Pure and Mixed Multi-Step Strategies for Rendezvous Search on the Platonic Solids

Xiao Xie and John C. Wierman
Department of Applied Mathematics and Statistics
Johns Hopkins University

Abstract

The *astronaut problem* is an open problem in the field of rendezvous search. The premise is that two astronauts randomly land on a planet and want to find one another. Research explores what strategies accomplish this in the least expected time. To investigate this problem, we create a discrete model which takes place on the edges of the Platonic solids. Some baseline assumptions of the model are: (1) The agents can see all of the faces around them. (2) The agents travel along the edges from vertex to vertex and cannot jump. (3) The agents move at a rate of one edge length per unit time. The 3-dimensional nature of our model makes it different from previous work. We explore multi-step strategies, which are strategies where both agents move randomly for one step, and then follow a pre-determined sequence. For the cube and octahedron, we are able to prove optimality of the "Left Strategy", in which the agents move in a random direction for the first step and then turn left. In an effort to find lower expected times, we explore mixed strategies. Mixed strategies incorporate an asymmetric case which under certain conditions can result in lower expected times. Most of the calculations were done using first-step decompositions for Markov chains.

1 Introduction

1.1 Background

The general premise behind all search theory problems is that an agent is looking for a target (which could be mobile or immobile).

Thus, the question we are naturally led to ask is "What is the optimal strategy?" Optimal is traditionally defined to mean the strategy that minimizes expected meeting time. From this basic premise, a wide variety of problems have been proposed. Search theory originally found its roots in military operations during World War II. During the period from 1965 to 1975, Lawrence Stone pioneered mathematical results for searches involving a single agent and an immobile target and made important progress on problems involving a moving target [9]. Another class of problems which has been thoroughly researched are search games which involve a searcher, who is looking to minimize time, and a hider, who is looking to maximize time [5].

Moreover, Thomas Schelling presents an interesting problem of "tacit coordination" in his book, *The Strategy of Conflict*. Two parachutists randomly land in area with defining landmarks such as roads, buildings, a river, and a bridge. The goal for the parachutists is to find one another. Schelling makes the claim that the crux of the problem is the two parachutists must meet at a unique "focal point". He then goes on to say that the problem cannot be properly defined (and thus cannot be solved) when the search area is homogeneous i.e. there are no distinguishing focal points [7]. We will see that rendezvous search, a subset of search theory, addresses this exact scenario.

Rendezvous search is a branch of search theory that has been gaining attention in recent years. The premise of rendezvous search problems is that two (or more) agents are in different locations, and they want to find one another. In this case, both agents want to minimize the time it takes to achieve this goal. Rendezvous search can be further broken down into two categories: asymmetric and symmetric. In the asymmetric case, the players are not bound to the same strategy; in the symmetric case, they are. Steve Alpern, a pioneer in the field of rendezvous search, proposed ten open rendezvous search problems in *Search Theory: A Game Theoretic Perspective* [2, Ch. 14]. Our research is inspired by the astronaut problem.

The astronaut problem is as follows: two astronauts randomly land on a spherical planet, each with the same detection range in which they can see each other, and their goal is to find one another. No significant progress has been made on this problem. However, there are some results for rendezvous search on two-dimensional objects, such as graphs and networks, which could potentially be used to approximate the sphere. We present results for the asymmetric case and the symmetric case.

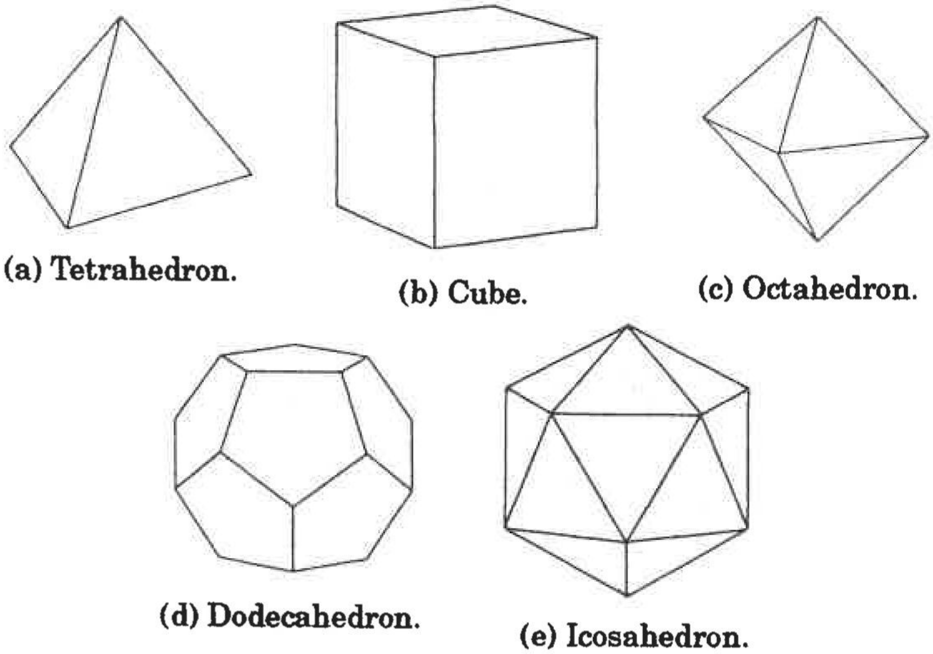
Significant work has been done for n discrete locations on a connected graph by Anderson and Weber. Results for the asymmetric case have found that a "Wait for Mommy" strategy is optimal [4]. This strategy entails that one agent waits while the other searches all the locations. For the symmetric case, one notable result is the Anderson-Weber Strategy for rendezvous search on n discrete locations in a complete graph. In this strategy, the agents have a probability of staying in the same location for $n-1$ steps and a probability of visiting all other $n-1$ locations for $n-1$ steps. If one agent waits and the other moves, then the agents are guaranteed to find one another. Meanwhile, if both agents move, there is no such guarantee. If both players wait, then they will definitely not meet. This strategy drew inspiration from the optimal "Waiting for Mommy" strategy in the asymmetric case, and has been proven optimal for two locations and three locations [4, 10].

Another variation of the problem that has been considered is the rendezvous search problem on the circle. Two agents are on an undirected circle and want to find one another. One symmetric strategy that has been studied by Alpern is the *Coin Half Tour*. In this strategy, each agent flips a fair coin. If the coin is heads, then the agent will walk $\frac{1}{2}$ around the circle in the right direction. If the coin is tails, then the agent will walk $\frac{1}{2}$ around the circle in the left direction. The agents repeat this process until they meet. On a circle with a circumference of 1, this strategy yields an expected time of $\frac{3}{4}$. Alpern also looked at the asymmetric "Wait for Mommy" strategy on the circle. Under the same conditions as before, this strategy yields an expected time of $\frac{1}{2}$ [1]. The "Coin Half Tour" and "Wait for Mommy" strategy are thought to be the best available strategy for the symmetric and asymmetric case respectively.

Along with the results presented in the mathematics community, the computer science community has published results on variations of this problem. In the field of computer science, this problem considers two mobile agents who are located on two vertices of a network. The agents' goal is to meet at the same vertex. A plethora of variants have been considered. For example, in one version of asymmetric rendezvous, a map of the graph and the agent's starting position is available for reference. Algorithms, called FOCAL strategies, have been created to get the agents to an agreed upon meeting location. In another version, the agents do not have access to a map, but there is a distinguishing focal point in the graph. In one version of symmetric rendezvous, agents are allowed to leave tokens to mark vertices as desired [6]. In addressing efficiency of strategies, researchers have considered both the expected time it takes for the agents to meet and the worst-case scenarios. Overall, the computer science work seems to emphasize utilizing all known information about the network in order to determine 1) Is the problem solvable? 2) What is the best solution?

1.2 The Discrete Model

In investigating the astronaut problem, we simplify and approximate this problem with the five platonic solids: the tetrahedron, the octahedron, the cube, the icosahedron, the dodecahedron. Recall that the platonic solids are polyhedrons composed of congruent, regular faces with the same number of faces meeting at each vertex. These solids seem to be a unique model among the rendezvous search literature, which currently focuses on two-dimensional graphs as opposed to three-dimensional objects.



(a) Tetrahedron. (b) Cube. (c) Octahedron.

(d) Dodecahedron. (e) Icosahedron.

Each platonic solid has a unique set of properties. Table 1 lists the face shape, number of vertices, number of edges, number of faces, and degree of vertices for each solid.

Solid	Face Shape	Vertices	Edges	Faces	Vertex Degree
Tetrahedron	Triangular	4	6	4	3
Cube	Square	8	12	6	3
Octahedron	Triangular	6	12	8	4
Dodecahedron	Pentagonal	20	30	12	3
Icosahedron	Triangular	12	30	20	5

Table 1: Properties of the platonic solids.

One property that they share in common is they all are vertex-transitive, and thus from any vertex, the surrounding area looks the same. Therefore when analyzing movements, we only need to consider the distance between the agents. We hope to find strategies that can be adapted to the original astronaut problem on a sphere. Our intuition is that "larger" solids will more closely approximate the sphere, though it is still helpful to compare both "larger" and "smaller" solids.

This article considers a rendezvous search problem with the following constraints:

1. The agents start on vertices and move along the edges from vertex to vertex. The agents cannot travel a fraction of an edge.
2. The agents both move at a rate of one edge length per unit of time.
3. The agents each have the same given detection range for which they can see each other.
4. The agents start in positions where they cannot see each other.
5. The agents cannot jump from one vertex to a non-adjacent vertex.
6. The search ends once the agents see each other.

Throughout this paper, we assume that the agents have full-face visibility. Full-face visibility means agents can see all faces (and their vertices) adjacent to their current vertex. For example, an agent on the icosahedron can see five triangular faces, along with their edges and vertices, and an agent on the cube can see three square faces, along with their edges and vertices.

Our ultimate goal is to minimize the expected time for the agents to see each other.

1.3 Previous Results

1.3.1 Unbiased Random Walk Strategy

Our previous work [11] explores simpler strategies to get a preliminary bound for the optimal expected time. The simplest strategy we considered is an unbiased random walk strategy. The two agents begin at vertices where they cannot see each other, and, at each iteration, move along a randomly chosen edge incident to their current vertex. By vertex transitivity, the agents have no preference for any direction, and thus even the first move is random. The agents must move at each turn, and the search will end when the agents can see each other. Intuitively, an unbiased random walk strategy might be considered "mindless" and would be expected to give longer expected meeting times than more intentional strategies. For all of the solids, calculations for this strategy can be done using first-step Markov chain decompositions. However, you can also use geometric probability mass functions. Examples of these calculations can be found in West, Xie, and Wierman [11]. We present a summary of the results.

Let $T(solid)$ denote the expected time for the search to end on the given solid.

$$T(tetrahedron) = 0.$$

$$T(cube) = \frac{4}{3} \approx 1.3333.$$

$$T(octahedron) = \frac{3}{2} = 1.5.$$

$$T(dodecahedron) = \frac{549}{140} \approx 3.9214.$$

$$T(icosahedron) = \frac{5}{2} = 2.5.$$

1.3.2 Multi-Step Strategies on the Cube and Octahedron

Next, we consider multi-step strategies which incorporate a deterministic component. A multi-step strategy is a symmetric strategy where both agents first randomly choose a direction in which to move one step, and then move in a predetermined n-step sequence consisting of steps in relative directions, creating an $n+1$-step strategy. The search ends whenever the agents see each other, including if they see each other in the middle of their sequence of movements. If the agents do not see each other after completing the $n+1$ steps, then they repeat the process until they do see each other.

The simplest multi-step strategy is that the agents first randomly choose a direction in which to move one step, and then move one step in the left direction relative to the edge that they just came from. We call this strategy the Left Strategy. On the cube and octahedron, the Left Strategy guarantees the agents see each other by the completion of the second step. We use this property to prove optimality of this strategy for those solids. See [11] for the proof. Through this proof, we are introduced to the idea of strategies having a guaranteed ending time, giving us a new way to define "best" strategy.

1.4 Outline

In Section 2, we discuss multi-step strategies on the larger solids. Given our previous results, we consider the relationship between optimality and guaranteeing a finite, bounded end time, and what it means to be "optimal". A low expected meeting time does not necessarily guarantee that the search will not take an arbitrarily long time. On the other hand, we can have a strategy with a longer expected time, but the probability that the search will end by a specific time is one. We also discuss the use of Markov chains in calculations.

In Section 3, we then consider mixed strategies. The agents can choose from a set of n strategies, all of the same length, and with each strategy there is an associated probability of picking it. In this chapter, we are primarily interested in finding the optimal expected time for the set of mixed strategies and determining whether there exists a mixed strategy that has a lower expected time than the lowest value obtained previously. The calculations for mixed strategies are more complex. However, we present some simplifications that allow us to make significant progress on the previously mentioned goals.

Finally in Section 4, we conclude by comparing all strategies we have explored alongside a discussion of optimality, and in Section 5, we discuss future research.

2 Pure Multi-Step Strategies

We also consider longer multi-step strategies on the dodecahedron and icosahedron. The icosahedron is unique among the solids because it has "hard" and "soft" turns. For the dodecahedron, we investigated strategies up to seven steps in length, and for the icosahedron, we investigated strategies up to five steps in length. Given our optimality results for the cube and octahedron, we investigated some strategies that guarantee a finite, fixed end time. For the dodecahedron, the shortest strategy with this property has seven steps, and for the icosahedron, it has five steps.

The calculations for these results are set up similarly to first-step Markov chain decompositions. We present the calculation for the expected time of the Hard Left Strategy on the icosahedron as an example. For the icosahedron, the agents cannot see each other when they are either two units or three units apart.

Given that the agents are two units apart, there is a $\frac{4}{10}$ probability of seeing each other on the first step and a $\frac{7}{25}$ probability of seeing each other on the second step. Thus there is a $\frac{8}{25}$ probability that the agents will not see each other during the iteration of the sequence. More specifically, there is a $\frac{7}{25}$ probability that they will end up two units apart and a $\frac{1}{25}$ probability that they will end up three units apart.

Given that the agents are three units apart, there is a $\frac{4}{10}$ probability of seeing each other on the first step and a $\frac{4}{10}$ probability of seeing each other on the second step. There is a $\frac{2}{10}$ probability that the agents will not see each other during the iteration of the sequence. More specifically, there is a $\frac{2}{10}$ probability that they will end up two units apart.

Moreover, given that the agents cannot see each other, there is a $\frac{5}{6}$ conditional probability of starting two units apart and a $\frac{1}{6}$ conditional probability of starting three units apart.

Using this information, we construct the following system of equations:

$$T_{HL}(icosahedron) = \frac{5}{6}E_2 + \frac{1}{6}E_3.$$

$$E_2 = \frac{4}{10}(1) + \frac{7}{25}(2) + \frac{7}{25}(E_2 + 2) + \frac{1}{25}(E_3 + 2).$$

$$E_3 = \frac{4}{10}(1) + \frac{2}{10}(2) + \frac{2}{10}(E_2 + 2).$$

Solving this system of equations yields the approximate expected time of 2.2921.

We are unable to claim an optimal strategy for all multi-step strategies of any given length on these solids. However, by investigating the longer multi-step strategies in our set through exhaustive search, we can lower our upper bound for the optimal expected time. For the dodecahedron, the lowest expected time we have found is 2.8. This expected time was attained by the strategy {Random, Left, Left, Right, Left, Left, Right} and its reflection {Random, Right, Right, Left, Right, Right, Left}. These strategies guarantee the agents see each other by the completion of the fifth step. For the icosahedron, the lowest expected time we have found is approximately 2.1333. This expected time was attained by the following strategies and their reflections:

{random, hard left, soft left, soft left, soft left},
{random, hard left, soft left, hard right, soft right},
{random, hard left, soft right, hard left, soft right},
{random, hard left, soft right, hard right, soft left},
{random, hard left, hard right, hard left, soft left},
{random, hard left, hard right, soft right, soft right}

All of these strategies also guarantee the agents see each other by the completion of the fifth step. More results are presented in [11].

3 Mixed Strategies

We next consider expanding the set of pure multi-step strategies to mixed strategies. Previous work and our own numerical results suggest that asymmetric cases yield lower expected times than symmetric cases. Mixed strategies are a way to incorporate these asymmetric cases into a symmetric strategy. The way a mixed strategy works is that there is a set of n strategies that agents can choose from. All of the strategies are the same length. For each strategy i, there is an associated probability, p_i, of choosing that strategy. If the players do not see each other after running through their strategies, then they pick again with the same probabilities. The pure strategies are a subset of the mixed strategies where one probability is set equal to 1 and the others to 0.

3.1 Lower Bound for Expected Time

Allowing the agents the option to choose again complicates the calculations. Therefore, in order to simplify the computations, we derive a lower bound on the expected time. To do this, we assume that if the agents have not seen each other by the last step, then they see each other on the next step.

Here, we present an example calculation. Suppose the agents are on the dodecahedron and both use the Left Strategy. Recall that under the Left Strategy, the agent moves one step in a randomly chosen direction, and then one step in the left direction.

Given that the agents are three units apart, there is a $\frac{1}{3}$ probability that they will see each other on the first step and a $\frac{2}{9}$ probability that they will see each other on the second step. Thus, there is a $\frac{4}{9}$ probability that the agents do not see each other during the iteration of the sequence.

Given that the agents are four units apart, there is a $\frac{2}{9}$ probability that they will see each other on the first step and a $\frac{1}{3}$ probability that they will see each other on the second step. Thus, there is a $\frac{4}{9}$ probability that the agents will not see each other during the iteration of the sequence.

Given that the agents are five units apart, there is zero probability that the agents will see each other on the first step and a $\frac{2}{3}$ probability that they will see each other on the second step. Thus, there is a $\frac{1}{3}$ probability that the agents do not see each other during the iteration of the sequence.

Moreover, given that the agents initially cannot see each other, there is a $\frac{6}{10}$ probability of the agents being three units apart, a $\frac{3}{10}$

probability of the agents being four units apart, and a $\frac{1}{10}$ probability of the agents being five units apart.

In this bounded expected time calculation, we assume that if the agents do not see each other during the iteration of the sequence, then they will see each other on the next step. Thus we assume the following:

Given that the agents start three units apart, there is a $\frac{4}{9}$ probability of the agents seeing each other on the third step. Given that the agents start four units apart, there is a $\frac{1}{3}$ probability of the agents seeing each other on the third step. Given that the agents start five units apart, there is a $\frac{2}{3}$ probability of the agents seeing each other on the third step.

Let E_i^B denote the bounded conditional expected time given the agents are i units apart.

Using the information above, we calculate:

$$E_3^B = \frac{1}{3}(1) + \frac{2}{9}(2) + \frac{4}{9}(3) = \frac{19}{9}.$$

$$E_4^B = \frac{2}{9}(1) + \frac{1}{3}(2) + \frac{4}{9}(3) = \frac{20}{9}.$$

$$E_5^B = \frac{1}{3}(2) + \frac{2}{3}(3) = \frac{8}{3}.$$

$$E_B(T) = \frac{6}{10}E_3^B + \frac{3}{10}E_4^B + \frac{1}{10}E_5^B = \frac{11}{5}.$$

Note that for strategies that guarantee the two agents see each other by the end of the sequence, the true expected time is equal to the bounded expected time.

Let $\mathcal{S} = \{S_1, S_2, ...S_n\}$ be a set of n pure strategies. Suppose the agents use a mixed strategy where they choose their strategies from set $\mathcal{S}$. Let B_{ij} denote the bounded expected time given that agent 1 uses strategy i and agent 2 uses strategy j. Note that $B_{ij} = B_{ji}$. After calculating all of the B_{ij} terms, to calculate a lower bound for the overall expected time of the mixed strategy, we need to optimize with regards to the constraints and objective function:

$$min \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j B_{ij}.$$

$$s.t. \sum_{k=1}^{n} p_k = 1.$$

$$\forall k, p_k \geq 0.$$

The objective function is an application of the Law of Total Expectation. Each term in the sum represents the probability that agent 1 uses strategy i and agent 2 uses strategy j, multiplied by the bounded expected time, given that strategies i and j are used. The constraints ensure that our optimal p_i values satisfy the probability axioms.

While these calculations do not result in exact expected times, we are still able to make some claims using these lower bounds. Specifically for the case when we are mixing two strategies, there are some interesting results that relate the overall expected time to the expected time of specific cases.

3.2 Reducing the Number of Strategies

We present two lemmas which reduce the number of mixed strategies that we need to analyze, given that the goal is to find a strategy with a lower expected time.

Lemma 1. *Let $E^*(T)$ be the optimal time for the set of pure strategies. Let $\{S_1, S_2, \ldots\ S_n\}$ be a set of n pure strategies. If for all $i, j, B_{ij} \geq E^*(T)$, then for any associated mixed strategy M, $E_M(T) \geq E^*(T)$, where $E_M(T)$ is the expected time of the strategy M.*

Proof. Let p_i denote the probability of choosing S_i for $i = 1, 2, \ldots n$. To satisfy the probability axioms, we require $\sum_{i=1}^{n} p_i = 1$ and for all i, $p_i \geq 0$.

We have the following inequality for the expected time of the mixed strategy M:

$$E_M(T) \geq \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j B_{ij}.$$

Recall that the right hand side is our bounded expected time for the mixed strategy.

Given that for all i, j, $B_{ij} \geq E^*(T)$, we can say

$$E_M(T) \geq \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j B_{ij} \geq \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j E^*(T).$$

Simplifying this expression, we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j E^*(T) = E^*(T)\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j = E^*(T)\left(\sum_{i=1}^{n} p_i\right)^2 = E^*(T).$$

Thus, $E_M(T) \geq E^*(T)$. □

From this lemma, we can conclude that given for all $i, j, B_{ij} \geq E^*(T)$, if there exists i, j such that $B_{ij} > E^*(T)$, then the inequality becomes a strict inequality, meaning the strategy is not optimal.

Also when $n = 2$, Lemma 1 is a statement about what pairs of pure strategies create mixed strategies that perform better. More specifically, the only mixed strategies that perform better than pure strategies are those with $B_{12} < E^*(T)$.

Lemma 2. *Let $S = \{S_1, S_2, \ldots S_n\}$ be a set of n pure strategies and let M denote a mixed strategy that consists of the n strategies in the set. Suppose there exists some U such that $E_M(T) < U$ and for some strategy S_i, $B_{ij} \geq U$ (and $B_{ji} \geq U$) for $j = 1,2,\ldots n$. Then there exists a mixed strategy $\hat{M}$ consisting of the strategies $\{S_1, S_2, \ldots S_{i-1}, S_{i+1}, \ldots S_n\}$ such that $E_{\hat{M}}(T) < E_M(T)$*

Proof. Let X: $S \times S \to \mathbb{R}$ be a random variable denoting the bounded expected time corresponding to the pair of randomly chosen strategies S_i and S_j.

Let A be the event that S_i is used by at least one of the agents. We will only consider the nontrivial case where $P(A) \neq 0$.

The possible values for our random variable X are B_{ij} for all combinations of i and j. The probability that $X = B_{ij}$ for a given i, j is equal to the probability of agent 1 choosing strategy S_i and agent 2 choosing strategy S_j. Thus, the expected value of X is equal to the bounded expected time for the mixed strategy. i.e. $E_M(T) = E(X)$.

Partition the expected value of X using the event A. By the Law of Total Expectation,

$$E(X) = P(A)E(X \mid A) + (1 - P(A))E(X \mid A^c).$$

Given that for S_i, $B_{ij} \geq U\ \forall j$, we can conclude $E(X \mid A) \geq U$. Since $E(X) = E_M(T) < U$, consider the following claim: $E(X \mid A^c) < E(X)$. We will prove this claim by way of contradiction.

Claim: $E(X \mid A^c) < E(X)$.

Proof. For sake of contradiction, assume that $E(X \mid A^c) \geq E(X)$.

Recall $E(X) = E(X \mid A)P(A) + E(X \mid A^c)P(A^c)$.

We concluded previously that $E(X \mid A) > U$. Since $E(X) < U$, this implies that $E(X \mid A) > E(X)$. Using this result and our claim, we write

$$E(X) = E(X \mid A)P(A) + E(X \mid A^c)P(A^c) > E(X)P(A) + E(X)P(A^c).$$

Simplifying the expression, we arrive at a contradiction.

$$E(X) > E(X)\,[P(A) + P(A^c)] = E(X).$$

Thus $E(X \mid A^c) < E(X)$. □

Now, suppose that S_i is no longer in the set of pure strategies from which the agents can choose. Therefore, the probability of choosing strategy S_i is now zero. Thus, $P(A) = 0$ (in other words, we set $P(X = B_{kj}) = 0$ for $k{=}i$ or $j{=}i$). In addition, for the pairs of strategies S_k and S_j in A^c, set $P(X = B_{kj}) = P(X = B_{kj} \mid A^c)$. Let $E_{new}(X)$ denote the expected time of X calculated with these new probabilities.

$E_{new}(X) = E(X \mid A^c)$. Recall $E(X \mid A^c) < E_M(T)$. Thus $E_{new}(X) < E_M(T)$.

Let C_j denote the event that pure strategy S_j is chosen. We can write $E_{new}(X)$ as

$$E_{new}(X) = \sum_x xP(X = x \mid A^c)$$

where the possible x values are the possible B_{kj} values. We know $P(X = B_{kj} \mid A^c) = \sum_{\{l,m:B_{lm}=B_{kj}\}} P(C_l \mid A^c)P(C_m \mid A^c)B_{kj}$. Thus,

$$E_{new}(X) = \sum_{k\neq i}\sum_{j\neq i} P(C_k \mid A^c)P(C_j \mid A^c)B_{kj}.$$

Consider $P(C_j \mid A^c)$ to be p_j^{new}, the probability of choosing strategy S_j. Then $E_{new}(X)$ is equal to the expected time of a mixed strategy consisting of the pure strategies, $\{S_1, S_2, ...S_{i-1}, S_{i+1}, ...S_n\}$. Let $E_{\hat{M}}(T)$ denote the optimal expected time of mixed strategies consisting of the pure strategies, $\{S_1, S_2, ...S_{i-1}, S_{i+1}, ...S_n\}$. $E_{\hat{M}}(T) \leq E_{new}(X)$ because the probabilities associated with $E_{new}(X)$ do not necessarily minimize the expected time.

Thus, $E_{\hat{M}}(T) < E_{new}(X) < E_M(T)$. □

3.3 Results for the Dodecahedron

For the dodecahedron, we consider mixing two 7-step strategies. Using the previous lemmas, we are able to reduce the number of strategies that we need to analyze. There are sixty-four 7-step strategies, meaning that besides pure strategies, there are $\binom{64}{2} = 2016$ mixed strategies.

Using Lemma 1, we can reduce our set of strategies to 162 strategies. Since we are only mixing two strategies, Lemma 2 is unable to further reduce our set. When mixing two strategies, Lemma 2 states that a mixed strategy is not optimal if there exists some U such that $E_M(T) < U$ and there exists some strategy S_i such that $B_{ij} \geq U$ for $j = 1, 2$. More specifically, the expected time of the pure strategy that is not S_i is less than or equal to that of the mixed strategy.

In Lemma 1, we restricted $B_{12} < 2.8$. Moreover, through exhaustive search we know that for all pure strategies, $B_{ii} \geq 2.8$. Therefore, $B_{11} \geq 2.8$ and $B_{22} \geq 2.8$, regardless of what strategies they are. Thus, $E_M(T) \geq B_{12}$. Since $E_M(T) \geq B_{12}$, there does not exist a U such that $E_M(T) < U$, $B_{12} \geq U$ and either $B_{11} \geq U$ or $B_{22} \geq U$. Thus there are no further reductions from Lemma 2.

Given that for our remaining strategies, $B_{ij} < 2.8$ for $i \neq j$, we can apply the following lemma:

Lemma 3. *For mixed strategies composed of two 7-step strategies on the dodecahedron, if* $B_{12} < 2.8$*, then it is guaranteed that the absolute minimum of the expected time function is attained and the optimal probability,* p*, will be between 0 and 1, exclusive.*

Proof. Let $E_B(T)$ denote the bounded expected time for the mixed strategy.

Recall that from the Law of Total Expectation,

$$E_B(T) = p^2 B_{11} + 2p(1-p)B_{12} + (1-p)^2 B_{22}.$$

Expanding this expression, we obtain

$$(B_{11} - 2B_{12} + B_{22})p^2 + (2B_{12} - 2B_{22})p + B_{22}.$$

For any parabola of the form $f(x) = ax^2 + bx + c$, where a, b, c are real constants, the parabola has a finite minimum only when a is strictly greater than zero. (When a is strictly greater than zero, the parabola is convex.) Therefore, the bounded expected time function has a finite minimum only when $B_{11} - 2B_{12} + B_{22} > 0$. This is what we will attempt to prove.

Given $B_{12} < 2.8$ and for all i, $B_{ii} \geq 2.8$, we conclude $B_{11} + B_{22} \geq 2 * 2.8$ and $2B_{12} < 2 * 2.8$. Thus $B_{11} + B_{22} > 2B_{12}$. (i.e. $B_{11} - 2B_{12} + B_{22} > 0$). Therefore, the function has a finite minimum.

Moreover, for any parabola of the form $f(x) = ax^2 + bx + c$, where $a > 0, b, c$ are real constants, the vertex of the parabola (the x value associated to the minimum functional value) is $\frac{-b}{2a}$. For the bounded expected time function, the vertex is equal to

$$\frac{-2(B_{12} - B_{22})}{2(B_{11} - 2B_{12} + B_{22})}.$$

We can write $(B_{11} - 2B_{12} + B_{22})$ as $-(B_{12} - B_{22}) + (B_{11} - B_{12})$. Because $B_{11} \geq 2.8$ and $B_{12} < 2.8$, $(B_{11} - B_{12}) > 0$.

Thus $-(B_{12} - B_{22}) + (B_{11} - B_{12}) > -(B_{12} - B_{22})$ and

$$\frac{-(B_{12} - B_{22})}{(B_{11} - 2B_{12} + B_{22})} < 1.$$

Furthermore, $-(B_{12} - B_{22}) > 0$ because $B_{12} < 2.8$ and $B_{22} \geq 2.8$, and $(B_{11} - 2B_{12} + B_{22}) > 0$ because $-(B_{12} - B_{22}) > 0$ and $(B_{11} - B_{12}) > 0$. Thus

$$\frac{-(B_{12} - B_{22})}{(B_{11} - 2B_{12} + B_{22})} > 0.$$

Therefore, the p-coordinate of the vertex is between 0 and 1 inclusive. □

This result reaffirms the potential for mixed strategies to yield lower expected times than pure strategies. In addition, we can come up with a set of conditions for when it is better to use a pure strategy versus a mixed strategy. For example, when $B_{11} - 2B_{12} + B_{22} \leq 0$, the function goes to negative infinity. Thus, the p value corresponding to the minimized bounded expected time will be at the boundary points: $p = 0$ or $p = 1$. Moreover, if the vertex of the graph is outside of the interval $[0,1]$, the p value will be at the boundary points.

3.4 Bounds for the Optimal Expected Time

Our ultimate goal is to find the optimal expected time for the set of mixed strategies. Since we are not calculating exact expected times, performing exhaustive search on current numerical results will not give an answer to this question. However, upper and lower bounds to this number can be found.

An upper bound of the optimal expected time is the expected time of our current optimal pure strategy, 2.8. We find an improved lower bound by finding the lowest of the lower bound expected times.

To find the lowest lower bound, we use the bounded expected times of the asymmetric cases to partition the strategy set. We then utilize the following lemmas to gradually increase and check the lower bound until we cannot further increase it.

Lemma 4. *Let $E^*(T)$ denote the optimal expected time for the set of mixed strategies. Let B_{12}^L denote the smallest B_{ij} value for $i \neq j$. Let B_{11}^L denote the smallest B_{ii} value and B_{22}^L denote the second smallest B_{ii} value (B_{11}^L can equal B_{22}^L if more than one pure strategy corresponds to the lowest B_{ii} value.) Let $E_{opti}(T)$ be the minimized value of $p^2 B_{11}^L + 2p(1-p)B_{12}^L + (1-p)^2 B_{22}^L$ with respect to p. $E_{opti}(T) < E^*(T)$.*

Proof. For the optimal expected time for the set of mixed strategies, $E^*(T) = p^2 B_{11} + 2p(1-p)B_{12} + (1-p)^2 B_{22}$ for some two strategies and some optimal value p.

Let p^* denote the corresponding optimal value of p for $E^*(T)$. Also, let B_{11}^*, B_{12}^*, and B_{22}^* be the corresponding B_{ij} values for $E^*(T)$. Thus we have

$$E^*(T) = (p^*)^2 B_{11}^* + 2p^*(1-p^*)B_{12}^* + (1-p^*)^2 B_{22}^*.$$

$B_{11}^L > 0$ is the smallest B_{ii} value. Thus, $B_{11}^L \leq B_{11}^*$. Replacing B_{11}^* with B_{11}^L in the above expression and making analogous replacements for B_{12}^* and B_{22}^*, we get the following inequality:

$$E^*(T) \geq (p^*)^2 B_{11}^L + 2p^*(1-p^*)B_{12}^L + (1-p^*)^2 B_{22}^L.$$

$E_{opti}(T)$ is the minimized value of $p^2 B_{11}^L + 2p(1-p)B_{12}^L + (1-p)^2 B_{22}^L$ with respect to p. Therefore,

$$E_{opti}(T) \leq (p^*)^2 B_{11}^L + 2p^*(1-p^*)B_{12}^L + (1-p^*)^2 B_{22}^L.$$

Connecting these inequalities, we get

$$E_{opti}(T) < (p^*)^2 B_{11}^L + 2p^*(1-p^*)B_{12}^L + (1-p^*)^2 B_{22}^L \leq E^*(T).$$

Thus $E_{opti}(T)$ is a lower bound for the true optimal expected value. □

Lemma 5. *Let $T(\gamma)$ denote the set of bounded expected times for mixed strategies such that $B_{12} = \gamma$ for a given γ. Let $E_\gamma(T)$ denote the minimized value of $p^2 B_{11}^L + 2p(1-p)\gamma + (1-p)^2 B_{22}^L$ with respect to p. Then $E_\gamma(T) \leq min\{T(\gamma)\}$*

Proof. Let $E_{min}(T)$ denote the minimum value in the set $T(\gamma)$. With this value, there is an associated mixed strategy, and thus an associated B_{11} and B_{22} value. Suppose $B_{11} \leq B_{22}$. Let's call these values B_{11}^{min} and B_{22}^{min} respectively. Also associated with $E_{min}(T)$ is the optimal p value, which we will call p^*.

Recall that B_{11}^L and B_{22}^L are the two smallest B_{ii} values overall. Thus, $B_{11}^L \leq B_{11}^{min}$ and $B_{22}^L \leq B_{22}^{min}$. Thus,

$$(p^*)^2 B_{11}^L + 2p^*(1-p^*)\gamma + (1-p^*)^2 B_{22}^L$$

$$< (p^*)^2 B_{11}^{min} + 2p^*(1-p^*)\gamma + (1-p^*)^2 B_{22}^{min}.$$

Given that $E_\gamma(T)$ is the minimized value of $p^2 B_{11}^L + 2p(1-p)\gamma + (1-p)^2 B_{22}^L$ with respect to p, we conclude

$$E_\gamma(T) \leq (p^*)^2 B_{11}^L + 2p^*(1-p^*)\gamma + (1-p^*)^2 B_{22}^L.$$

Connecting these inequalities, we conclude $E_\gamma(T) \leq E_{min}(T)$. □

Lemma 6. *Let $E_j(T)$ denote the minimized value of $p^2 B_{11}^L + 2p(1-p)j + (1-p)^2 B_{22}^L$ with respect to p. Suppose $j_1 < j_2$. Then $E_{j_1}(T) \leq E_{j_2}(T)$.*

Proof. For sake of contradiction, suppose that $E_{j_1}(T) > E_{j_2}(T)$.

Let p^* denote the associated optimal value of p for $E_{j_1}(T)$ and p' denote the associated optimal value of p for $E_{j_2}(T)$.

Since $E_{j_2}(T) < E_{j_1}(T)$,

$$(p')^2 B_{11}^L + 2p'(1-p')j_2 + (1-p')^2 B_{22}^L < (p^*)^2 B_{11}^L + 2p^*(1-p^*)j_1 + (1-p^*)^2 B_{22}^L.$$

Moreover, since $0 < j_1 < j_2$, we can conclude that

$$(p')^2 B_{11}^L + 2p'(1-p')j_1 + (1-p')^2 B_{22}^L < (p')^2 B_{11}^L + 2p'(1-p')j_2 + (1-p')^2 B_{22}^L.$$

Connecting these inequalities, we get

$$(p')^2 B_{11}^L + 2p'(1-p')j_1 + (1-p')^2 B_{22}^L < (p^*)^2 B_{11}^L + 2p^*(1-p^*)j_1 + (1-p^*)^2 B_{22}^L.$$

In other words,

$$(p')^2 B_{11}^L + 2p'(1-p')j_1 + (1-p')^2 B_{22}^L < E_{j_1}(T).$$

However, $E_{j_1}(T)$ is supposed to be the minimized value of $p^2 B_{11}^L + 2p(1-p)j_1 + (1-p)^2 B_{22}^L$ with respect to p. We arrive at a contradiction. □

For the dodecahedron, when optimizing $p^2 B_{11} + 2p(1-p)B_{12} + (1-p)^2 B_{22}$ with respect to p, we use the lower bound value 2.6333 for B_{12}. This value was found through exhaustive search. For B_{11} and B_{22} we use the two lowest lower bounds for 7-step pure strategies, which are 2.8 and 2.8. This calculation yields our initial lower bound of 2.7165.

From Lemma 5, we know that for all the mixed strategies with $B_{12} = 2.6333$, the bounded expected times are greater than or equal to 2.7165. After analyzing the mixed strategies that satisfy $B_{12} = 2.63333$, we find that none of the pure strategy combinations have expected times 2.8 and 2.8. This means the bounded lower bound is never attained, and thus there is room for improvement. Looking at the set of mixed strategies with $B_{12} = 2.6333$, the minimum value is 2.7537037. Consider this value to be our new tentative lower bound.

Next, consider the second lowest B_{ij} value, 2.6667. Optimizing the expected time (still using 2.8 and 2.8 as the expected times for the pure strategies), yields the value 2.733345. From Lemma 5, we conclude that all of the mixed strategies in this set have bounded expected times greater than or equal to 2.733345. Analyzing the subset of strategies where $B_{12} = 2.6667$, we find that none of the mixed strategies have a combination of pure strategies with expected times 2.8 and 2.8, and the minimum value is 2.75897436. This value is greater than the tentative lower bound, 2.7537037. Therefore, the tentative lower bound holds.

This process can be repeated until we reach a B_{ij} value, B^*_{ij}, such that the optimized expected time is greater than the lower bound. Lemma 6 states that all strategies with $B_{ij} > B^*_{ij}$ will have a larger expected time.

After executing this process, the final result for the lower bound is 2.7537037. Therefore our current bounds for the optimal expected time for the set of mixed strategies are 2.7537037 and 2.8. This interval is relatively small, with an approximate length of 0.05.

4 Summary and Conclusion

We have investigated expected meeting times for our simplified version of the astronaut problem on the platonic solids. Our previous results explore random walks and multi-step strategies. This paper focuses mainly on the extension of multi-step strategies to mixed strategies. These results are the first of their kind in modeling the expected time for two astronauts to see each other in the astronaut problem. Overall, there are some more intuitive results that are consistent with the current literature, and there are more surprising results that illustrate thought-provoking ideas in the problem.

Previous results investigate an unbiased random walk strategy. The unbiased random walk strategy is considered to be the most basic strategy since the agents are not trying to strategize in any way. These baseline cases are used as comparisons for other strategies as we attempt to decrease the current optimal expected meeting time. More work on the unbiased random walk strategy can be found in [11] and [12].

The Left Strategy serves as an introduction to multi-step strategies. It gives the agents a path that is not fully random. It decreases the expected time on the three solids examined, the octahedron, cube, and dodecahedron under face visibility. The Soft/Hard Left Strategy on the icosahedron also yield expected times shorter than that of the unbiased random walk strategy. This verifies that non-random, multi-step strategies can perform better than random strategies. In fact, we prove that the Left Strategy is optimal on the octahedron and cube due to the property that it guarantees the agents see each other by the end of the second step.

On the icosahedron and dodecahedron, longer multi-step strategies are explored. More specifically, we consider strategies up to seven steps in length and strategies up to five steps in length on the dodecahedron and icosahedron respectively. Similar to the unbiased random walk strategy, first-step Markov chain decompositions are used to calculate expected times. We are unable to prove optimality for a multi-step strategy. However, in the subsets that we have investigated through exhaustive search, some of these strategies have much smaller expected meeting times than that of the unbiased random walk strategy. In addition, some strategies have the added property that they guarantee the agents see each other by the end of the sequence.

Mixed strategies offer a way to insert an asymmetric case. We consider mixed strategies composed of two 7-step strategies on the dodecahedron. Results on the dodecahedron suggest that there is potential for a mixed strategy to have a lower expected time than all pure strategies. However, there are many combinations of pure strategies that do not result in a more beneficial mixed strategy. In other words, the asymmetric case does not help lower the expected time. In these cases, it is more advantageous for the agents to use a pure strategy as opposed to a mixed strategy. In addition to numerical results, we have results pertaining to the process of finding the optimal expected time of the set of mixed strategies. The original set of strategies can be reduced to a smaller set of interest. Also, upper and lower bounds for the optimal expected time can be calculated. These methods are not specific to the dodecahedron, and can be applied to other solids.

5 Future Research

There are many options for future research on this topic. Topics that we are particularly interested in pursuing are mixed strategies on the other solids, multi-step and mixed strategies under adjacent-vertex visibility (under this visibility, the agents can only see the vertices that are adjacent to the one they are standing on) and no visibility (the agents must meet), and making conjectures for the sphere.

Another possibility for future research is to add more edges and vertices to the faces of the solids to closer approximate the sphere. One obstacle we have found in the first approach is that the solids are no longer vertex-transitive, so there are many cases to be explored and calculations become much more complicated. Therefore, another possibility is to consider other vertex-transitive solids that are larger than the platonic solids. For more information on this work, see [12].

Another possibility is to consider strategies with a waiting component. West [12] has considered the case where in the unbiased random walk strategy, the agents also have the option to wait at each step. It would be interesting to see how inserting a waiting component in multi-step strategies would affect results. In addition, one could adapt the Anderson-Weber strategy for our search and see if it is as effective.

Overall, there are many more strategies to be explored on the platonic solids and other vertex-transitive solids. Comparing these strategies will hopefully point us in the right direction for more complex approximations on the sphere.

Acknowledgement: The authors thank the Acheson J. Duncan Fund for the Advancement of Research in Statistics for funding undergraduate research.

References

[1] Alpern, Steve. (1995) The rendezvous search problem. *SIAM Journal on Control and Optimization*, **33**, 673–683.

[2] Alpern, Steve, and Gal, Shmuel. (2011) *The Theory of Search Games and Rendezvous.* New York: Springer.

[3] Alpern, Steve; Fokkink, Robert; Gasieniec, Leszek; Lindelauf, Roy; and Subrahmanian, V. S. (2015) *Search Theory: A Game Theoretic Perspective.* Springer.

[4] Anderson, E. J., and Weber, R. R. (1990) The rendezvous problem on discrete locations. *Journal of Applied Probability*, **27**, 839–851.

[5] Gal, Shmuel. (1980) *Search Games.* Academic Press, New York.

[6] Kranakis, E., Krizanc, D., Santoro, N., and Sawchuk, C. (2003) Mobile agent rendezvous in a ring. *Proc. 23rd International Conference on Distributed Computing Systems (ICDCS 2003)*, 592-599.

[7] Schelling, T. C. (1960) *The Strategy of Conflict.* Harvard University Press, Cambridge.

[8] Stone, Lawrence D. (1989) OR Forum - What's happened in search theory since the 1975 Lanchester Prize? *Operations Research*, **37**, 501–506.

[9] Stone, Lawrence D. (1975) *Theory of Optimal Search.* Academic Press, New York.

[10] Weber, Richard. (2012) Optimal Symmetric rendezvous search on three locations. *Mathematics of Operations Research*, **37**, 111–122.

[11] West, E., Xie X., and Wierman, J. C. (2018) Rendezvous Search on the Edges of Platonic Solids: Optimal Waiting Probabilities and Multi-Step Strategies. *Congressus Numerantium*, to appear.

[12] West, E. (2018) *Rendezvous Search on the Edges of Vertex-Transitive Solids* (Master's Thesis). Johns Hopkins University, Baltimore, MD.