A SHORT NOTE ON THE LARGE STEINER k-DIAMETER OF GRAPHS

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ABSTRACT. **In a graph G, the Steiner distance $d(S)$ of the vertex subset $S \subseteq V(G)$ is the minimum size among all connected subgraphs whose vertex sets contain S, and the Steiner k-diameter of a connected graph G is the maximum $d(S)$ among all k-element vertex subsets $S \subseteq V(G)$. In this paper we examine the Steiner k-diameter for large k and then discuss the applications of the results.**

Keywords: Steiner diameter, hypercubes, star graphs, hyper-stars, split stars, alternating group graphs, augmented cubes, generalized Petersen graphs

1. INTRODUCTION

In a finite, simple, undirected graph, the ***Steiner distance***, introduced by Chartrand et al. [1], is a natural generalization of the classical graph distance. For a graph $G(V,E)$ and a set $S \subseteq V(G)$ of at least two vertices, an ***S-Steiner tree*** is a subgraph $T(V',E')$ of G that is a tree with $S \subseteq V'$. In a graph $G(V,E)$ with at least 2 vertices, for any $S \subseteq V$ with $|S| \geq 2$ the Steiner distance $d_G(S)$ (or simply $d(S)$ if the graph G is clear from the context) is the minimum size of an S-Steiner tree; alternatively, it is the minimum size connected subgraph whose vertex set contains S. We set $d_G(S) = \infty$ when there is no S-Steiner tree in G. We note that if $S = \{u,v\}$, then $d_G(S)$ is simply classical distance $d(u,v)$. If G is a graph on n vertices, then for any k with $2 \leq k \leq n$, we define the ***Steiner k-diameter*** of G, $\text{sdiam}_k(G)$, as the maximum value of $d_G(S)$ over all vertex subsets S with $|S| = k$. Again, we note that when $k = 2$, we recover the classical diameter. We may observe that in a connected graph, a spanning tree of the graph is an S-Steiner tree for any set S. This leads to the following basic bound on the Steiner k-diameter.

Theorem 1.1. *[2] Let k,n be two integers with $2 \leq k \leq n$, and let G be a connected graph of order n. Then $k-1 \leq sdiam_k(G) \leq n-1$. Moreover, the upper and lower bounds are sharp.*

In [3], the graphs with a given Steiner k-diameter were characterized by connectivity and the structure of vertex cut-sets.

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Theorem 1.2. *[3] Let G be a connected graph of order n, and let* $1 \leq k \leq n-2$ *and* $1 \leq t \leq k$. *Then* $sdiam_{n-k}(G) \geq n-t$ *if and only if there exists* $B \subseteq V(G)$ *with* $|B| = k$ *such that each t-element subset of B is a vertex cut-set of G.*

This result, with some additional analysis, can be used to give a full classification of the graphs with a given Steiner k-diameter. In [3], such classifications were given for $k = n-1, n-2, n-3$.

Corollary 1.3. *Let G be a connected graph of order* $n \geq 3$. *Then*
(1) $sdiam_{n-1}(G) = n-2$ *if and only if G is 2-connected.*
(2) $sdiam_{n-1}(G) = n-1$ *if and only if G has a cut-vertex.*

Corollary 1.4. *Let G be a connected graph of order* $n \geq 4$. *Then*
(1) $sdiam_{n-2}(G) = n-3$ *if and only if G is 3-connected.*
(2) $sdiam_{n-2}(G) = n-2$ *if and only if* $\kappa(G) = 2$ *or G has exactly one cut-vertex.*
(3) $sdiam_{n-2}(G) = n-1$ *if and only if G has at least two cut-vertices.*

Corollary 1.5. *Let G be a connected graph of order* $n \geq 5$. *Then*
(1) $sdiam_{n-3}(G) = n-4$ *if and only if G is 4-connected.*
(2) $sdiam_{n-3}(G) = n-3$ *if and only if one of the following is true:*
(2.1) $\kappa(G) = 3$.
(2.2) $\kappa(G) = 2$, *and for every minimum vertex cut-set* $\{u, v\}$ *of G and* $w \in V(G)\backslash\{u, v\}$, *at least one of* $\{u, w\}$ *and* $\{v, w\}$ *is not a vertex cut-set of G.*
(2.3) $\kappa(G) = 1$, *G has exactly one cut-vertex u, and every 2-element vertex cut-set of G contains u.*
(3) $sdiam_{n-3}(G) = n-2$ *if and only if one of the following is true:*
(3.1) $\kappa(G) = 2$, *and there are three vertices* $u, v, w \in V(G)$ *such that* $\{u, v\}$, $\{u, w\}$, *and* $\{v, w\}$ *are all vertex cut-sets of G.*
(3.2) $\kappa(G) = 1$, *and one of the following is true:*
(3.2.1) G has exactly two cut-vertices.
(3.2.2) G has exactly one cut-vertex u and a vertex cut-set $\{v, w\}$ *that does not contain u.*
(4) $sdiam_{n-3}(G) = n-1$ *if and only if G has at least three cut-vertices.*

This note serves two purposes. The first is to find a complete classification of graphs with a given Steiner $(n-4)$ diameter in the same manner as the above results. The second is to discuss the possible application of these results to particular families of graphs. Although the proof is straightforward, it illustrates that it is feasible to obtain such a complete classification even though the statement is complicated.

2. The Steiner $(n-4)$-Diameter

Theorem 2.1. *Let G be a connected graph of order* $n \geq 6$. *Then*

(1) $sdiam_{n-4}(G) = n-5$ if and only if $\kappa(G) \geq 5$.
(2) $sdiam_{n-4}(G) = n-4$ if and only if one of the following are true:
(2.1) $\kappa(G) = 4$
(2.2) $\kappa(G) = 3$ and for all cut-sets $\{u,v,w\}$ in G and $x \in G \setminus \{u,v,w\}$, at least one of $\{x,u,v\}$, $\{x,u,w\}$, and $\{x,v,w\}$ is not a cut-set.
(2.3) $\kappa(G) = 2$ and for all vertex cut-sets $\{u,v\}$ of G and sets $\{w,x\} \in V(G) \setminus \{u,v\}$, at least one of $\{u,w,x\}$ and $\{v,w,x\}$ is not a cut-set.
(2.4) $\kappa(G) = 1$ and G has exactly one cut-vertex u and all 2-element and 3-element vertex cut-sets contain u.
(3) $sdiam_{n-4}(G) = n-3$ if and only if one of the following are true:
(3.1) $\kappa(G) = 3$ and there exists some $\{u,v,w,x\} \in V(G)$ such that $\{u,v,w\}$, $\{u,v,x\}$, $\{u,w,x\}$, and $\{v,w,x\}$ are all cut-sets.
(3.2) $\kappa(G) = 2$ and all two element cut-sets of G $\{u,v\}$ and $\{w,x\}$ are disjoint.
(3.3) $\kappa(G) = 1$ and at least one of the following conditions is true
(3.3.1) There are exactly two cut-vertices, and all two element cut-sets contain at least one of these vertices.
(3.3.2) There is exactly one cut-vertex u and some three element cut-set $\{v,w,x\} \in G \setminus u$ such that at least one of $\{v,w\}$, $\{v,x\}$, and $\{w,x\}$ is not a cut-set.
(4) $sdiam_{n-4}(G) = n-2$ if and only if one of the following are true:
(4.1) $\kappa(G) = 2$ and G has some elements $\{u,v,w,x\}$ such that all two element subsets of $\{u,v,w,x\}$ are cut-sets.
(4.2) $\kappa(G) = 1$ and G has exactly three cut-vertices or exactly two cut-vertices x,u and a cut-set $\{v,w\} \in G \setminus \{x,u\}$.
(5) $sdiam_{n-4}(G) = n-1$ if and only if G has at least 4 cut-vertices.

Proof. In each case, we apply Theorem 1.2 to suppose the existence of the set B for the given value of t and the nonexistence of such a B for $t-1$, and enumerate all possibilities.

For (1), we note that there are no cut-sets with 4 vertices in a 5-connected graph. The case (5) is also clear.

For (2), we must include exactly one vertex of B in the S-Steiner tree. If $\kappa(G) = 4$, choose B as a 4-element cut-set. If $\kappa(G) = 1, 2$, or 3, there must be some three element subset of B that is not a vertex cut-set, but B must be a cut-set of G. The conditions in (2.2) and (2.3) enumerate the possibilities. For $\kappa(G) = 1$, there can only be one cut-vertex, and the other three vertices in B must not be a cut-set. The conditions in (2.4) follow.

For (3), there must be some two element subset of B that is not a vertex cut-set, but every three element subset of B is a vertex cut-set. Cases (3.1) and (3.2) follow directly. For $\kappa(G) = 1$, there at least one cut-vertex. Then, the other three vertices in B must form a cut-set. However, adding one more vertex must connect G. Thus if B has exactly two cut-vertices the remaining two vertices in B are not a cut-set, as in (3.3.1). If B has exactly one cut-vertex, the other three vertices, must form a cut-set, but at least one two-element subset of these three vertices must connect G. This describes (3.3.2).

For (4), there must be some element of B that is not a cut-vertex, but every two element subset of B is a cut-set. Cases (4.1) and (4.2) follow directly.

□

3. Applications

Given the characterizations described in the results above, a natural goal is to use them to investigate various families of graphs. This is complicated by the fact that the conditions we have given only characterize the Steiner k-diameter when $k \geq n-4$. As many families of graphs, especially those suitable for use as interconnection networks, have high connectivity, the application of these results is relatively uninteresting in many cases. For example, the n-dimensional hypercube is n-regular, n-connected, and has 2^n vertices. Thus, our results are only informative in the case where $n < 5$.

While ad-hoc methods may lead to a calculation of the Steiner diameter for more values of k in such families of graphs, more general methods for $k < n-4$ have not yet been described. As such, to illustrate the use of these particular results, we examine the family of generalized Petersen graphs, which are 3-regular and 3-connected. The generalized Petersen graph $P(n,k)$, $1 \leq k < \frac{n}{2}$ consists of n outer vertices labeled $a_0, a_1, ..., a_{n-1}$ and n inner vertices labeled $b_0, b_1, ..., b_{n-1}$. $a_i = a_j$ and $b_i = b_j$ if $i \equiv j \pmod{n}$. Connect edges a_i with a_{i+1}, b_i with b_{i+k}, and a_i with b_i. For more on generalized Petersen graphs, refer to [5].

Theorem 3.1. *Let $n \geq 2$. Then $sdiam_{2n-1}(P(n,k)) = 2n-2$, $sdiam_{2n-2}(P(n,k)) = 2n-3$, $sdiam_{2n-3}(P(n,k)) = 2n-3$, and $sdiam_{2n-4}(P(n,k)) = 2n-4$.*

Proof. Note that $\kappa(P(n,k)) = 3$, so $\mathrm{sdiam}_{2n-1}(P(n,k)) = 2n-2$, $\mathrm{sdiam}_{2n-2}(P(n,k)) = 2n-3$, and $\mathrm{sdiam}_{2n-3}(P(n,k)) = 2n-3$. In addition, all three element cut-sets of a generalized Petersen graph are of the form $\{a_{i-1}, a_{i+1}, b_i\}$, $\{b_{i-k}, b_{i+k}, a_i\}$, or $\{a_i, a_{i+k}, a_{i+2k}\}$ (when $n = 3k$). From this, we see that no four element set can contain four three element subsets that are all cut-sets. Thus, by Theorem 2.1, $\mathrm{sdiam}_{2n-4}(P(n,k)) = 2n-4$. □

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