than 2 vertices from any three consecutive copies of P_2. If the set $\{ v_{i,1}, v_{i,2}, v_{i+1,1}, v_{i+1,2}, v_{i+2,1}, v_{i+2,2} \}$ contains 3 vertices from S, then from Lemma 3.1 we know that each copy of P_2 contains exactly one vertex from S. Any cycle containing a vertex from the middle P_2 must also contain either both $v_{i,1}$ and $v_{i,2}$ or both $v_{i+2,1}$ and $v_{i+2,2}$ and one of each pair is in S. A similar argument shows that the 4-cycles at either end of the graph can

contain at most one vertex in S. This gives the upper bound $\nabla_m(P_2 \times P_n) \leq \left\lfloor \frac{2n}{3} \right\rfloor$ which proves the theorem.

Theorem 3.3: For $n \geq 4$, $\nabla_m(P_3 \times P_n) = 2\left\lfloor \frac{2n}{3} \right\rfloor$.

Proof: The sets $\{v_{1,i} : i \equiv 0 \text{ or } 1 \text{ } mod\ 3\} \cup \{v_{3,i} : i \equiv 0 \text{ or } 1 \text{ } mod\ 3\}$ when $n \equiv 0$ or 2 mod 3 or the set $\{v_{1,i} : i \equiv 0 \text{ or } 1 \text{ } mod\ 3 \text{ and } i < n - 2\} \cup \{v_{1,n-1}\} \cup \{v_{3,i} : i \equiv 0$ or $1 \text{ } mod\ 3 \text{ and } i < n - 2\} \cup \{v_{3,n-1}\}$ when $n \equiv 1$ mod 3 are ∇-critical sets. Are there larger ∇-critical sets?

Note that no ∇-critical set may contain 5 vertices from any three consecutive copies of P_3. Let S be a ∇-critical set. Assume the set $\{ v_{i,1}, v_{i,2}, v_{i,3}, v_{i+1,1}, v_{i+1,2}, v_{i+1,3}, v_{i+2,1}, v_{i+2,2}, v_{i+2,3} \}$ contains 5 vertices from S. Then S must intersect the 3 copies of P_3 in either a 2-1-2 pattern or a 1-2-2 (2-2-1) pattern by Lemma 3.1. For the 2-1-2 pattern S is not ∇-critical since every cycle containing a vertex in the middle copy of P_3 must contain at least two vertices from one of the other two copies of P_3. For the 1-2-2 (2-2-1) pattern call u the single vertex of S in the leftmost (rightmost) P_3 and v a vertex of S closest to u in the middle P_3. If $d(u, v) = 1$, any cycle containing v must contain either two vertices of S in the rightmost (leftmost) P_3, or u, so S is not minimal. If $d(u, v) = 2$, any cycle containing v must contain either two vertices of S in the rightmost (leftmost) P_3, u, or the other vertex of S in the middle P_3, so S is not minimal. Thus $\frac{4n}{3} = 2 \cdot \frac{2n}{3} \geq 2 \cdot \left\lfloor \frac{2n}{3} \right\rfloor$ is an upper bound as well as a lower bound.

Theorem 3.4: For $n \geq 4$, $\nabla_m(P_4 \times P_n) \geq \begin{cases} 6 & n = 4 \\ 8 & n = 5 \\ 11 & n = 7 \\ \left\lfloor \frac{5n-3}{3} \right\rfloor & otherwise \end{cases}$.

Proof: For $n = 4$ $\{v_{1,1}, v_{3,1}, v_{4,2}, v_{1,3}, v_{2,3}, v_{4,4}\}$, $n = 5$ $\{ v_{1,1}, v_{3,1}, v_{4,2}, v_{1,3}, v_{2,3}, v_{4,4}, v_{1,5}, v_{3,5} \}$, and $n = 7$ $\{ v_{1,1}, v_{3,1}, v_{4,2}, v_{1,3}, v_{2,3}, v_{4,4}, v_{1,5}, v_{2,5}, v_{4,5}\ v_{2,7}, v_{3,7} \}$, and $\{ v_{3,i} : 1 \leq i \leq n - 1\} \cup \{ v_{1,i} : i \equiv 0 \text{ or } 1 \text{ } mod\ 3 \}$ when $n \equiv 0$ or 2 mod 3 or $\{ v_{1,i} : i \equiv 0 \text{ or } 1 \text{ } mod\ 3 \text{ and } i < n - 2 \} \cup \{ v_{1,n-1} \}$ when $n \equiv 1$ mod 3 are ∇-critical sets of the appropriate order.

Theorem 3.5: For $5 \leq m \leq n$, $\nabla_m(P_m \times P_n) \geq \begin{cases} \left\lceil \frac{2n}{3} \right\rceil + \frac{m-2}{2}(n-1) & m \text{ even} \\ 2\left\lceil \frac{2n}{3} \right\rceil + \frac{m-3}{2}(n-1) & m \text{ odd} \end{cases}$

Proof: The sets $\{v_{1,i} : i \equiv 0 \text{ or } 1 \bmod 3\} \cup \{v_{k,j} : 3 \le k \le m - 1, \text{ and } k \equiv 1 \bmod 2, j = 1, 2, \ldots, n - 1\}$ when m is even or $\{v_{1,i} : i \equiv 0 \text{ or } 1 \bmod 3\} \cup \{v_{m,i} : i \equiv 0 \text{ or } 1 \bmod 3\} \cup \{v_{k,j} : 3 \le k \le m - 2, \text{ and } k \equiv 1 \bmod 2, j = 1, 2, \ldots, n - 1\}$ when m is odd are ∇-critical sets which attain the bound.

When investigating $\nabla_m(C_m \times C_n)$ [3] an upper bound is easily shown. Then, we will only need to give an example to prove each lower bound.

Lemma 3.6: For a ∇-critical set S of $C_m \times C_n$, $|S| \le \left\lceil \frac{mn}{2} \right\rceil$.

Proof: For the graph $C_m \times C_n$, $p = mn$ and $q = 2mn$. To find the decycling number we want to choose vertices for S that destroy as many edges as possible, but here we want to choose vertices that destroy as few new edges as possible. The first vertex we choose will destroy 4 edges, but if we pick new vertices adjacent to vertices already in S each will destroy at most 3 edges. If each new vertex destroys exactly 3 edges, then $< S >$ (the subgraph induced by S) contains no cycles, so $C_m \times C_n - S$ has only one component, and $p - |S| - 1 = q - 3|S| - 1$. Solving for $|S|$ we get $|S| = \left\lceil \frac{mn}{2} \right\rceil$, which is an upper bound as we may destroy fewer edges.

Theorem 3.7: For $n \ge 4$, $\nabla_m(C_3 \times C_n) = \left\lceil \frac{3n}{2} \right\rceil$

Proof: Let $S_1 = \{v_{1,i} : i \le n - 1 \text{ and } i \equiv 0, 1, 2 \bmod 4\} \cup \{v_{2,j} : j \le n - 1 \text{ and } j \equiv 0, 2, 3 \bmod 4\} \cup \{v_{3,n}\}$, and $S_2 = \emptyset$ when $n \equiv 1 \text{ or } 3 \bmod 4$, $\{v_{1,n}\}$ when $n \equiv 0 \bmod 4$, and $\{v_{3,n-1}\}$ when $n \equiv 2 \bmod 4$. Then $S_1 \cup S_2$ is a decycling set of the appropriate order.

4 Hypercubes

We next look at the hypercube Q_n. We will use the standard [2] description of Q_n (vertices are n-bit binary strings and an edge connects two vertices if they differ in exactly one bit). For any set of vertices T, we will use the notation $0T$ (resp. $1T$) for the vertices in T whose first bit is 0 (resp. 1) and ET (resp. $\overline{ET}$) for the vertices in T with even weight (resp. odd weight). Recall that in each cycle the parities of the vertices will alternate.

Observation 4.1: $\nabla_m(Q_2) = \nabla_m(C_4) = 1$

Theorem 4.2: $\nabla_m(Q_3) = 3$

Proof: From [1] we know $\nabla(Q_3) = 3$, could $\nabla_m(Q_3) \ge 4$? Let S be a decycling set with 4 vertices. If the vertices all have the same parity S would not be minimal. If 3 of the vertices have same parity S would not be minimal either (the vertex of the other parity is irrelevant). Thus, S must contain exactly two vertices of each parity. If we remove two vertices of even parity from Q_3, the only remaining cycle must be a 4-cycle containing both odd vertices that are neighbors of the remaining even pair, hence the decycling set could be completed with only one vertex. Therefore, any maximal minimal decycling set contains at most 3 vertices, and $\nabla_m(Q_3) = 3$.

Theorem 4.3: $\nabla_m(Q_4) = 7$

Proof: The vertex set {0000, 0001, 0101, 0111, 1100, 1010, 1111} is a ∇-critical set of order 7.
Could $\nabla_m(Q_4) \geq 8$? Let S be a ∇-critical sets of order 8. Neither $S \cap 0Q_4$ nor $S \cap 1Q_4$ could contain exactly 0, 1, 2, 6, 7, or 8 vertices or S would not be a decycling set.
Case 1: $S \cap 0Q_4$ contains 5 vertices. Then (WOLOG) $EQ_4 \cap S \cap 0Q_4$ contains either 4 or 3 vertices.
Case 1A: $EQ_4 \cap S \cap 0Q_4$ contains 4 vertices. All of the remaining cycles are in $1Q_4$, so the odd degree vertex is irrelevant.
Case 1B: $EQ_4 \cap S \cap 0Q_4$ contains 3 vertices. One of the odd degree vertices is adjacent to the 3 even vertices, so it cannot be in S. Regardless of how the two odd vertices are chosen to complete the set, they induce a P_5 in $0Q_4$. From Lemma 3.1 there is an induced P_3 in $1Q_4$ which cannot be in S. The other vertex in $1Q_4$ adjacent to the middle vertex in that P_3 cannot be in S either, or S will not be minimal. The remaining 4 vertices in $1Q_4$ (from which you need to choose 3 to complete S) induce a $K_{1,3}$. If you choose the central vertex and 2 others, then S is not a decycling set. If you choose the 3 pendant vertices, then S is not minimal.
Case 2: S intersects both $0Q_4$ and $1Q_4$ in 4 vertices.
Case 2A: If all the vertices in $S \cap 0Q_4$ are even (odd), then all the remaining cycles are in $1Q_4$, so S can be completed with at most 3 vertices.
Case 2B: Three of the vertices in $S \cap 0Q_4$ are (WOLOG) even. One odd degree vertex is adjacent to the 3 even vertices, so it cannot be in S. Whichever odd vertex we choose to complete the set will be the middle vertex in an induced P_3 in $0Q_4$. The even vertex in $1Q_4$ adjacent to that odd vertex cannot be in S (Observation 2.1). From Case 1 we know that no copy of Q_3 can contain 5 vertices of S, so 3 of the remaining vertices in S must be in the C_4 in $1Q_4$ with 3 of its vertices adjacent to the induced P_3 from above. For the same reason, the last vertex in S is now determined. No matter how S is completed, it is not minimal.
Case 2C: From cases (2A) and (2B), all of the sets $EQ_4 \cap S \cap 0Q_4$, $\overline{EQ4} \cap S \cap 0Q_4$, $EQ_4 \cap S \cap 1Q_4$, and $\overline{EQ4} \cap S \cap 1Q_4$ contain exactly 2 vertices. From Case (2B) no copy of Q_3 can contain 3 even vertices of S, so once we know the two even vertices in $0Q_4$ the even vertices in $1Q_4$ are determined. One odd vertex in S from $0Q_4$ and $1Q_4$ must be adjacent to exactly one even vertex in S otherwise S is not a decycling set. Regardless of how these two odd vertices are chosen, there is only one cycle remaining, so S can be completed with a single vertex.
In any of the cases, S cannot contain 8 vertices and be minimal.

Theorem 4.4: $\nabla_m(Q_5) = 20$

The words of weight 2 and weight 3, form a ∇-critical set of order 20.
Could $\nabla_m(Q_5) \geq 21$? Let S be a ∇-critical sets of order 21 and $T = Q_5 - S$. If $\langle T \rangle$ is connected, it has 10 edges, and there are $2 \cdot 4 + 9 \cdot 3 = 35$ edges between T and S, hence 35 edges in S. However, $\frac{2 \cdot 35}{21} > 3$, so some vertex in S has degree 4 in $\langle S \rangle$, which contradicts Observation 2.1. If we reduce the number of edges in T, then we increase the number of edges between S and T, and decrease the number of edges in S. The first time that brings the degree condition in line is when the number of edges in T is 6. For S to be minimal, every vertex v in S must be in some cycle in $T \cup v$. Since every cycle in Q_5 must be even, then each vertex v must be adjacent to the endpoints of an induced P_3 or P_5 in T. Six or fewer edges is not enough to accommodate the 21 vertices in S.

Theorem 4.5: $\nabla_m(Q_6) \geq 42$

The vertices in $0Q_6$ of weight 2 and weight 3 along with the vertices in $1Q_6$ of weight 3 and weight 4 along with 000000 and 111111 form a ∇-critical set of order 42.

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