Tight Bounds for the Split Domination Number of a Nearly Regular Tournament

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Abstract. A set of vertices, S, in a digraph D, is split dominating provided it is: 1) dominating and 2) $D[V(D)\backslash S]$ is either trivial or has a lower level of connection than D. In this paper, we consider split dominating sets in strongly connected tournaments. The split domination number of a strongly connected tournament T, denoted by $\gamma_s(T)$, is the minimum cardinality of a split dominating set for that tournament. The authors previously gave a tight lower bound for $\gamma_s(T)$ when T is regular. In this paper, we show that when T is a nearly regular $2k$-tournament, then $\gamma_s(T) \geq \lceil \frac{2k}{3} \rceil$ and this bound is tight.

Keywords: split domination, nearly regular tournament

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. A set $S \subseteq V(D)$, is considered *dominating* provided for each $v \in V(D)$, either $v \in S$, or there is an arc $(s, v) \in A(D)$ for some $s \in S$. See [6] for an introduction to domination in graphs. For more recent work on domination in digraphs, see [2] and [5]. Many variations on domination in both graphs and digraphs have been considered. Recent examples include those found in [1] and [8].

In this paper, we consider *split domination*, introduced for graphs by Kulli and Janakiram [9]. A set of vertices, S, in a connected graph G is called *split dominating* if S is dominating and the subgraph induced by $V(G)\backslash S$ is either trivial (a single vertex) or not connected. More recently, this problem was explored in graphs by Hedetniemi, Knoll, and Laskar [7] and in digraphs by Factor and Merz [4]. Connection in digraphs is nuanced, making split dominating in digraphs an interesting extension.

A digraph D is *strongly connected* (or simply *strong*) provided for all $x, y \in V(D)$, there is a directed path from x to y. A digraph D is *unilaterally connected* provided for all $x, y \in V(D)$, there is either a directed path from x to y or from y to x. A digraph is *weakly connected* provided its underlying graph (the graph obtained by ignoring the direction of the

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arcs) is connected. Thus, digraphs have levels of connectedness. Listed in decreasing order, the levels are: strong, unilaterally connected, weakly connected, and not weakly connected. Since strong connection implies unilateral connection, which implies weak connection, we can talk about the maximum level of connectedness of a digraph. For example, a digraph that is unilaterally connected but not strongly connected attains unilateral connection as its maximum level of connectedness. Given a digraph D, we say a set of vertices S is *split dominating* provided S is dominating and the subdigraph of D induced by $V(D)\backslash S$ is trivial or has a lower maximum level of connection than D. The *split domination number* of a digraph D, denoted by $\gamma_s(D)$, is the minimum size of a split dominating set in D.

In this paper, we consider split domination in tournaments. T is a tournament provided for all $x, y \in V(T)$, $x \neq y$, either (x, y) or $(y, x) \in A(T)$, but not both. Since every tournament is unilaterally connected, the most interesting case is to consider the split domination number of strong tournaments. An *n-tournament* is a tournament with n vertices. If an n-tournament is not strongly connected, then the split domination number will automatically be $n - 1$.

If $v \in V(D)$, the *outset* of v is $N^+(v) = \{u : (v, u) \in A(D)\}$ and the *inset* of v is $N^-(v) = \{u : (u, v) \in A(D)\}$. We write $N_S^+(v)$ to denote $N^+(v) \cap S$, where $S \subseteq V(D)$. Similarly, $N_S^-(v) = N^-(v) \cap S$. The *outdegree* of a vertex x is $|N^+(x)|$ while $|N^-(x)|$ is the *indegree* of x. A tournament is *k-regular* provided each vertex has outdegree (and consequently, indegree) equal to k. Thus, a k-regular tournament must be a $2k+1$-tournament. Tight upper and lower bounds for $\gamma_s(T)$ when T is a k-regular tournament are given in [3].

In this paper, we consider $\gamma_s(T)$ when T is a nearly regular tournament. A tournament is *nearly regular* provided half the vertices have outdegree k and indegree $k - 1$, while the remaining vertices have indegree k and outdegree $k - 1$. Such a tournament will necessarily have $2k$ vertices. It is known that every nearly regular $2k$-tournament is strong (proving this statement is a nice problem for an undergraduate graph theory course). Next, we give an upper bound on the split domination number.

Lemma 1 *If T is a nearly regular $2k$-tournament, then $\gamma_s(T) \leq k$.*

Proof. If $k = 1$, the statement is true so assume $k \geq 2$. Let $x \in V(T)$ with $|N^+(x)| = k - 1$. Consider $S = N^+(x) \cup \{y\}$ where y is any vertex from $N^-(x)$. Since $|N^+_{V(T)\backslash S}(x)| = 0$, $T[V(T)\backslash S]$, the subgraph of T induced by $V(T)\backslash S$, is not strong. Suppose S is not dominating. Then there is some vertex, call it v such that $v \in N^-(x)$ and v has an arc to every vertex in S. But then the outdegree of v is at least $|S| + 1 = k + 1$, a contradiction. □

Furthermore, this bound is tight. If T is a 2-tournament, $\gamma_s(T) = 1 = k$. Otherwise, $k \geq 2$ and a nearly regular $2k$-tournament with $\gamma_s(T) = k$ is given in [4]. See Figure 1 for an example.

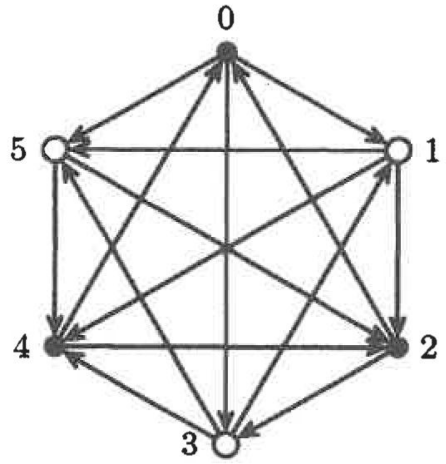

Figure 1: A nearly regular $2k$-tournament with $k = 3$. The black vertices form a split dominating set of minimum size. This example is generalized to all $k \geq 2$ in [4].

Next, we establish a lower bound for $\gamma_s(T)$ when T is a strong nearly regular $2k$-tournament. Since any n-tournament has $n(n-1)/2$ arcs, the average outdegree (and average indegree) of the vertices is $(n-1)/2$. This means there is at least one vertex of outdegree at least $(n-1)/2$ in any tournament, and if n is even, there is a vertex with outdegree at least $n/2$. Likewise, in any tournament with n vertices, there is a vertex with indegree at least $(n-1)/2$ and if n is even we know there is a vertex with indegree at least $n/2$.

Theorem 2 *If T is a nearly regular $2k$-tournament, then*

$$\gamma_s(T) \geq \left\lceil \frac{2k}{3} \right\rceil.$$

Proof. There is just one 2-tournament and it has split domination number 1. So assume $k \geq 2$. Therefore T is strong. Let S be a minimum size split dominating set of T. Since $T[V(T)\backslash S]$ is not strong, we may partition $V(T)\backslash S$ into X and Y such that (x, y) an arc for all $x \in X$ and all $y \in Y$. Observe that $|S| + |X| + |Y| = 2k$.

Consider $T[X]$. There must be at least one vertex x' in X with $|N_X^+(x')| \geq (|X| - 1)/2$. Since x' is directed toward every vertex in Y,

$$k \geq |N^+(x')| \geq \frac{|X| - 1}{2} + |Y|. \tag{1}$$

Similarly, there is some vertex y' in Y such that

$$k \geq |N^-(y')| \geq \frac{|Y|-1}{2} + |X|.$$

Since S is dominating we know there is at least one arc from a vertex in S to y'. Thus,

$$k \geq |N^-(y')| \geq \frac{|Y|-1}{2} + |X| + 1. \tag{2}$$

Adding inequalities (1) and (2), we see that

$$\frac{|X|-1}{2} + |Y| + \frac{|Y|-1}{2} + |X| + 1 \leq 2k \Rightarrow \frac{3|X|}{2} + \frac{3|Y|}{2} \leq 2k.$$

Thus, $|X| + |Y| \leq (4k)/3$. Since $|S| + |X| + |Y| = 2k$, we conclude that

$$|S| + \frac{4k}{3} \geq 2k \Rightarrow |S| \geq \frac{2k}{3}.$$ □

Theorem 3 *For all k, there exists a nearly regular $2k$-tournament such that*

$$\gamma_s(T) = \left\lceil \frac{2k}{3} \right\rceil.$$

Proof. There is just one 2-tournament and it has split domination number 1. So assume $k \geq 2$. There are three cases to consider: either $3|(2k)$, $3|(2k+1)$ or $3|(2k+2)$. Suppose $3|(2k)$. Partition $V(T)$ into $W \cup X \cup Y$. Let $W = \{w_0, w_1, \ldots, w_{(2k/3)-2}\}$,, $X = \{x_0, \ldots, x_{(2k/3)}\}$, and $Y = \{y_1, \ldots, y_{(2k/3)}\}$. Let $T[W]$ be any regular tournament with $(2k/3) - 1$ vertices. Let $T[X]$ be any regular tournament with $2k/3 + 1$ vertices. Let $T[Y]$ be any nearly regular tournament with $2k/3$ vertices with the property that $|N_Y^+(y_1)| < |N_Y^-(y_1)|$.

For all $w \in W$ and $x \in X$, let $(w, x) \in A(T)$. For all $y \in Y$ and $w \in W$, let $(y, w) \in A(T)$. For all $i \in \{1, \ldots, 2k/3\}$, let $(y_i, x_i) \in A(T)$. Let arc $(y_1, x_0) \in A(T)$. Finally, for all remaining arcs between vertices of X and Y, direct them from the vertex in X toward the vertex in Y. See Figure 2 for an example.

Observe that Y is split dominating and $|Y| = 2k/3$. It remains to be verified that T is nearly regular. Let $w \in W$. Then,

$$|N_W^+(w)| = \frac{2k/3 - 2}{2}, |N_Y^+(w)| = 0, \text{ and } |N_X^+(w)| = \frac{2k}{3} + 1,$$

so $|N^+(w)| = k$ (and therefore $|N^-(w)| = k - 1$). Let $x \in X$. Then,

$$|N_X^+(x)| = \frac{2k/3}{2}, |N_Y^+(x)| = \frac{2k}{3} - 1, \text{ and } |N_W^+(x)| = 0,$$

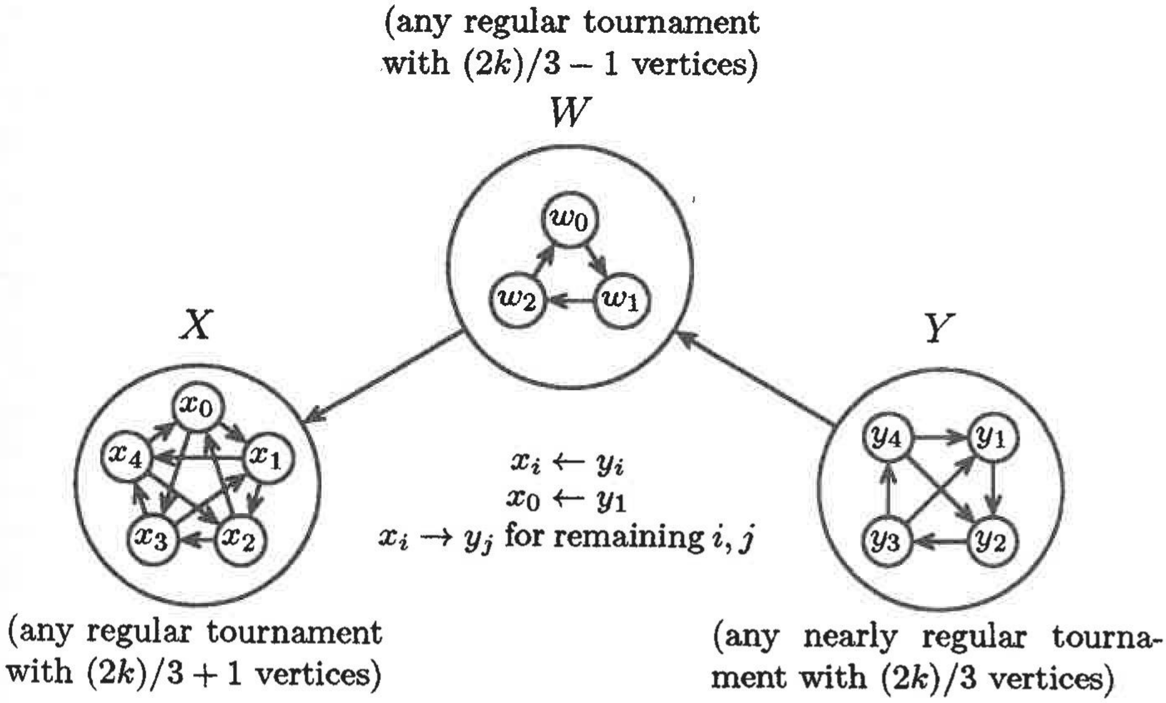

Figure 2: An example ($k = 6$) of the construction when $3|(2k)$.

so $|N^+(x)| = k - 1$ (and therefore $|N^-(x)| = k$). The number of vertices in X is two more than the number of vertices in W. So if we can show that, in Y, the number of vertices with degree k is two more than the number of vertices with degree $k - 1$, we will be done. Observe that

$$|N_Y^+(y_1)| = \frac{2k/3}{2} - 1, |N_W^+(y_1)| = \frac{2k}{3} - 1, \text{ and } |N_X^+(y_1)| = 2.$$

Thus $|N^+(y_1)| = k$. If y is one of the $k/3 - 1$ other vertices in Y with outdegree in Y one less than the indegree in Y, then

$$|N_Y^+(y)| = \frac{2k/3}{2} - 1, |N_W^+(y)| = \frac{2k}{3} - 1, \text{ and } |N_X^+(y)| = 1.$$

Therefore $|N^+(y)| = k - 1$. The remaining $k/3$ vertices $y \in Y$ have $|N_Y^+(y)| = |N_Y^-(y)| + 1$. For such y,

$$|N_Y^+(y)| = \frac{2k/3}{2}, |N_W^+(y)| = \frac{2k}{3} - 1, \text{ and } |N_X^+(y)| = 1.$$

So $|N^+(y)| = k$. In other words, Y has $k/3 + 1$ vertices with outdegree k and $k/3 - 1$ vertices with outdegree $k - 1$. Thus T is a nearly regular $2k$-tournament.

For the second case, assume that $3|(2k+1)$. Let $|W| = (2k+1)/3 - 1$ and let $|X| = |Y| = (2k+1)/3$. Notice that $|V(T)| = 2k$. Let $T[W]$ be any nearly regular tournament and let $T[X]$ and $T[Y]$ be any regular tournaments. Let every vertex in W beat every vertex X. Let every vertex in Y beat every vertex of W.

All that remains is the arcs between vertices in X and Y. Label $X = \{x_0, x_1, \ldots, x_{r-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{r-1}\}$ where $r = (2k+1)/3$. For $i = 0, \ldots, r-1$, let $(y_i, x_i) \in A(T)$. For $i \neq j$, let $(x_i, y_j) \in A(T)$.

Observe that Y is split dominating and $|Y| = (2k+1)/3 = \lceil 2k/3 \rceil$. Having already established that $|V(T)| = 2k$, all that remains is to verify that T is nearly regular. Let $x \in X$. Then

$$|N_X^+(x)| = \frac{(2k+1)/3 - 1}{2}, |N_Y^+(x)| = \frac{2k+1}{3} - 1, \text{ and } |N_W^+(x)| = 0.$$

Thus $|N^+(x)| = k - 1$ (and so $|N^-(x)| = k$). Let $y \in Y$. Then

$$|N_Y^+(y)| = \frac{(2k+1)/3 - 1}{2}, |N_X^+(y)| = 1, \text{ and } |N_W^+(y)| = \frac{2k+1}{3} - 1.$$

Thus, $|N^+(y)| = k$ (and so $|N^-(y)| = k - 1$).

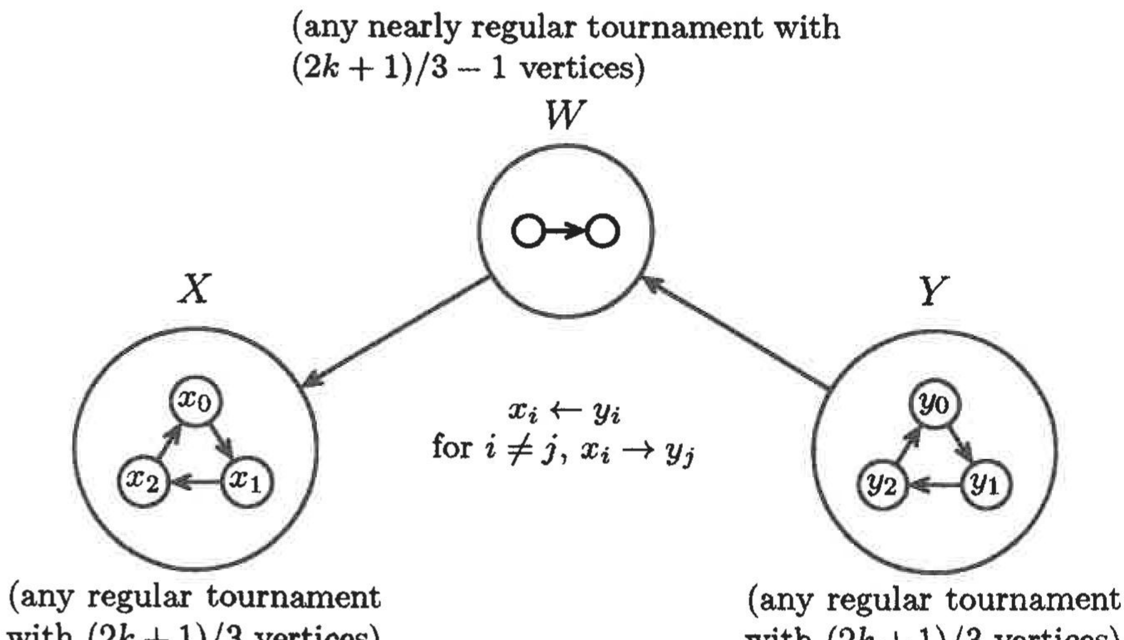

Figure 3: An example ($k = 4$) of the construction when $(2k+1)$ is a multiple of 3.

Finally, let $w \in W$. Since $T[W]$ is nearly regular with $(2k+1)/3 - 1$ vertices, $|N_W^+(w)| = p$ or $p - 1$ where $2p = (2k+1)/3 - 1$. In the former case,

$$|N_W^+(w)| = \frac{(2k+1)/3-1}{2}, |N_X^+(w)| = \frac{2k+1}{3}, \text{ and } |N_Y^+(w)| = 0,$$

so $|N^+(w)| = k$. But, if $|N_W^+(w)| = p-1$, then $|N^+(w)| = k-1$.

Thus half the vertices in T have outdegree k, specifically the vertices in Y and half the vertices in W. The remaining vertices in T have outdegree $k-1$, making T nearly regular. See Figure 3 for the example where $k = 4$.

Finally, we consider the case where $3|(2k+2)$. Let $T[W]$ be any nearly regular tournament on $(2k+2)/3$ vertices. Let $T[X]$ and $T[Y]$ be any regular tournaments on $(2k+2)/3 - 1$ vertices. Notice that T has $2k$ vertices. For each $y \in Y$ and $w \in W$, let $(y, w) \in A(T)$. For each $w \in W$ and $x \in X$, let $(w, x) \in A(T)$. Finally, for each $x \in X$ and $y \in Y$, let $(x, y) \in A(T)$.

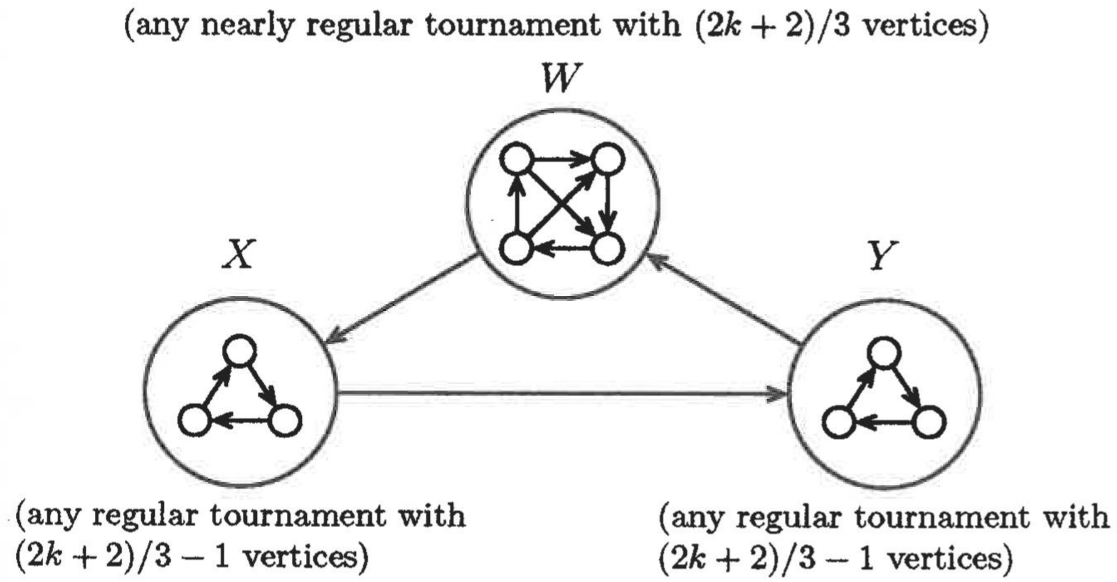

Figure 4: An example ($k = 5$) of the construction when $(2k+2)$ is a multiple of 3.

Notice that if we let $S = Y \cup \{w\}$ for any $w \in W$, then S is split dominating and of size $(2k+2)/3 = \lceil 2k/3 \rceil$. If T is nearly regular, the proof is complete. Every vertex in X has an arc to every vertex in Y and half of the other vertices in X. So for all $x \in X$,

$$|N^+(x)| = \frac{(2k+2)/3-2}{2} + \frac{2k+2}{3} - 1 = k-1.$$

Every vertex in Y has an arc to every vertex in W and half of the other vertices in Y. So for all $y \in Y$,

$$|N^+(y)| = \frac{2k+2}{3} + \frac{(2k+2)/3-2}{2} = k.$$

Of the vertices considered so far, an equal number have outdegree k and $k-1$. There are $|W| = (2k+2)/3$ vertices unexamined as of yet. Each of these vertices has an arc to every vertex in X and an arc from each vertex in Y. Exactly half of the vertices in W have outdegree p in W, while the other half of the vertices in W have outdegree $p-1$ in W, where $p = \frac{(2k+2)/3}{2}$. Let $w \in W$. If $|N^+_W(w)| = p$, then

$$|N^+(w)| = p + \frac{2k+2}{3} - 1 = k.$$

Therefore, if $|N^+_W(w)| = p-1$, then $|N^+(w)| = k-1$. Thus half of the vertices in T have outdegree k (and indegree $k-1$), while the other half have outdegree $k-1$ (and indegree k). See Figure 4 for an example. □

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