PROPERTIES OF PARTIAL DOMINATING SETS OF GRAPHS

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ABSTRACT. A set $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ of G equals the minimum cardinality of a dominating set S in G; we say that such a set S is a γ*-set*. A generalization of this is partial domination which was introduced in 2017 by Case, Hedetniemi, Laskar, and Lipman [3, 2] . In *partial domination* a set S is a *p-dominating set* if it dominates a proportion p of the vertices in V. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality of a p-dominating set in G. In this paper, we investigate further properties of partial dominating sets, particularly ones related to graph products and locating partial dominating sets. We also introduce the concept of a *p-influencing set* as the union of all p-dominating sets for a fixed p and investigate some of its properties.

Keywords: partial domination, dominating set, partial domination number, domination number, influencing set, graph parameters, Vizing's conjecture

1. INTRODUCTION

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and *order* $n = |V|$. The *open neighborhood* of a vertex v is the set $N(v) := \{u \mid uv \in E\}$ of vertices u that are adjacent to v; the *closed neighborhood* of v is $N[v] := N(v) \cup \{v\}$. A set $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ is adjacent to at least one vertex in S, or equivalently, if $N[S] := \bigcup_{u \in S} N[u] = V$. The *domination number* $\gamma(G)$ of G equals the minimum cardinality of a dominating set S in G; we say that such a set S is a γ*-set*. Domination has been a well studied area for many years [1, 4, 7, 8, 9].

For any graph $G = (V, E)$ and proportion $p \in [0, 1]$, a set $S \subseteq V$ is a *p-dominating set* if

$$\frac{|N[S]|}{|V|} \geq p.$$

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The *p-domination number* $\gamma_p(G)$ equals the minimum cardinality of a p-dominating set in G. Partial domination was first introduced by Case, Hedetniemi, Laskar, and Lipman [3, 2] in 2017. Around the same time, in an independent work of Das the same concept was introduced [6, 5].

As noted in [3], a γ_p-set is not in general related to a γ-set. In particular, a γ-set does not necessarily contain a γ_p-set. Equivalently, a γ_p-set cannot necessarily be extended to a γ-set. To see this, it is helpful to revisit the subdivided star graph in Figure 1, where the γ-set denoted by triangles is disjoint from $\gamma_{1/2}$-set consisting of just the square vertex.

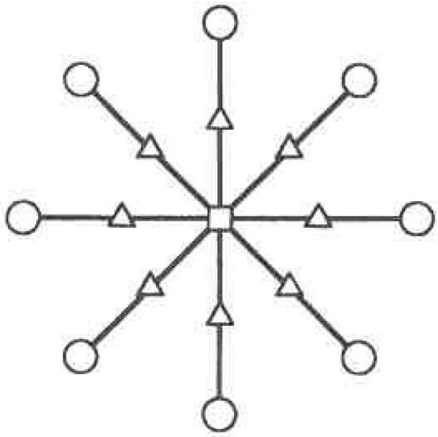

FIGURE 1. The γ-set denoted by triangles is disjoint from the $\gamma_{1/2}$-set consisting of just the square vertex.

Organization of the paper. In section 2 we generalize Vizing's conjecture about domination in graph products to the setting of partial domination and prove some special cases. In Section 3 we investigate some results related to finding a partial dominating set in a graph. In Section 4 we introduce p-influencing sets and consider some properties and examples.

2. Graph Products

We investigate properties of partial dominating sets related to graph products. In 1986, Vadim G. Vizing conjectured that for domination

$$\gamma(G\Box H) \geq \gamma(G)\gamma(H).$$

Here we conjecture that

$$\gamma_p(G\Box H) \geq \gamma_p(G)\gamma_p(H).$$

We will primarily be interested in the case when $p = 1/2$.

Proposition 2.1. *For paths P_m and P_n, with $n \geq m \geq 2$*

$$\gamma_{1/2}(P_m\Box P_n) \geq \gamma_{1/2}(P_m)\gamma_{1/2}(P_n).$$

Proof. The claim follows directly from Theorem 2.8 in [2], which says

(1) $\gamma_{1/2}(P_n) = \lceil n/6 \rceil$,
(2) for $m = 2$, $\gamma_{1/2}(P_2 \square P_n) = \lceil n/4 \rceil$,
(3) for $m \geq 3$, $\gamma_{1/2}(P_m \square P_n) = \lceil mn/10 \rceil$.

□

Note that if $\gamma_p(G) = \gamma_p(H) = 1$, then the conjecture holds trivially (e.g., if G and H are both complete graphs).

Proposition 2.2. *For complete graphs, K_m and K_n,*

$$\gamma_{1/2}(K_m \square K_n) = \left\lceil \frac{m + n - \sqrt{m^2 + n^2}}{2} \right\rceil .$$

Proof. We consider choosing vertices for a $\gamma_{1/2}$-set S. Say the first vertex added to S is $v_1 = (v, v')$. Note that v dominates exactly $m+n-1$ vertices. If $m + n - 1 \geq mn/2$, we are done. If not, we must add another vertex u to S. Note we essentially have three options here:

$$(1)\ v_2 = (v, u') \qquad (2)\ v_2 = (u, v') \qquad (3)\ v_2 = (u, u')$$

where $u \neq v$ and $u' \neq v'$. In the first case, v_2 dominates exactly $m - 1$ vertices not already dominated by S. In the second case v_2 dominates $n - 1$ such vertices, and in the last case v_2 dominates $m + n - 3$ such vertices.

Now in general, when adding the next vertex to S, it is helpful to consider which copies of K_m and K_n the vertex is coming from relative to the last vertex chosen. We have these three options: (1) choose the same K_m but a different K_n, (2) choose a different K_m but the same K_n, (3) choose a different K_m and K_n. In general option (3) is optimal, followed by option (1), then option (2). When option (3) is used every time in such a way that each vertex added to S comes from a copy of K_m and K_n not previously used, then the kth vertex dominates $m + n - 2k + 1$ vertices not already dominated by S. We claim this is the optimal strategy in choosing vertices for a $\gamma_{1/2}$-set. We refer to it as $*$

To see this, consider alternative choice of vertices, where we execute option (2) optimally ℓ times, and then execute option (3) optimally until we have a 1/2-dominating set. We refer to this as $**$. Now compare the number of vertices dominated by $*$ and $**$ at each step. There is no difference for the first vertex. For the second, $**$ is dominates $n - 2$ fewer than $*$. In general, for the kth vertex added in $**$ with $2 \leq k \leq \ell + 1$, $**$ dominates $(k - 1)(n - k)$ fewer vertices than $*$. So at step $\ell + 1$, $**$ dominates $\ell(n - \ell - 1)$ fewer vertices than $*$. Now for each vertex added after step $\ell + 1$, $**$ dominates ℓ more vertices than $*$. Thus, it will take $**$ $(n - \ell - 1) + 1 + \ell = n$ steps to dominate the same number of vertices as $*$. However, $*$ will have produced a 1/2-dominating set well before the nth step. Using a similar argument, we can show that any choice of vertices different than $*$ will be suboptimal.

Now the cardinality of the $\gamma_{1/2}$-set found by $*$ is

$$\min\{k \mid \sum_{i=1}^{k} (m+n-(2i-1)) \geq \frac{mn}{2}\}.$$

But note that $\sum_{i=1}^{k} (m+n-(2i-1)) = k(m+n)-k^2$. So the inequality to solve is

$$k(m+n)-k^2 \geq \frac{mn}{2}$$

which has minimum positive integer solution

$$k = \left\lceil \frac{m+n-\sqrt{m^2+n^2}}{2} \right\rceil.$$

□

Proposition 2.3. *For a path P_n and complete graph K_m,*

$$\gamma_{1/2}(P_n \square K_m) = \left\lceil \frac{mn}{2(m+2)} \right\rceil.$$

Proof. The maximum degree of a vertex in $P_n \square K_m$ is $m+2$. Consider the set of vertices S of maximum cardinality such that each vertex in S dominates exactly $m+2$ vertices, and no two vertices in S dominate the same vertex. The cardinality of S is $\lceil \frac{n-2}{2} \rceil$. Now write $n = 2k$ $(n = 2k-1)$ for even (odd) n and $k \in \mathbb{N}$. Note that in either case

$$\left\lceil \frac{n-2}{2} \right\rceil = k-1.$$

That is S has cardinality $k-1$.

Now suppose $(m+2)(k-1) \geq mn/2$. Then either S or a subset of S is a $\gamma_{1/2}$-set, and

$$\gamma_{1/2}(P_n \square K_m) = \left\lceil \frac{mn}{2(m+2)} \right\rceil.$$

On the other hand, suppose $(m+2)(k-1) < mn/2$. Then neither S nor any subset of S is a $\gamma_{1/2}$-set. However, we may add a vertex v to S so $S \cup \{v\}$ dominates at least k copies of K_m. That is, $S \cup \{v\}$ is a $\gamma_{1/2}$-set. Moreover, we have

$$k(m+2) > \frac{mn}{2} > (k-1)(m+2) \implies k = \left\lceil \frac{mn}{2(m+2)} \right\rceil.$$

□

Proposition 2.4. *Let G be a graph of order n,*

$$\gamma_{1/2}(G \square P_2) \geq \gamma_{1/2}(G).$$

Proof. Let S be a $\gamma_{1/2}(G\Box P_2)$-set. Now think of $G\Box P_2$ as two copies of G, say G_1 and G_2. Then since $|N[S]| \geq n$, we have WLOG that $|N[S] \cap G_1| \geq n/2$. Denote $S \cap G_i$ as S_i. Now consider the vertex set

$$S' = S_1 \cup \{v \mid v \in N[S_2] \cap G_1\}.$$

Then S' dominates at least half of G, and $|S'| \leq |S|$.

Thus, given a $\gamma_{1/2}(G\Box P_2)$-set, we can find a vertex set of G of size at most $\gamma_{1/2}(G\Box P_2)$ which dominates $1/2$ of G. So $\gamma_{1/2}(G\Box P_2) \geq \gamma_{1/2}(G)$. □

Proposition 2.5. *If* $\gamma_{1/2}(G) = 1$, *then*

$$\gamma_{1/2}(G\Box P_m) \geq \gamma_{1/2}(P_m).$$

Proof. Let $|V(G)| = n$. Since $\gamma_{1/2}(G) = 1$, there exists some $v \in V(G)$ such that $|N[v]| \geq \lceil \frac{n}{2} \rceil$. Now at best, $|N[v]| = n$, in which case, $\max\{|N[v]| : v \in V(G\Box P_m)\} = n + 2$. Thus we have

$$\gamma_{1/2}(G\Box P_m) \geq \left\lceil \frac{mn}{n+2} \right\rceil \geq \left\lceil \frac{m}{3} \right\rceil \geq \left\lceil \frac{m}{6} \right\rceil = \gamma_{1/2}(P_m).$$

□

Proposition 2.6. *If* $\gamma_{1/2}(G) = 2$, *then*

$$\gamma_{1/2}(G\Box P_m) \geq 2\gamma_{1/2}(P_m).$$

Proof. Let $|V(G)| = n$. Since $\gamma_{1/2}(G) = 2$, then $\max\{|N[v]| : v \in V(G)\} \leq \lceil \frac{n}{2} \rceil - 1$. Therefore, $\max\{|N[v]| : v \in V(G\Box P_m)\} \leq \lceil \frac{n}{2} \rceil + 1$.

Thus we have

$$\gamma_{1/2}(G\Box P_m) \geq \left\lceil \frac{mn}{2\left(\lceil \frac{n}{2} \rceil + 1\right)} \right\rceil \geq \left\lceil \frac{mn}{n+3} \right\rceil.$$

Now, since $\gamma_{1/2}(G) = 2$, we must have $n \geq 7$. Thus $\gamma_{1/2}(G\Box P_m) \geq \lceil 0.7m \rceil$.

Now say $m = 6a + b$. Then

$$\gamma_{1/2}(P_m) = \left\lceil \frac{m}{6} \right\rceil = \begin{cases} a & b = 0 \\ a+1 & b \neq 0 \end{cases}.$$

Thus for all $m \geq 2$,

$$\gamma_{1/2}(G\Box P_m) \geq \lceil 0.7m \rceil = \lceil 4.2a + 0.7b \rceil \geq 2(a+1) \geq 2\gamma_{1/2}(P_m).$$

□

Proposition 2.7. *If* $\gamma_{1/2}(G) = 3$, *then*

$$\gamma_{1/2}(G\Box P_m) \geq 3\gamma_{1/2}(P_m).$$

Proof. Let $|V(G)| = n$. Since $\gamma_{1/2}(G) = 3$, we must have $n \geq 13$. Moreover, we have $\max\{|N[v]| : v \in V(G)\} \leq \lceil \frac{n}{2} \rceil - 3$. Thus, $\max\{|N[v]| : v \in V(G\square P_m)\} \leq \lceil \frac{n}{2} \rceil - 1$.

Therefore we have

$$\gamma_{1/2}(G\square P_m) \geq \left\lceil \frac{mn}{2\left(\lceil \frac{n}{2} \rceil - 1\right)} \right\rceil \geq \left\lceil \frac{mn}{n-1} \right\rceil \geq m.$$

Now say $m = 6a + b$. Then

$$\gamma_{1/2}(P_m) = \left\lceil \frac{m}{6} \right\rceil = \begin{cases} a & b = 0 \\ a+1 & b \neq 0 \end{cases}.$$

Thus for $m \geq 3$,

$$\gamma_{1/2}(G\square P_m) \geq m = 6a + b \geq 3(a+1) \geq 3\gamma_{1/2}(P_m).$$

Lastly, when $m = 2$, we have

$$\gamma_{1/2}(G\square P_2) \geq \left\lceil \frac{n}{\lceil \frac{n}{2} \rceil - 1} \right\rceil = 3 = 3\gamma_{1/2}(P_2).$$

□

3. Locating Partial Dominating Sets in Graphs

When we look a graph we want some tools (theorems and/or algorithms) that will help us locate a partial dominating set or elements of a partial dominating set. A γ_p set does not have to be unique; and for many applications having anyone of them would work. The first intuitive idea in looking for a partial dominating set is to consider vertices with high degrees. The following several results will explore what can and cannot be achieved following this idea.

Lemma 3.1. *For any $p \in [0,1]$, if a vertex $v \in G$ has the highest degree, then there is a γ_p-set that contains at least one of the following:*

1) *v,*
2) *a neighbor of v, i.e. an element of $N(v)$,*
3) *a distance two neighbor of v, i.e. an element of $N(N[v])$.*

Proof. Suppose none of these vertices is in any γ_p set. Let $v \in G$ be a vertex of highest degree. If $S \subseteq G$ is a p-dominating set of G, consider any $s \in S$. The degree of s is less than or equal to the degree of v. So the set

$$S' = \{v\} \cup (S \setminus \{s\})$$

dominates at least as many vertices as S. Thus, S' is a p-dominating set that contains v. □

Now we illustrate with some examples that in the preceding theorem, it may be the case that 2) or 3) hold and not 1).

Example 3.2. Let $p = 8/9$ consider the graph in Figure 2. The set of boxed vertices is the only p–dominating set. This shows that the highest degree vertex may not be in any of the p-dominating sets of the graph but that instead some of its distance 2 neighbors are.

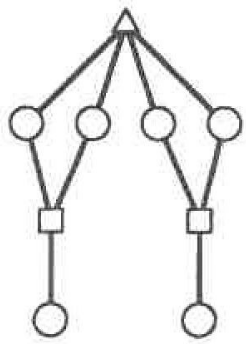

FIGURE 2. The highest degree vertex v (triangle) is not in any $\gamma_{8/9}$-set; rather the distance 2 neighbors of v (rectangles) form the only $\gamma_{8/9}$-set.

Example 3.3. Let $p = 7/9$ and consider the graph in Figure 3. The triangle vertex, v, is a maximum degree vertex in the graph. Any two vertices chosen from $N(v)$ form a $\gamma_{7/9}$-set. Thus it may be that a highest degree vertex is not in any p-dominating set, but that instead some its neighbors are.

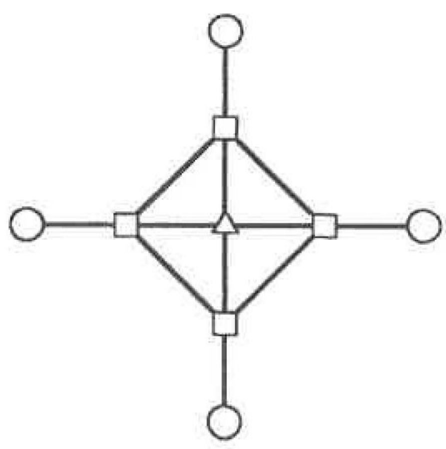

FIGURE 3. A highest degree vertex v (triangle) is not in any $\gamma_{7/9}$-set; rather from the neighbors of v, any two together form a $\gamma_{7/9}$-set.

Lemma 3.4. *Let $p \in [0, 1]$ be fixed. If $v \in V$ has highest degree and v is not in any p-dominating set, then $|S \cap N[N[v]]| \geq 2$ for every γ_p-set S.*

Proof. Prove by contradiction; suppose $|S \cap N[N[v]]| < 2$ for some γ_p-set S. In the case $|S \cap N[N[v]]| = 0$, we can swap any $s \in S$ with v to make a p-dominating set since v has highest degree and none of its neighbors would be dominated. This contradicts v not being in any p-dominating set.

In the case $|S \cap N[N[v]]| = 1$, suppose $\{u\} = S \cap N[N[v]]$. In the case $u \in N(v)$, we can swap u and v and still have a p-dominating set. In the case $u \in N(N(v))$, we can again swap u and v and still have a p-dominating set. In either case this again contradicts v not being in any p-dominating set. □

These results and examples together show that being greedy for the highest degree vertices does not work by itself in finding you elements from a p-dominating set, but that this greedy mindset can get you looking in the right area of the graph. Observe that neither of graphs in Figures 2 or 3 were trees. The graph if Figure 4 also shows that in a tree, a highest degree vertex need not be in a p-dominating set.

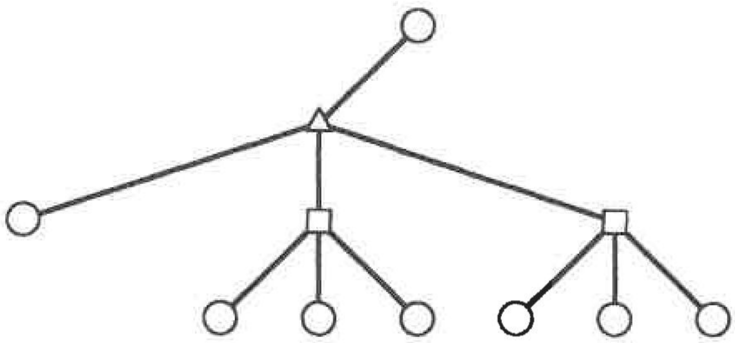

FIGURE 4. A highest degree vertex v (triangle) is not in any $\gamma_{9/11}$-set; rather two of its neighbors (rectangles) form a $\gamma_{9/11}$-set.

4. p-INFLUENCING SET

Now we introduce a related definition by considering the union of all p-dominating sets of a graph. We will call this union the influencing set.

Definition 4.1. Let $p \in [0, 1]$ be fixed. The union of all γ_p-sets of G is called the *p−influencing set* of G.

Note that there are $n = |V|$ interesting proportions p that can be considered for a graph

$$1/n, 2/n, 3/n, ..., 1.$$

Also we've allowed $p = 0$, but here the p-dominating set is just $\emptyset$. For the smallest of these above $p = 1/n$ the p-influencing set is all of V.

Lemma 4.2. *Let $p = 1/n$, then the p-influencing set of G is all of V. If G is a connected graph and $p = 2/n$, then the p-influencing set of G is all of V.*

Proof. For the first, every individual vertex in V is a $1/n$-dominating set. Thus the union of all $1/n$-dominating sets contains every vertex in V.

For the second, every individual vertex in V is a $2/n$-dominating set, since it dominates itself and at least one neighbor. Thus the union of all $2/n$-dominating sets is all of V. □

This lemma can be generalized further as follows.

Lemma 4.3. *Let $\delta(G)$ be the smallest degree of any vertex in G. If $p \leq \frac{\delta(G)+1}{n}$, then the p-influencing set of G is all of V.*

Proof. Every vertex is a $\frac{\delta(G)+1}{n}$ dominating set, thus the $\frac{\delta(G)+1}{n}$-influencing set is all of G. □

As we see above, for the smaller interesting proportions the p-influencing set is as large as possible. One might ask if the other p-influencing sets have any containment properties as the proportion increases or decreases, or if the size of p-influencing set only decreases as p increases. In general, this does not happen. Considering the graph in Figure 2, one can see that as p runs from $1/9$ to 1 the p—influencing sets change from all the vertices down to one vertex and then back to all the vertices. Also in this example the intersection of all the p-influencing sets with $p > 0$ is $\emptyset$.

Lemma 4.4. *Let $\Delta(G)$ be the highest degree of any vertex in G. If $p = \frac{\Delta(G)+1}{n}$, then the p-influencing set is made up of exactly those vertices in G with degree equal to $\Delta(G)$.*

Proof. When $p = \frac{\Delta(G)+1}{n}$, each vertex of degree $\Delta(G)$ is a p-dominating set. Furthermore, there are no p-dominating sets with more than one vertex, since they would not be of minimum size. Thus, the p-influencing set consists exactly of the degree $\Delta(G)$ vertices. □

The previous results gave the p-influencing sets of any graph for a fixed p. We now find all p-influencing sets for two common graphs, namely complete bipartite graphs and paths.

Proposition 4.5. *Consider the complete bipartite graph $K_{m,n} = (V_1, V_2, E)$ with $|V_1| = m$, $|V_2| = n$, and $m > n$. The p-influencing sets of $K_{m,n}$ are*

- *$V_1 \cup V_2$ for $0 < p \leq \frac{n+1}{m+n}$ and $p \geq \frac{m+2}{m+n}$*
- *V_2 for $\frac{n+2}{m+n} \leq p \leq \frac{m+1}{m+n}$*

Moreover, if $m = n$, then the p—influencing set of $K_{m,n}$ is $V_1 \cup V_2$ for all $p > 0$.

Proof. Suppose $m > n$. If $0 < p \leq \frac{n+1}{m+n}$, then any vertex of $K_{m,n}$ is a γ_p-set. Thus the p-influencing set is $V_1 \cup V_2$. If $p \geq \frac{m+2}{m+n}$, then $\{v_1, v_2\}$ where $v_1 \in V_1$ and $v_2 \in V_2$, is a γ_p-set. Thus the p-influencing set is $V_1 \cup V_2$. Lastly, if $\frac{n+2}{m+n} \leq p \leq \frac{m+1}{m+n}$, then any single vertex from V_2 is a γ_p-set, and no vertex from V_1 can be in a γ_p-set.

The argument is similar when $m = n$. □

Corollary 4.6. *Consider the complete bipartite graph $K_{m,n} = (V_1, V_2, E)$ with $|V_1| = m$, $|V_2| = n$, and $m > n$. The intersection of all the p-influencing sets ($p > 0$) of $K_{m,n}$ is V_2. Moreover, if $m = n$, the intersection is $V_1 \cup V_2$.*

For the following proposition and corollary, we consider a path P_n where the vertices are labeled v_i, $i \in \{1, 2, \ldots, n\}$ with v_1, v_n as leaves and $N(v_i) = \{v_{i-1}, v_{i+1}\}$ for $i \in \{2, 3, \ldots, n-1\}$.

Proposition 4.7. *Consider a path P_n with the vertices labeled as described above. The p-influencing sets of P_n are given below. Note k is a nonnegative integer chosen so that $0 < p \leq 1$ unless otherwise stated.*

(1) For $n \equiv 0 \mod 3$
- $V(P_n)$ *if* $p = \frac{3k+1}{n}$ *or* $p = \frac{3k+2}{n}$
- $\{v_2, v_3, \ldots, v_{n-1}\}$ *if* $p = \frac{3k}{n} < 1$
- $\{v_2, v_5, v_8, \ldots, v_{n-1}\}$ *if* $p = 1$

(2) For $n \equiv 1 \mod 3$
- $V(P_n)$ *if* $p = \frac{3k+1}{n}$ *or* $p = \frac{3k+2}{n}$
- $\{v_2, v_3, \ldots, v_{n-1}\}$ *if* $p = \frac{3k}{n}$ *and* $3k \neq n-1$
- $V(P_n) \setminus \{v_1, v_4, v_7, \ldots, v_n\}$ *if* $p = \frac{n-1}{n}$

(3) For $n \equiv 2 \mod 3$
- $V(P_n)$ *if* $p = \frac{3k+1}{n}$ *or* $p = \frac{3k+2}{n} < 1$
- $\{v_2, v_3, \ldots, v_{n-1}\}$ *if* $p = \frac{3k}{n}$
- $V(P_n) \setminus \{v_3, v_6, v_9, \ldots, v_{n-2}\}$ *if* $p = 1$

Proof. Suppose $n \equiv 0 \mod 3$. Partition $V(P_n)$ into three sets

$$V_1 = \{v_1, v_4, \ldots, v_{n-2}\}, V_2 = \{v_2, v_5, \ldots, v_{n-1}\}, V_3 = \{v_3, v_6, \ldots, v_n\}.$$

Now consider $p = \frac{3k+1}{n}$ or $p = \frac{3k+2}{n}$. Then $\gamma_p(P_n) = k+1$, and any $k+1$ vertices of V_1, V_2, or V_3 comprise a γ_p-set.

Now consider $p = \frac{3k}{n} < 1$. Then $\gamma_p(P_n) = k$, and any k vertices of $V_1 \setminus \{v_1\}, V_2$ or $V_3 \setminus \{v_n\}$ comprises a γ_p-set.

Lastly, if $p = 1$, then $\gamma_1(P_n) = \frac{n}{3}$, and V_2 is the only γ_1-st.

Suppose $n \equiv 1 \mod 3$. Partition $V(P_n)$ into three sets

$$V_1 = \{v_1, v_4, \ldots, v_n\}, V_2 = \{v_2, v_5, \ldots, v_{n-2}\}, V_3 = \{v_3, v_6, \ldots, v_{n-1}\}.$$

Note that $|V_1| = \frac{n-1}{3} + 1$, $|V_2| = |V_3| = \frac{n-1}{3}$.

Now consider $p = \frac{3k+1}{n}$ or $p = \frac{3k+2}{n}$, then $\gamma_p(P_n) = k+1$. Now if $p < 1$, so $k < \frac{n-1}{3}$, then any $k+1$ vertices of V_1, V_2, or V_3 comprise a γ_p-set. If $p = 1$, so $k = \frac{n-1}{3}$, then V_1, $V_2 \cup \{v_n\}$, and $V_3 \cup \{v_1\}$ are $\gamma_1(P_n)$-sets.

Now if $p = \frac{3k}{n}$ and $3k \neq n-1$, so $k < \frac{n-1}{3}$. Then $\gamma_p(P_n) = k$, and any k vertices from $V_1 \setminus \{v_1, v_n\}, V_2$, or V_3 comprise a γ_p-set.

Lastly, if $p = \frac{n-1}{n}$, then $\gamma_p(P_n) = \frac{n-1}{3}$. Then note that V_2 and V_3 are the only $\gamma_p(P_n)$-sets.

Suppose $n \equiv 2 \mod 3$. Partition $V(P_n)$ into three sets

$$V_1 = \{v_1, v_4, \ldots, v_{n-1}\}, V_2 = \{v_2, v_5, \ldots, v_n\}, V_3 = \{v_3, v_6, \ldots, v_{n-2}\}.$$

Note that $|V_1| = |V_2| = \frac{n+1}{3}$, and $|V_3| = \frac{n-2}{3}$.

Now consider $p = \frac{3k+1}{n}$ or $p = \frac{3k+2}{n} < 1$, so $k < \frac{n-2}{3}$. Then $\gamma_p(P_n) = k+1$ and any $k+1$ vertices from V_1, V_2, or V_3 comprise a γ_p-set.

Now if $p = \frac{3k}{n}$, so $k \leq \frac{n-2}{3}$, then $\gamma_p(P_n) = k$. Any k vertices from $V_1 \setminus v_1, V_2 \setminus v_n$, or V_3 will be a γ_p-set.

Lastly, if $p = 1$, then $\gamma_1(P_n) = \frac{n+1}{3}$. Note that V_1 and V_2 are the only $\gamma_1(P_n)$-sets. □

Corollary 4.8. *The intersection of all p-influencing sets ($p > 0$) for a path P_n is*

(1) $\{v_{2+3k} | 0 \leq k \leq \frac{n-3}{3}\}$ *if* $n \equiv 0 \mod 3$

(2) $\{v_{2+3k}, v_{3+3k} | 0 \leq k \leq \frac{n-4}{3}\}$ *if* $n \equiv 1 \mod 3$

(3) $\{v_{1+3k}, v_{2+3j} | 0 < k \leq \frac{n-2}{3}, 0 \leq j < \frac{n-2}{3}\}$ *if* $n \equiv 2 \mod 3$.

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