On s-Bipartite Ramsey Numbers
Stars, Matchings and Double Stars

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Abstract

For bipartite graphs F and H and a positive integer s, the s-bipartite Ramsey number $BR_s(F, H)$ of F and H is the smallest integer t with $t \geq s$ such that every red-blue coloring of $K_{s,t}$ results in a red F or a blue H. We evaluate this number for all positive integers s when F and H are both stars, are both matchings or one is a star and the other is a matching as well as when $F = H$ is an arbitrary double star.

Key Words: Ramsey number, bipartite Ramsey number, s-bipartite Ramsey number, star, matching, double star.

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1 Introduction

In a *red-blue coloring* of a graph G, every edge of G is colored red or blue. For two graphs F and H, the *Ramsey number* $R(F, H)$ of F and H is the smallest positive integer n such that every red-blue coloring of the complete graph K_n of order n results in either a subgraph isomorphic to F all of whose edges are colored red (a *red* F) or a subgraph isomorphic to H all of whose edges are colored blue (a *blue* H). A graph (subgraph) all of whose edges are colored the same is called a *monochromatic graph* (*subgraph*).

In [2] Beineke and Schwenk introduced a bipartite version of Ramsey numbers. For two bipartite graphs F and H, the *bipartite Ramsey number* $BR(F, H)$ of F and H is the smallest positive integer r such that every red-blue coloring of the r-regular complete bipartite graph $K_{r,r}$ results in either a red F or a blue H. Consequently, if $BR(F, H) = r$ for bipartite graphs F and H, then every red-blue coloring of $K_{r,r}$ results in a red F or a blue H, while there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red F nor a blue H. Beineke and Schwenk [2] showed that $BR(F, H)$ exists for every two bipartite graphs F and H.

Thus, if $BR(F, H) = r$ for bipartite graphs F and H, then every red-blue coloring of $K_{r,r}$ results in a red F or a blue H and there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red F nor a blue H. In [1], red-blue colorings of the intermediate graph $K_{r-1,r}$ were considered, which led to the concept of the 2-Ramsey number. For bipartite graphs F and H, the *2-Ramsey number* $R_2(F, H)$ of F and H is the smallest positive integer n such that every red-blue coloring of the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ of order n results in a red F or a blue H.

In [3], red-blue colorings of complete bipartite graphs were considered where the numbers of vertices in the two partite sets need not differ by at most 1. Let F and H be two bipartite graphs. For a positive integer s, the *s-bipartite Ramsey number* $BR_s(F, H)$ of F and H is the smallest integer t with $t \geq s$ such that every red-blue coloring of $K_{s,t}$ results in a red F or a blue H. In [3, 4, 6, 7, 10], the numbers $BR_s(F, H)$ were studied when $F, H \in \{K_{2,2}, K_{2,3}, K_{3,3}\}$. In [5, 10], the s-bipartite Ramsey numbers of small paths (those of order 8 or less) are determined and s-bipartite Ramsey numbers are investigated for paths versus stars. In this work, we determine $BR_s(F, H)$ for all positive integers s when F and H are stars, matchings or one is a star and the other is a matching as well as when $F = H$ is a double star. We refer to the book [8] for graph theory notation and terminology not described in this paper.

2 Stars and Matchings

In this section, we determine the s-bipartite Ramsey numbers $BR_s(F, H)$ of bipartite graphs F and H for all positive integers s where each of F and H is either a star or a matching (also referred to as *stripes*). We beginning with the case when F and H are both stars. For an integer $n \geq 2$, a *star of size n* is denoted by $K_{1,n}$.

Theorem 2.1 *For integers $m, n, s \geq 2$,*

$$BR_s(K_{1,m}, K_{1,n}) = \begin{cases} m + n - 1 & \text{if } 2 \leq s \leq m + n - 2 \\ s & \text{otherwise.} \end{cases}$$

Proof. We consider two cases, according to whether $2 \leq s \leq m + n - 2$ or $s \geq m + n - 1$.

Case 1. $2 \le s \le m+n-2$. Let there be given a red-blue coloring of $H = K_{s,m+n-1}$ resulting in the red subgraph H_R and the blue subgraph H_B. Let U and W be the partite sets of H with $|U| = s$ and $|W| = m+n-1$. Now let $u \in U$. If $\deg_{H_R} u \ge m$, then H contains a red $K_{1,m}$; while if $\deg_{H_R} u \le m-1$, then $\deg_{H_B} u \ge (m+n-1)-(m-1) = n$ and so H contains a blue $K_{1,n}$. Therefore, $BR_s(K_{1,m}, K_{1,n}) \le m-n+1$.

Next, we show that there exists a red-blue coloring of $G = K_{s,m+n-2}$ that avoids both a red $K_{1,m}$ and a blue $K_{1,n}$. Let $U = \{u_1, u_2, \dots, u_s\}$ and $W = \{w_1, w_2, \dots, w_{m+n-2}\}$ be the partite sets of G. For each integer i with $1 \le i \le s$, assign the color red to the $m-1$ edges $u_iw_i, u_iw_{i+1}, \dots, u_iw_{i+m-2}$ incident with u_i, where the subscripts of vertices are expressed as integers modulo $m+n-2$, and assign the color blue to the remaining edges of G. Let G_R and G_B be the resulting red and blue subgraphs of G, respectively. If $u \in U$, then $\deg_{G_R} u = m-1$ and $\deg_{G_B} u = n-1$. Thus, this red-blue coloring of G produces neither a red $K_{1,m}$ nor a blue $K_{1,n}$ whose central vertex belongs to U. By this construction,

$$\max\{\deg_{G_R} w : w \in W\} = \deg_{G_R} w_s \le m-1 \tag{1}$$

$$\min\{\deg_{G_R} w : w \in W\} = \deg_{G_R} w_{m+n-2} \ge 0. \tag{2}$$

Hence, $0 \le \delta(G_R) \le \Delta(G_R) \le m-1$ and so there is no red $K_{1,m}$ whose central vertex belongs to W. Let $\deg_{G_R} w_{m+n-2} = k$. Since $\deg_{G_R} w + \deg_{G_B} w = s$ for each $w \in W$, it follows that

$$\deg_{G_B} w_{m+n-2} = s - \deg_{G_R} w_{m+n-2} = s-k. \tag{3}$$

First, suppose that $k \ge 1$. Since

$$N_{G_R}(u_{s-k+1}) = \{w_{s-k+1}, w_{s-k+2}, \dots, w_{m+n-2}\}$$

and $\deg_{G_R} u_{s-k+1} = m-1$, it follows that $(m+n-2)-(s-k+1)+1 = m-1$ and so $s = n+k-1$. It then follows by (3) that $\deg_{G_B} w_{m+n-2} = n-1$ and so $\deg_{G_B} w \le n-1$ for each $w \in W$ by (2). Next, suppose that $k = 0$. Since u_s is adjacent to $w_s, w_{s+1}, \dots, w_{s+(m-2)}$, it follows that $s+(m-2) < m+n-2$ and so $s \le n-1$. Hence, $\deg_{G_B} w \le \deg_G w = s \le n-1$ for each $w \in W$. Therefore, there is no blue $K_{1,n}$ whose central vertex belongs to W.

Hence, there is neither a red $K_{1,m}$ nor a blue $K_{1,n}$ in G. Therefore, $BR_s(K_{1,m}, K_{1,n}) \ge m-n+1$ and so $BR_s(K_{1,m}, K_{1,n}) = m-n+1$ when $2 \le s \le m+n-2$.

Case 2. $s \ge m+n-1$. We show that every red-blue coloring of $H = K_{s,s}$ produces either a red $K_{1,m}$ or a blue $K_{1,n}$. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let U and W be the partite sets of H with $|U| = |W| = s$. Let v be any vertex of H, say $v \in U$. If $\deg_{H_R} v \ge m$, then H contains a red $K_{1,m}$; while if $\deg_{H_R} v \le m-1$, then

$$\deg_{H_B} v \geq s-(m-1) \geq (m+n-1)-(m-1)=n$$

and so H contains a blue $K_{1,n}$. Therefore, $BR_s(K_{1,m}, K_{1,n}) = s$. ∎

Next, we determine the s-bipartite Ramsey numbers $BR_s(F, H)$ when F and H are both matchings. For an integer $n \geq 2$, a *matching of size n* is denoted by nK_2, which consists of n independent edges. For two disjoint sets X and Y of vertices of a graph G, the set of edges joining a vertex of X and a vertex of Y in G is denoted by $G[X, Y]$ or, more simply, by $[X, Y]$ if the graph G under discussion is clear.

Theorem 2.2 *For integers $m, n, s \geq 2$,*

$$BR_s(mK_2, nK_2) = \begin{cases} \text{does not exist} & \text{if } 2 \leq s \leq m+n-2 \\ s & \text{otherwise.} \end{cases}$$

Proof. First, suppose that $2 \leq s \leq m+n-2$. Let t be an integer where $t \geq s$. We show that there is a red-blue coloring of $G = K_{s,t}$ that produces neither a red mK_2 nor a blue nK_2. If $s \leq m-1$, then assign the color red to each edge of G. This produces a red-blue coloring that avoids both a red mK_2 and a blue nK_2. Thus, we may assume that $s \geq m$. Let U and W be the partite sets of G with $|U| = s$ and $|W| = t$. Now partition the set U into two subsets U_1 and U_2 where $|U_1| = m-1$ and $|U_2| = s-m+1$. Assign the color red to each edge in $[U_1, W]$ and the color blue to each edge in $[U_2, W]$. This red-blue coloring results in the red subgraph $G_R = K_{m-1,t}$ and the blue subgraph $G_B = K_{s-m+1,t} \subseteq K_{n-1,t}$ (since $s-m+1 \leq (m+n-2)-m+1 = n-1$). Hence, there is neither a red mK_2 nor a blue nK_2 and so $BR_s(mK_2, nK_2)$ does not exist.

Next, suppose that $s \geq m+n-1$. We show that every red-blue coloring of $H = K_{s,s}$ produces either a red mK_2 or a blue nK_2. Let there be given a red-blue coloring of H and let M be a perfect matching in H. Thus, $|M| = s$. If there are m edges in M that are colored red, then there is a red mK_2; otherwise, at most $m-1$ edges in M are red and so at least $s-(m-1) \geq (m+n-1)-(m-1) = n$ edges in M are blue, producing a blue nK_2. Therefore, $BR_s(mK_2, nK_2) = s$. ∎

Bipartite Ramsey numbers $BR(F, H)$ when one of F and H is a star and the other is a matching were studied in [9] and the following was obtained.

Theorem 2.3 [9] *For integers $m, n \geq 2$, $BR(K_{1,m}, nK_2) = m + \lfloor \frac{n-1}{2} \rfloor$.*

We now determine $BR_s(F, H)$ when one of F and H is a star and the other is a matching, beginning with conditions under which these numbers do not exist.

Proposition 2.4 *For integers $m, n, s \geq 2$, if $s \leq n-1$ or $s \leq m-1 \leq n-1$, then $BR_s(K_{1,m}, nK_2)$ does not exist.*

Proof. Suppose that $2 \le s \le n-1$. For an arbitrary integer t, the red-blue coloring of $K_{s,t}$ that assigns the color blue to each edge of $K_{s,t}$ produces neither a red $K_{1,m}$ nor a blue nK_2. Therefore, $BR_s(K_{1,m}, nK_2)$ does not exist when $s \le n-1$ as well as when $s \le m-1 \le n-1$. ∎

Under any other conditions, the numbers $BR_s(K_{1,m}, nK_2)$ always exist.

Proposition 2.5 *If m, n, s are integers with $2 \le n \le s \le m-1$, then*

$$BR_s(K_{1,m}, nK_2) = m + n - 1.$$

Proof. First, we show that $BR_s(K_{1,m}, nK_2) \ge m+n-1$; that is, there is a red-blue coloring of $G = K_{s,m+n-2}$ that produces neither a red $K_{1,m}$ nor a blue nK_2. Let U and W be the partite sets of G with $|U| = s$ and $|W| = m+n-2$. Partition the partite set W into two subsets W_1 and W_2 with $|W_1| = m-1$ and $|W_2| = n-1$. Define a red-blue coloring of G by assigning the color red to each edge in $[U, W_1]$ and the color blue to each edge in $[U, W_2]$. Then the red subgraph is $G_R = K_{s,m-1}$ and the blue subgraph is $G_B = K_{s,n-1}$. Since $s \le m-1$ and the maximum matching in G_B has size $n-1$, there is neither a red $K_{1,m}$ in G_R nor a blue nK_2 in G_B. Therefore, $BR_s(K_{1,m}, nK_2) \ge m+n-1$.

To verify that $BR_s(K_{1,m}, nK_2) \le m+n-1$, we show every red-blue coloring of $H = K_{s,m+n-1}$ results in a red $K_{1,m}$ or a blue nK_2. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_{m+n-1}\}$ be the partite sets of H. Let M be a maximum matching in H_B. If $|M| \ge n$, then we obtain a blue nK_2. So we may assume that $|M| \le n-1$. Suppose that $M = \{u_1w_1, u_2w_2, \ldots, u_{|M|}w_{|M|}\}$. Let $U_1 = \{u_1, u_2, \ldots, u_{|M|}\}$ and $W_1 = \{w_1, w_2, \ldots, w_{|M|}\}$. Now, let $U_2 = U - U_1$ and $W_2 = W - W_1$. If there is a blue edge in $[U_2, W_2]$, then we obtain a matching by adding this blue edge to M, which contradicts the maximality of M. Hence, we may assume that $H[U_2, W_2] = K_{s-|M|,\, m+n-1-|M|} \subseteq H_R$. Since $|M| \le n-1$, it follows that $m+n-1-|M| \ge m+n-1-(n-1) = m$. So there is a red $K_{1,m}$ in H. Thus, every red-blue coloring of $K_{s,m+n-1}$ results in a red $K_{1,m}$ or a blue nK_2 and so $BR_s(K_{1,m}, nK_2) \le m+n-1$. Therefore, $BR_s(K_{1,m}, nK_2) = m+n-1$. ∎

Proposition 2.6 *If m, n, s are integers with $m, n \ge 2$ and $s \ge m + \lfloor \frac{n-1}{2} \rfloor$, then $BR_s(K_{1,m}, nK_2) = s$.*

Proof. By the definition of s-bipartite Ramsey number,

$$BR_s(K_{1,m}, nK_2) \ge s.$$

Hence, we need only show that $BR_s(K_{1,m}, nK_2) \le s$, that is, every red-blue coloring of $H = K_{s,s}$ results in a red $K_{1,m}$ or a blue nK_2. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of H. Let M be a maximum matching in H_B. If $|M| \geq n$, then we obtain a blue nK_2. If $|M| \leq \lfloor \frac{n-1}{2} \rfloor$, then we may assume that $M = \{u_1w_1, u_2w_2, \ldots, u_{|M|}w_{|M|}\}$. Let $U_1 = \{u_1, u_2, \ldots, u_{|M|}\}$ and $W_1 = \{w_1, w_2, \ldots, w_{|M|}\}$. Now, let $U_2 = U - U_1$ and $W_2 = W - W_1$. If there is a blue edge in $[U_2, W_2]$, then we obtain a matching by adding this blue edge to M, which contradicts the maximality of M. Hence, we may assume that $H[U_2, W_2] = K_{s-|M|,\, s-|M|} \subseteq H_R$. Since $|M| \leq \lfloor \frac{n-1}{2} \rfloor$ and $s \geq m + \lfloor \frac{n-1}{2} \rfloor$, it follows that $s - |M| \geq m + \lfloor \frac{n-1}{2} \rfloor - \lfloor \frac{n-1}{2} \rfloor = m$. So there is a red $K_{1,m}$ in H. Thus, we may assume that $\lfloor \frac{n-1}{2} \rfloor + 1 \leq |M| \leq n - 1$. For each vertex $w \in W_2$,

$$\deg_{H_R} w \geq s - |M| \geq m + \lfloor \tfrac{n-1}{2} \rfloor - |M| = m - 1 - (|M| - \lfloor \tfrac{n-1}{2} \rfloor - 1).$$

If w is joined to at least $|M| - \lfloor \frac{n-1}{2} \rfloor$ vertices in U_1 by red edges, then there is a red $K_{1,m}$ in H. Otherwise, each vertex in W_2 is joined to at most $|M| - \lfloor \frac{n-1}{2} \rfloor - 1$ vertices in U_1 by red edges; so each vertex in W_2 is joined to at least $\lfloor \frac{n-1}{2} \rfloor + 1$ vertices in U_1 by blue edges. Assume, without loss of generality, that $u_iw_{|M|+1}$ is blue for each i with $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor + 1$. If there is an integer j with $1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor + 1$ such that $u_{|M|+1}w_j$ is blue, say $u_{|M|+1}w_1$ is blue, then there is a matching

$$M' = \{u_{|M|+1}w_1, u_1w_{|M|+1}\} \cup \{u_iw_i : 2 \leq i \leq |M|\}$$

whose size is larger than M, a contradiction. Hence, $u_{|M|+1}w_j$ is red for all j with $1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor + 1$. This implies that

$$\begin{aligned}
\deg_{H_R} u_{|M|+1} &\geq s - |M| + \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \\
&\geq (m + \left\lfloor \frac{n-1}{2} \right\rfloor) - (n-1) + \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \\
&= m - n + 2 + 2\left\lfloor \frac{n-1}{2} \right\rfloor \geq m - n + 2 + (n-2) = m.
\end{aligned}$$

Thus, there is a red $K_{1,m}$ whose central vertex is $u_{|M|+1}$ in H. Consequently, every red-blue coloring of $K_{s,s}$ results in a red $K_{1,m}$ or a blue nK_2 and so $BR_s(K_{1,m}, nK_2) \leq s$. Therefore, $BR_s(K_{1,m}, nK_2) = s$. ∎

For two vertex-disjoint graphs G and H, let $G + H$ denote the *union* of G and H.

Theorem 2.7 *If m, n, s are integers with $3 \leq n < m \leq s \leq m + \lfloor \frac{n-1}{2} \rfloor - 1$, then*

$$BR_s(K_{1,m}, nK_2) = 2(m-1) + n - s.$$

Proof. Since $m \leq s \leq m + \lfloor \frac{n-1}{2} \rfloor - 1$, we can write $s = m + j$ for some integer j with $0 \leq j \leq \lfloor \frac{n-1}{2} \rfloor - 1$. Then $2(m-1) + n - s = m + n - 2 - j$.

First, we show that $BR_s(K_{1,m}, nK_2) \geq m + n - 2 - j$; that is, we show that there is a red-blue coloring of $G = K_{s,m+n-3-j}$ that produces neither a red $K_{1,m}$ nor a blue nK_2. Let U and W be the partite sets of G with $|U| = s = m + j$ and $|W| = m + n - 3 - j$. Partition the partite set U into three subsets U_1, U_2 and U_3 and the partite set W into three subsets W_1, W_2 and W_3, where

$$\begin{aligned}
&|U_1| = |W_1| = n - 1 - (j+1) = n - j - 2 \\
&|U_2| = |W_2| = j + 1 \\
&|U_3| = s - (n-1) = m + j - (n-1) = m + j - n + 1 \\
&|W_3| = m + n - 3 - j - (n-1) = m - j - 2.
\end{aligned}$$

Define a red-blue coloring of G by assigning the color blue to each edge in the set $[U_1 \cup U_3, W_1] \cup [U_2, W_2 \cup W_3]$ and the color red to the remaining edges of G. Let G_B and G_R be the resulting blue and red subgraphs of G. Observe that

$$\begin{aligned}
G_B &= G[U_1 \cup U_3, W_1] + G[U_2, W_2 \cup W_3] \\
&= K_{n-1-(j+1),\ m-1} + K_{j+1,\ m-1} \\
G_R &= G[U_1 \cup U_3, W_2 \cup W_3] + G[U_2, W_1] \\
&= K_{m-1,\ m-1} + K_{n-1-(j+1),\ j+1}.
\end{aligned}$$

Since there is neither a red $K_{1,m}$ in G_R nor a blue nK_2 in G_B, it follows that

$$BR_s(K_{1,m}, nK_2) \geq m + n - 2 - j.$$

To verify that $BR_s(K_{1,m}, nK_2) \leq m + n - 2 - j$, we show that every red-blue coloring of $H = K_{s,m+n-2-j}$ results in a red $K_{1,m}$ or a blue nK_2. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s = u_{m+j}\}$ and $W = \{w_1, w_2, \ldots, w_{m+n-2-j}\}$ be the partite sets of H. Let M be a maximum matching in H_B. If $|M| \geq n$, then we obtain a blue nK_2. If $|M| \leq n-j-2$, then we may assume that $M = \{u_1w_1, u_2w_2, \ldots, u_{|M|}w_{|M|}\}$. Let

$$U_1 = \{u_1, u_2, \ldots, u_{|M|}\} \text{ and } W_1 = \{w_1, w_2, \ldots, w_{|M|}\}.$$

Now, let $U_2 = U - U_1$ and $W_2 = W - W_1$. If there is a blue edge in $[U_2, W_2]$, then we obtain a matching by adding this blue edge to M, which contradicts the maximality of M. Hence, we may assume that

$$H[U_2, W_2] = K_{s-|M|,\ m+n-2-j-|M|} \subseteq H_R.$$

Since $|M| \leq n - j - 2$ and $j \leq \lfloor \frac{n-1}{2} \rfloor - 1$, it follows that

$$m + n - 2 - j - |M| \geq m + n - 2 - j - (n - j - 2) = m.$$

So there is a red $K_{1,m}$ in H. Thus, we may assume that $n-j-1 \leq |M| \leq n-1$. For each vertex $u \in U_2$, it follows that $\deg_{H_R} = m+n-2-j-|M|$. If u is joined to at least $|M| - (n-j-1) + 1$ vertices in W_1 by red edges, then there is a red $K_{1,m}$ in H. Thus, each vertex in U_2 is joined to at most $|M| - (n-j-1)$ vertices in W_1 by red edges; so each vertex in U_2 is joined to at least $n - j - 1$ vertices in W_1 by blue edges. Assume, without loss of generality, that $u_{|M|+1}w_i$ is blue for each i with $1 \leq i \leq n - j - 1$. If there is an integer i with $1 \leq i \leq n-j-1$ such that $w_{|M|+1}u_i$ is blue, say $w_{|M|+1}u_1$ is blue, then there is a matching

$$M' = \{w_{|M|+1}u_1, u_{|M|+1}w_1\} \cup \{u_iw_i : 2 \leq i \leq |M|\}$$

whose size is larger than $|M|$, a contradiction. Hence, $w_{|M|+1}u_i$ is red for all i with $1 \leq i \leq n - j - 1$. This implies that

$$\begin{aligned} \deg_{H_R} w_{|M|+1} &\geq m + j - |M| + n - j - 1 = m + n - 1 - |M| \\ &\geq m + n - 1 - (n-1) = m. \end{aligned}$$

So there is a red $K_{1,m}$ whose central vertex is $w_{|M|+1}$ in H. Thus,

$$BR_s(K_{1,m}, nK_2) \leq m + n - 2 - j,$$

and therefore, $BR_s(K_{1,m}, nK_2) = m + n - 2 - j$. ■

Theorem 2.8 *Let n and m be integers with $n \geq m \geq 3$. If s is an integer with $n \leq s \leq m + \lfloor \frac{n-1}{2} \rfloor - 1$, then*

$$BR_s(K_{1,m}, nK_2) = \begin{cases} 2(m-1) + n - s & \text{if } m \leq n \leq 2m-3 \\ s & \text{if } n \geq 2m-2. \end{cases}$$

Proof. We consider two cases, according to whether $m \leq n \leq 2m - 3$ or $n \geq 2m - 2$.

Case 1. $m \leq n \leq 2m-3$. First, observe that since $m+\lfloor \frac{n-1}{2} \rfloor -1 \geq s$, it follows that $2(m-1)+n-s \geq s+1$. Suppose that $s = n+j$ for some integer j with $0 \leq j \leq m+\lfloor \frac{n-1}{2} \rfloor -1-n$. We show that $BR_s(K_{1,m}, nK_2) = 2m-2-j$. First, we show that $BR_s(K_{1,m}, nK_2) \geq 2m - 2 - j$; that is, we show that there is a red-blue coloring of $G = K_{s,2m-3-j}$ resulting in neither a red $K_{1,m}$ nor a blue nK_2. Let U and W be the partite sets of G with $|U| = s = n+j$

and $|W| = 2m-3-j$. Partition the partite set U into three subsets U_1, U_2 and U_3 and the partite set W into three subsets W_1, W_2 and W_3, where

$$|U_1| = |W_1| = |W_3| = m-j-2$$
$$|U_2| = j+1+n-m$$
$$|U_3| = |W_2| = j+1.$$

Define a red-blue coloring of G by assigning the color blue to each edge in the set $[U_1 \cup U_3, W_1] \cup [U_2, W_2 \cup W_3]$ and the color red to the remaining edges of G. Let G_B and G_R be the resulting blue and red subgraphs of G. Observe that

$$\begin{aligned} G_B &= G[U_1 \cup U_3, W_1] + G[U_2, W_2 \cup W_3] \\ &= K_{m-j-2,\ m-1} + K_{j+1+n-m,\ m-1} \\ G_R &= G[U_1 \cup U_3, W_2 \cup W_3] + G[U_2, W_1] \\ &= K_{m-1,\ m-1} + K_{m-j-2,\ j+1+n-m}. \end{aligned}$$

Since $j \le m + \lfloor \frac{n-1}{2} \rfloor - 1 - n$ and $n \le 2m-3$, it follows that

$$\begin{aligned} j+1+n-m &\le m + \left\lfloor \frac{n-1}{2} \right\rfloor - 1 - n + 1 + n - m \\ &= \left\lfloor \frac{n-1}{2} \right\rfloor \le \left\lfloor \frac{2m-4}{2} \right\rfloor = \lfloor m-2 \rfloor < m. \end{aligned}$$

Thus, there is neither a red $K_{1,m}$ in G_R nor a blue nK_2 in G_B. Therefore,

$$BR_s(K_{1,m}, nK_2) \ge 2m-2-j.$$

To verify that $BR_s(K_{1,m}, nK_2) \le 2m-2-j$, we show that every red-blue coloring of $H = K_{s,2m-2-j}$ results in a red $K_{1,m}$ or a blue nK_2. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s = u_{n+j}\}$ and $W = \{w_1, w_2, \ldots, w_{2m-2-j}\}$ be the partite sets of H. Let M be a maximum matching in H_B. If $|M| \ge n$, then we obtain a blue nK_2. If $|M| \le j+n-m$, then we may assume that $M = \{u_1w_1, u_2w_2, \ldots, u_{|M|}w_{|M|}\}$. Let $U_1 = \{u_1, u_2, \ldots, u_{|M|}\}$ and $W_1 = \{w_1, w_2, \ldots, w_{|M|}\}$. Now, let $U_2 = U - U_1$ and $W_2 = W - W_1$. If there is a blue edge in $[U_2, W_2]$, then we obtain a matching by adding this blue edge to M, which contradicts the maximality of M. Hence, we may assume that $H[U_2, W_2] = K_{s-|M|,\ 2m-2-j-|M|} \subseteq H_R$. Since $|M| \le j+n-m$, it follows that

$$s - |M| = n+j-|M| \ge n+j-(j+n-m) = m.$$

So there is a red $K_{1,m}$ in H. Thus, we may assume that $j+1+n-m \le |M| \le n-1$. If there is $w \in W_2$ such that w is joined to at least

$|M|-(j+1+n-m)+1$ vertices in U_1 by red edges, then there is a red $K_{1,m}$ in H. Thus, each vertex in W_2 is joined to at most $|M|-(j+1+n-m)$ vertices in U_1 by red edges; so each vertex in W_2 is joined to at least $j+1+n-m$ vertices in U_1 by blue edges. Assume, without loss of generality, that $u_i w_{|M|+1}$ is blue for each i with $1 \le i \le j+1+n-m$. If there is an integer i with $1 \le i \le j+1+n-m$ such that $u_{|M|+1}w_i$ is blue, say $u_{|M|+1}w_1$ is blue, then there is a matching

$$M' = \{u_{|M|+1}w_1, u_1 w_{|M|+1}\} \cup \{u_i w_i : 2 \le i \le |M|\}$$

whose size is larger than $|M|$, a contradiction. Hence, $u_{|M|+1}w_i$ is red for all i with $1 \le i \le j+1+n-m$. This implies that

$$\begin{aligned} \deg_{H_R} u_{|M|+1} &\ge 2m-2-j-|M|+j+1+n-m \\ &= m-1-|M|+n \\ &\ge m-1-(n-1)+n = m. \end{aligned}$$

Thus, there is a red $K_{1,m}$ whose central vertex is $u_{|M|+1}$ in H. Hence, $BR_s(K_{1,m}, nK_2) \le 2m-2-j$. Therefore, if $n \le s \le m + \lfloor \frac{n-1}{2} \rfloor - 1$ and $s = n+j$, then $BR_s(K_{1,m}, nK_2) = 2m-2-j$.

Case 2. $n \ge 2m-2$. Since $BR_s(K_{1,m}, nK_2) \ge s$, we need only show that $BR_s(K_{1,m}, nK_2) \le s$, that is, every red-blue coloring of $H = K_{s,s}$ results in a red $K_{1,m}$ or a blue nK_2. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of H. Let M be a maximum matching in H_B. If $|M| \ge n$, then we obtain a blue nK_2. If $|M| \le m-2$, then we may assume that $M = \{u_1w_1, u_2w_2, \ldots, u_{|M|}w_{|M|}\}$. Let $U_1 = \{u_1, u_2, \ldots, u_{|M|}\}$ and $W_1 = \{w_1, w_2, \ldots, w_{|M|}\}$. Now, let $U_2 = U - U_1$ and $W_2 = W - W_1$. If there is a blue edge in $[U_2, W_2]$, then we obtain a matching by adding this blue edge to M, which contradicts the maximality of M. Hence, we may assume that

$$H[U_2, W_2] = K_{s-|M|,\ s-|M|} \subseteq H_R.$$

Since $|M| \le m-2$ and $s \ge n \ge 2m-2$, it follows that

$$s - |M| \ge 2m-2-(m-2) = m.$$

So there is a red $K_{1,m}$ in H. Thus, we may assume that $m-1 \le |M| \le n-1$. For each vertex $w \in W_2$, it follows that

$$\deg_{H_R} w \ge s - |M| \ge n - |M| \ge 2m-2-|M|.$$

If w is joined to at least $|M|-m+2$ vertices in U_1 by red edges, then there is a red $K_{1,m}$ in H. Thus, each vertex in W_2 is joined to at most $|M|-m+1$ vertices in U_1 by red edges; so each vertex in W_2 is joined to at least $m-1$ vertices in U_1 by blue edges. Assume, without loss of generality, that $u_i w_{|M|+1}$ is blue for each i with $1 \le i \le m-1$. If there is an integer i with $1 \le i \le m-1$ such that $u_{|M|+1} w_i$ is blue, say $u_{|M|+1} w_1$ is blue, then there is a matching

$$M' = \{u_{|M|+1} w_1, u_1 w_{|M|+1}\} \cup \{u_i w_i : 2 \le i \le |M|\}$$

whose size is larger than $|M|$, a contradiction. Hence, $u_{|M|+1} w_i$ is red for all i with $1 \le i \le m-1$. This implies that

$$\begin{aligned} \deg_{H_R} u_{|M|+1} &\ge s - |M| + m - 1 \ge n - |M| + m - 1 \\ &\ge n - (n-1) + m - 1 = m. \end{aligned}$$

Thus, there is a red $K_{1,m}$ whose central vertex is $u_{|M|+1}$ in H. Thus, every red-blue coloring of $K_{s,s}$ results in a red $K_{1,m}$ or a blue nK_2 and so $BR_s(K_{1,m}, nK_2) \le s$. Therefore, $BR_s(K_{1,m}, nK_2) = s$. ∎

The following result summarizes the values of $BR_s(F, H)$ for all positive integers s when F is a star and H is a matching.

Theorem 2.9 *Let m, n and s be integers with $m, n, s \ge 2$.*

1. *If $s \le n-1$ or $s \le m-1 \le n-1$, then $BR_s(K_{1,m}, nK_2)$ does not exist.*
2. *If $n \le s \le m-1$, then $BR_s(K_{1,m}, nK_2) = m + n - 1$.*
3. *If (i) $3 \le n < m \le s \le m + \lfloor \frac{n-1}{2} \rfloor - 1$ or (ii) $n \le s \le m + \lfloor \frac{n-1}{2} \rfloor - 1$ and $3 \le m \le n \le 2m-3$, then $BR_s(K_{1,m}, nK_2) = 2(m-1) + n - s$.*
4. *If (i) $s \ge m + \lfloor \frac{n-1}{2} \rfloor$ or (ii) $m \ge 3$ and $2m-2 \le n \le s \le m + \lfloor \frac{n-1}{2} \rfloor - 1$, then $BR_s(K_{1,m}, nK_2) = s$.*

3 Double Stars

For integers $a, b \ge 2$ where $a \le b$, let $S_{a,b}$ be the double star whose central vertices have degrees a and b. In this section, we determine the values of $BR_s(F, H)$ for all positive integers s when $F = H$ is a double star. In this case, we write $BR_s(S_{a,b}, S_{a,b})$ as $BR_s(S_{a,b})$.

Proposition 3.1 *Let a, b, s be integers with $a, b, s \ge 2$ and $a \le b$.*

If $s \le 2a-2$, then $BR_s(S_{a,b})$ does not exist.

Proof. For an integer t where $t \geq 2a-2$, the red-blue coloring of $K_{2a-2,t}$, in which both red and blue subgraphs are $K_{a-1,t}$, produces no monochromatic $S_{a,b}$. Since $K_{s,t} \subseteq K_{2a-2,t}$ for each integer s with $2 \leq s \leq 2a-2$, there is a red-blue coloring of $K_{s,t}$ that avoids a monochromatic $S_{a,b}$. Therefore, $BR_s(S_{a,b})$ does not exist. ■

We now show that $BR_s(S_{a,b})$ exists otherwise, beginning with the case where $2a-1 \leq s \leq 2b-2$.

Theorem 3.2 *Let a, b and s be integers with $2 \leq a \leq b$.*

$$\text{If } 2a-1 \leq s \leq 2b-2, \text{ then } BR_s(S_{a,b}) = 2b-1.$$

Proof. First, we show that $BR_s(S_{a,b}) \geq 2b-1$; that is, we show that there is a red-blue coloring of $G = K_{s,2b-2}$ that produces no monochromatic $S_{a,b}$. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_{2b-2}\}$ be the partite sets of G. Partition the set U into two subsets U_1 and U_2 with $|U_1| = \lceil s/2 \rceil$ and $|U_2| = \lfloor s/2 \rfloor$ and partition the set W into two subsets W_1 and W_2 with $|W_1| = |W_2| = b-1$. Define a red-blue coloring of G by assigning the color blue to each edge in $[U_1, W_1] \cup [U_2, W_2]$ and the color red to each edge in $[U_1, W_2] \cup [U_2, W_1]$. Let G_R and G_B be the resulting red and blue subgraphs of G, respectively. For each vertex x of G, it follows that $\deg_{G_R} x \leq b-1$ and $\deg_{G_B} x \leq b-1$. Therefore, there is no monochromatic $S_{a,b}$ in G and so $BR_s(S_{a,b}) \geq 2b-1$.

To show that $BR_s(S_{a,b}) \leq 2b-1$, we proceed by induction on $a \geq 2$. First, suppose that $a = 2$ and so $3 \leq s \leq 2b-2$. Let there be given a red-blue coloring of $H = K_{s,2b-1}$ resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_{2b-1}\}$ be the partite sets of H. Since u_1 is incident with $2b-1$ edges, at least b edges are colored the same, say u_1w_i is red for $1 \leq i \leq b$. Let $S = U - \{u_1\}$ and $T = \{w_1, w_2, \ldots, w_b\}$. If there is a red edge in $[S, T]$, then there is a red $S_{2,b}$; otherwise, $H[S,T]$ is a blue $K_{s-1,b}$. Since $s-1 \geq 2$, it follows that $H[S,T]$ contains a blue $S_{2,b}$. Hence, there is a monochromatic $S_{2,b}$ in H. Therefore, the statement is true when $a = 2$.

Next, suppose that the inequality $BR_s(S_{a,b}) \leq 2b-1$ holds for an integer $a-1 \geq 2$. Thus, for every integer c with $c \geq a-1$ and every integer s with $2a-3 \leq s \leq 2c-2$, it follows that $BR_s(S_{a-1,c}) \leq 2c-1$. We show next that the inequality holds for a. So, let b and s be integers such that $b \geq a$ and $2a-1 \leq s \leq 2b-2$. We show that $BR_s(S_{a,b}) \leq 2b-1$. Since $b \geq a$, it follows that $b \geq a-1$. Because $2a-1 \leq s \leq 2b-1$, it follows that $2a-3 \leq s \leq 2b-2$. Hence, $BR_s(S_{a-1,b}) \leq 2b-1$. Consequently, every red-blue coloring of $K_{s,2b-1}$ results in a monochromatic $S_{a-1,b}$. We show that every such coloring also results in monochromatic $S_{a,b}$.

Let there be given a red-blue coloring of $H = K_{s,2b-1}$ resulting in the red subgraph H_R and the blue subgraph H_B. Assume, to the contrary, that there is no monochromatic $S_{a,b}$. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_{2b-1}\}$ be the partite sets of H. Since $2a-1 \leq s \leq 2b-2$ and so $2a-3 \leq s \leq 2b-2$, it follows by the induction hypothesis that H contains a monochromatic $F = S_{a-1,b}$. We may assume, without loss of generality, that F is a blue $S_{a-1,b}$ whose central vertices are u_1 and w_1 such that u_1 is adjacent w_i for $1 \leq i \leq b$ and w_1 is adjacent u_j for $1 \leq j \leq a-1$. Thus, $\deg_F u_1 = b$ and $\deg_F w_1 = a-1$. Let $U_1 = \{u_a, u_{a+1}, \ldots, u_s\}$.

- ⋆ If there is a blue edge in $[\{w_1\}, U_1]$, then there is a blue $S_{a,b}$, a contradiction. Thus, each edge in $[\{w_1\}, U_1]$ is red. Hence,

$$\deg_{H_R} w_1 = s - (a-1) \geq (2a-1) - (a-1) = a. \qquad (4)$$

- ⋆ If there is $u \in U_1$ such that $\deg_{H_R} u \geq b$, then there is a red $S_{a,b}$ whose central vertices are u and w_1, a contradiction. Thus, $\deg_{H_R} u \leq b-1$ for each $u \in U_1$ and so

$$\deg_{H_B} u \geq (2b-1) - (b-1) = b \text{ for each } u \in U_1. \qquad (5)$$

- ⋆ If there is $w \in W$ such that $\deg_{H_B} w \geq a$, then w must be adjacent to some vertex $u \in U_1$ (as $|U - U_1| = a-1$). Since $\deg_{H_B} u \geq b$ by (5), there is a blue $S_{a,b}$ whose central vertices are u and w, a contradiction. Thus,

$$\deg_{H_B} w \leq a-1 \text{ for each } w \in W. \qquad (6)$$

It then follows by (6) that the size m_{H_B} of H_B is at most $(a-1)(2b-1)$ and

$$\deg_{H_R} w \geq s - (a-1) \geq a \text{ for each } w \in W. \qquad (7)$$

- ⋆ If there is $u \in U$ such that $\deg_{H_R} u \geq b$, it then follows by (7) that there is a red $S_{a,b}$, a contradiction. Thus, $\deg_{H_R} u \leq b-1$ for each $u \in U$ and so

$$\deg_{H_B} u \geq (2b-1) - (b-1) = b \text{ for each } u \in U. \qquad (8)$$

It then follows by (8) that $m_{H_B} \geq sb$.

Therefore, $sb \leq m_{H_B} \leq (2b-1)(a-1)$. Since $s \geq 2a-1 > 2a-2$, it follows that $(2a-2)b < sb \leq (2b-1)(a-1)$ and so $a < 1$, which is impossible.

It then follows by the Principle of Mathematical Induction that there is a monochromatic $S_{a,b}$ in H and so $BR_s(S_{a,b}) \leq 2b-1$. Therefore, $BR_s(S_{a,b}) = 2b-1$ when $2a-1 \leq s \leq 2b-2$. ■

Theorem 3.3 *Let a and b be integers with $2 \le a \le b$.*

If s is an integer with $s \ge 2b-1$, then $BR_s(S_{a,b}) = s$.

Proof. Since $BR_s(S_{a,b}) \ge s$, we need only show that $BR_s(S_{a,b}) \le s$. We proceed by induction on $a \ge 2$ to show that every red-blue coloring of $H = K_{s,s}$ produces a monochromatic $S_{a,b}$ for integers a and b with $2 \le a \le b$ where $s \ge 2b-1$.

First, suppose that $a = 2$. We show that H contains a monochromatic $S_{2,b}$. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of H. Since u_1 is incident with $s \ge 2b-1$ edges, at least b edges are colored the same, say u_1w_i is red for $1 \le i \le b$. Let $S = U - \{u_1\}$ and $T = \{w_1, w_2, \ldots, w_b\}$. If there is a red edge in $[S,T]$, then there is a red $S_{2,b}$; for otherwise, $H[S,T]$ is a blue $K_{s-1,b}$. Since $s-1 \ge 2b-2 \ge 2$, it follows that $K_{2,b} \subseteq H[S,T]$ and so $H[S,T]$ contains a blue $S_{2,b}$. Hence, there is a monochromatic $S_{2,b}$ in H.

Next, assume for an integer $a-1 \ge 2$ that for all integers c and s with $c \ge a-1$ and $s \ge 2c-1$, we have $BR_s(S_{a,c}) \le s$. We now show for integers b and s with $b \ge a$ and $s \ge 2b-1$ that $BR_s(S_{a,b}) \le s$. Assume, to the contrary, that there exists a red-blue coloring of $H = K_{s,s}$ for which there is no monochromatic $S_{a,b}$. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of H. Since $b \ge a$, it follows that $b \ge a-1$ and so by the induction hypothesis there is a monochromatic $F = S_{a-1,b}$ in H. We may assume, without loss of generality, that F is a blue $S_{a-1,b}$ whose central vertices are u_1 and w_1 such that u_1 is adjacent w_i for $1 \le i \le b$ and w_1 is adjacent u_j for $1 \le j \le a-1$. Thus, $\deg_F u_1 = b$ and $\deg_F w_1 = a-1$. Let $U_1 = \{u_a, u_{a+1}, \ldots, u_s\}$.

- ⋆ If there is a blue edge in $[\{w_1\}, U_1]$, then there is a blue $S_{a,b}$, a contradiction. Thus, each edge in $[\{w_1\}, U_1]$ is red. Hence,

$$\deg_{H_R} w_1 = s-(a-1) \ge (2b-1)-(a-1) = 2b-a \ge b. \qquad (9)$$

- ⋆ If there is $u \in U_1$ such that $\deg_{H_R} u \ge a$, then there is a red $S_{a,b}$ whose central vertices are u and w_1, a contradiction. Thus, $\deg_{H_R} u \le a-1$ for each $u \in U_1$ and so

$$\deg_{H_B} u \ge s-(a-1) \ge b \text{ for each } u \in U_1. \qquad (10)$$

- ⋆ If there is $w \in W$ such that $\deg_{H_B} w \ge a$, then w must be adjacent to some vertex $u \in U_1$ (as $|U - U_1| = a-1$). Since $\deg_{H_B} u \ge b$ by (10), there is a blue $S_{a,b}$ whose central vertices are u and w, a contradiction. Thus, $\deg_{H_B} w \le a-1$ for each $w \in W$ and so

$$\deg_{H_R} w \ge s-(a-1) \ge b \text{ for each } w \in W. \qquad (11)$$

This implies that the size m_{H_R} of H_R is at least $s(s-a+1)$.

⋆ If there is $u \in U$ such that $\deg_{H_R} u \geq a$, it then follows by (11) that there is a red $S_{a,b}$, a contradiction. Thus, $\deg_{H_R} u \leq a-1$ for each $u \in U$ and so $m_{H_R} \leq s(a-1)$.

Therefore, $s(s-a+1) \leq m_{H_R} \leq s(a-1)$ or $s \leq 2a-2$. Since $s \geq 2b-1$, it follows that $b \leq a - 1/2 < a$, which is impossible.

It then follows by the Principle of Mathematical Induction that there is a monochromatic $S_{a,b}$ in H and so $BR_s(S_{a,b}) \leq s$. Therefore, $BR_s(S_{a,b}) = s$ when $s \geq 2b-1$. ■

In summary, we have the following theorem which provides the values of $BR_s(S_{a,b})$ for all integers $a, b, s \geq 2$.

Theorem 3.4 *Let a, b, s be integers with $a, b, s \geq 2$ and $a \leq b$.*

1. *If $s \leq 2a-2$, then $BR_s(S_{a,b})$ does not exist.*
2. *If $2a-1 \leq s \leq 2b-2$, then $BR_s(S_{a,b}) = 2b-1$.*
3. *If $s \geq 2b-1$, then $BR_s(S_{a,b}) = s$.*

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