Chromatic Number of the Plane, Breakthroughs and Aspirations in Six Essays[1]

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[I] can't offer money for nice problems of other people because then I will really go broke... It is a very nice problem. If it were mine, I would offer $250 for it.

– Paul Erdős
Boca Raton, February 1992

Unfinished, a picture remains alive, dangerous.
A finished work is a dead work, killed.

– Pablo Picasso

1. Chromatic Number of the Plane: The Problem

I would like to bring to your attention a series of six sketches. As Picasso observes, unfinished, work remains alive. It is dangerous, for one can spend life looking for the chromatic number of the plane (CNP). And it is contagious!

On behalf of all enthusiasts of CNP, let me express gratitude to my friend, the late Edward Nelson, who created this problem at a tender age of 18 in November 1950:

> *What is the smallest number of colors sufficient for coloring the plane in such a way that no two points of the same color are unit distance apart?*

This number is called *the chromatic number of the plane*, or CNP, and is denoted by χ. In 1961, the Swiss geometer Hugo Hadwiger admitted that he was not the author of the problem, even though the name "Hadwiger-Nelson" got stuck to the problem, just as Cardano did not author the Cardano Formula, and Pythagoras Theorem was known a millennium before the great Greek was born. Such is life with credits in mathematics. Right at the problem's birth, Eddie Nelson determined the lower bound of 4, and his 20-year old friend John Isbell figured out the upper bound of 7:

[1] This is an essentially expanded version of [Soi2]

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$$\chi = 4, \text{ or } 5, \text{ or } 6, \text{ or } 7.$$

A very broad spread. Which one is the value of χ? Paul Erdős thought $\chi \geq 5$. On May 28, 2009, during the DIMACS Ramsey Theory International Workshop that I organized on request of DIMACS Director Fred Roberts and its Executive Committee, I asked the distinguished audience to determine the chromatic number of the plane by democratic means of a vote. Except for one young lady voting for 6, and I voting for 7, the rest of the workshop participants *equally* split between 4 (including Peter D. Johnson Jr.), and 5 (including Ronald L. Graham). I was therefore able to determine *the democratic value of the chromatic number of the plane*: 4.5.

Eddie Nelson, ca. 1950. Courtesy of Edward Nelson

Through the years, Ronald L. Graham offered to reward progress in this problem (I quote here from [Soi1]):

On Saturday, May 4, 2002, which by the way was precisely my friend Edward Nelson's 70th birthday, Ronald L. Graham gave a talk on Ramsey Theory at the Massachusetts Institute of Technology for about 200 participants of the USA Mathematical Olympiad. During the talk he offered $1,000 for the first proof or disproof of what he called, after Nelson, "Another 4-Color

Conjecture." The talk commenced at 10:30 AM (I attended the talk and took notes).

Another 4-Color $1000 Conjecture 3 (Graham, May 4, 2002). Is it possible to 4-color the plane to forbid a monochromatic distance 1?

In August 2003, in his talk *What is Ramsey Theory?* at the Mathematical Sciences Research Institute in Berkeley, California [Gra1], Graham asked for more work for $1000:

$1000 Open Problem 4 (Graham, August 2003). Determine the value of the chromatic number χ of the plane.

It seems that Ron came to believe that the chromatic number of the plane takes on an intermediate value, between its known boundaries, for in his two latest surveys [Gra2], [Gra3], he offers the following open problems:

$100 Open Problem 5 (Graham [Gra2], [Gra3]). Show that $\chi \geq 5$.

$250 Open Problem 6 (Graham [Gra2], [Gra3]). Show that $\chi \leq 6$.

2. A Train of Thought originated by a Paul Erdős' Conjecture

Paul Erdős in July 1975, (and published in 1976), who, as was usual with him, offered to "buy" the first solution – for $25.

Paul Erdős's $25 Problem 5.6. [E76.49]. Let *S* be a subset of the plane which contains no equilateral triangles of size 1. Join two points of *S* if their distance is 1. Does this graph have chromatic number 3?

If the answer is no – assume that the graph defined by *S* contains no C_l [cycles of length l] for $3 \leq l \leq t$ and ask the same question.

It appears that Paul Erdős was not sure of the outcome – which was rare for him. Moreover, from the next publication of the problem in 1979 [E79.04], it is clear that Paul expected that triangle–free unit distance graphs had chromatic number 3, or else chromatic number 3 can be forced by prohibiting all small cycles up to C_k for a sufficiently large k:

Paul Erdős's $25 Problem 5.6'. [E79.04]. "Let our *n* points [in the plane] be such that they do not contain an equilateral triangle of side 1. Then their chromatic number is probably at most 3, but I do not see how to prove this. If the conjecture would unexpectedly [sic] turn out to be false, the situation can perhaps be saved by the following new conjecture:

There is a k so that if the girth of $G(x_1,\ldots,x_n)$ is greater than k, then its chromatic number is at most three – in fact, it will probably suffice to assume that $G(x_1,\dots,x_n)$ has no odd circuit of length $\leq k$."[2]

Erdős's first surprise arrived in 1979 from Australia: Nicholas Wormald, then of the University of Newcastle, Australia, disproved the first, easier, triangle–free conjecture. Erdős paid $25 reward for the surprise, and promptly reported it in his next 1978 talk (published 3 years later [81.23]):

> Wormald in a recent paper (which is not yet published) disproved my original conjecture – he found a [set] S for which [the unit distance graph] $G_1(S)$ has girth 5 and chromatic number 4. Wormald's construction uses elaborate computations and is fairly complicated.

In his paper [Wor], Wormald proved the existence of a set S of 6448 (!) points without triangles and quadrilaterals with all sides 1, whose chromatic number was 4. He was aided by a computer.

Problem 12.14 (N. C. Wormald, [Wor]). Prove that the Wormald graph G is indeed a 4–chromatic girth 5 graph.

So what is so special about Nicholas Wormald's paper [Wor] of 1979? Even though independently discovered (I think), didn't he use the construction that was published 25 years earlier by Blanche Descartes [Des2]? The real Wormald's trick was to *embed* his huge 6448–vertex graph in the plane, i.e., draw his graph on the plane with all adjacent vertices, and only them, distance 1 apart. In my talk at the conference dedicated to Paul Erdős's 80th birthday in Keszthely, Hungary in July 1993, I presented Wormald's graph as a picture frame without a picture inside it, to indicate that Wormald proved the existence and did not actually draw his graph. Nick Wormald accepted my challenge and on September 8, 1993 mailed to me a drawing of the actual plane embedding of his graph. I am happy to share his drawing with you here. Ladies and Gentlemen, the Wormald Graph! (Figure 12.9).

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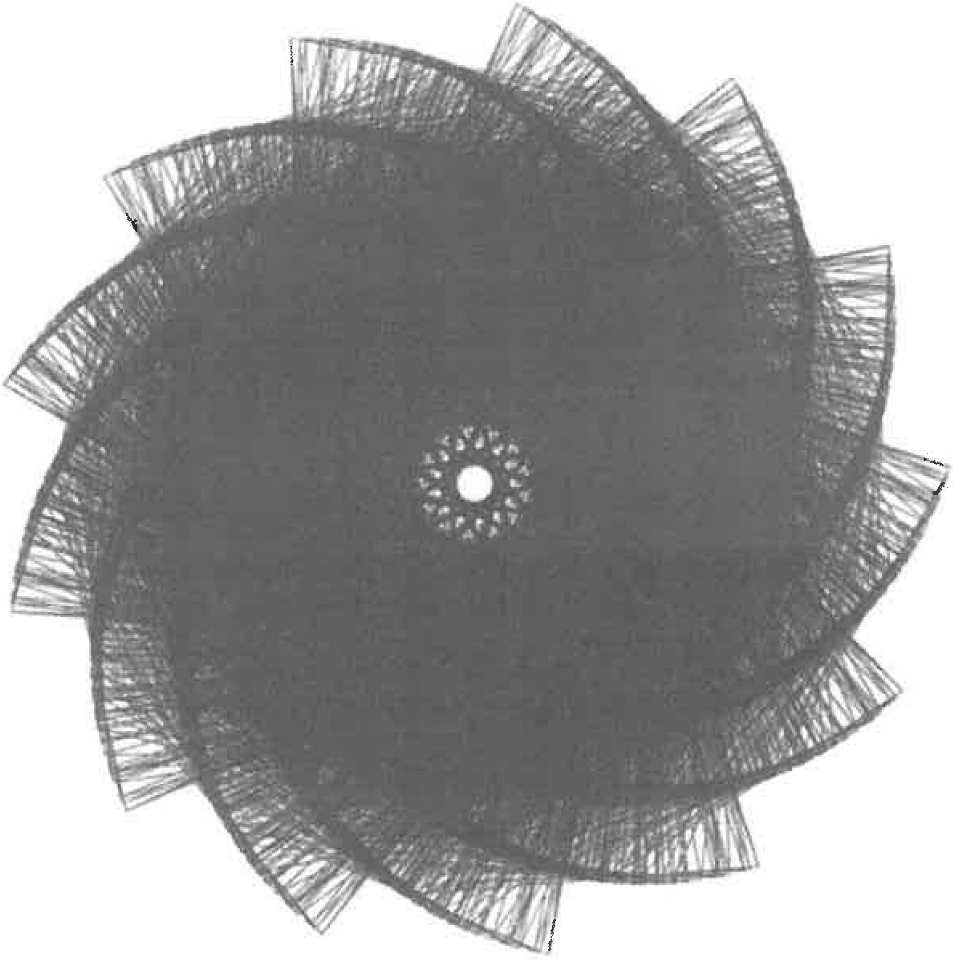
[2] The symbol $G(x_1,\dots,x_n)$ denotes the graph on the listed inside parentheses n vertices, with two vertices adjacent if and only if they are unit distance apart.

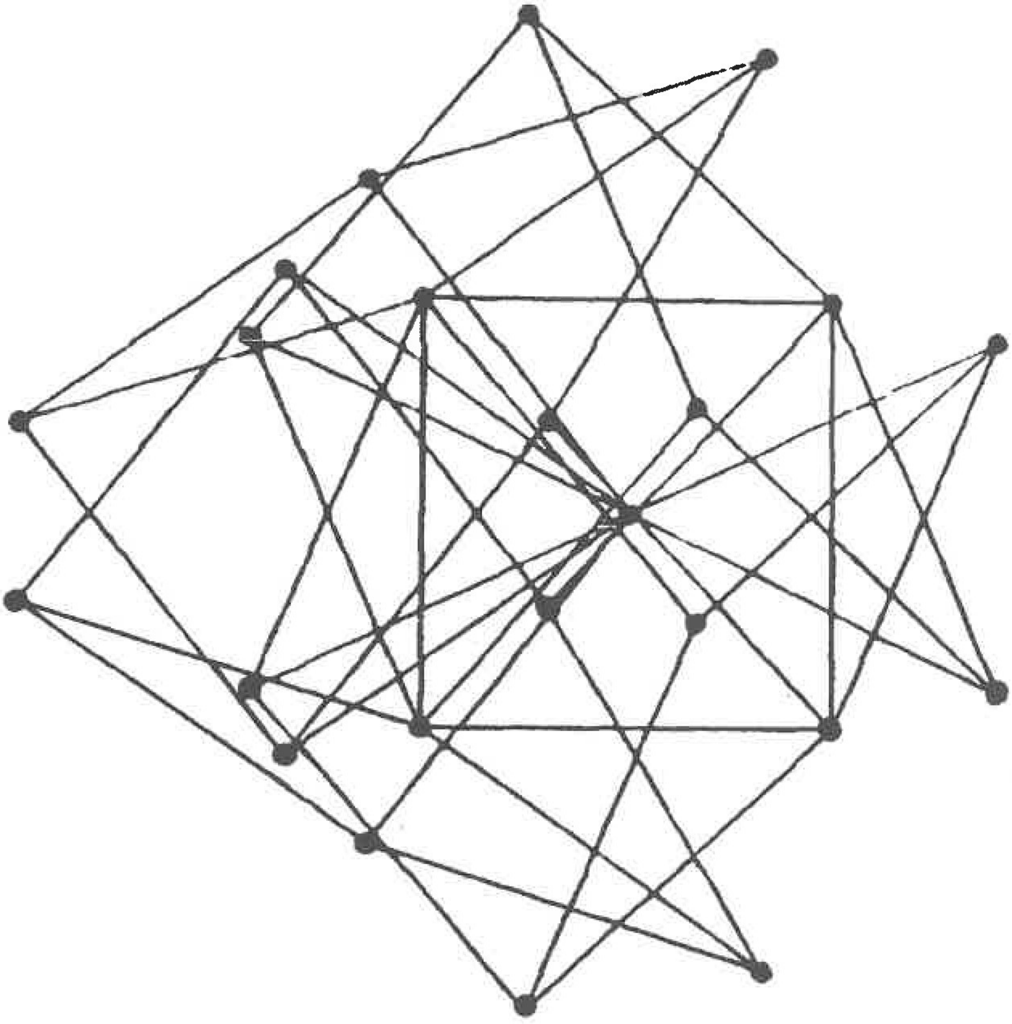


The 6448–vertex 4-chromatic of Wormald Graph of girth 5 embedded in the plane

The size of Wormald's example, of course, did not appear to be anywhere near optimal. Surely, it must have been possible to do the job with less than 6448 points! In my March–1992 talk at the Conference on Combinatorics, Graph Theory and Computing at Florida Atlantic University, I shared Paul Erdős's old question, but put it in a form of competition. In [Soi1] this problem was repeated:

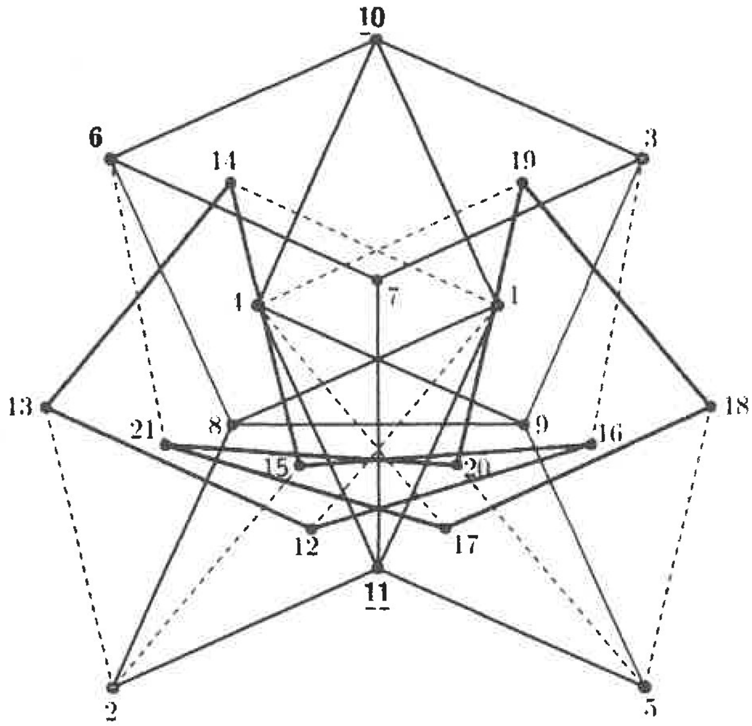
Open Problem 15.4. What is the *smallest* size of a 4–chromatic unit distance graph of girth 4?

In the Mathematical Coloring Book [Soi1] I describe the competition between Kiran Chilakamarri and Paul O'Donnell, which ended up in a remarkably small graph, constructed by the duo of the former Rutgers University roommates, Rob Hochberg and Paul O'Donnell, who in 1996 constructed what I named the Fish Graph [HO].

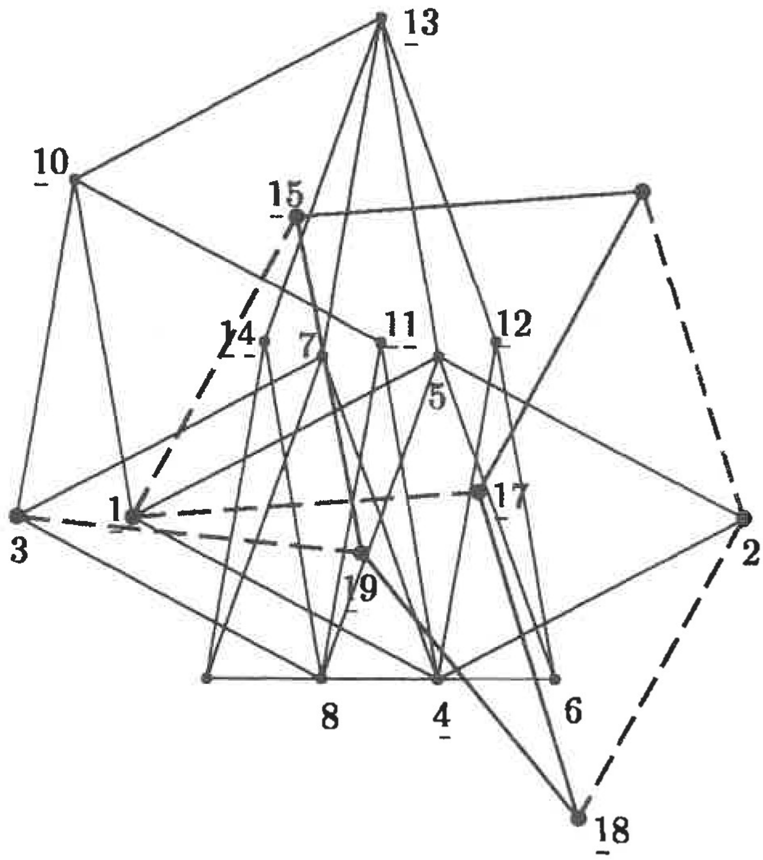


The Hochberg–O'Donnell's Fish Graph: a 23-vertex triangle-free 4-chromatic unit-distance graph

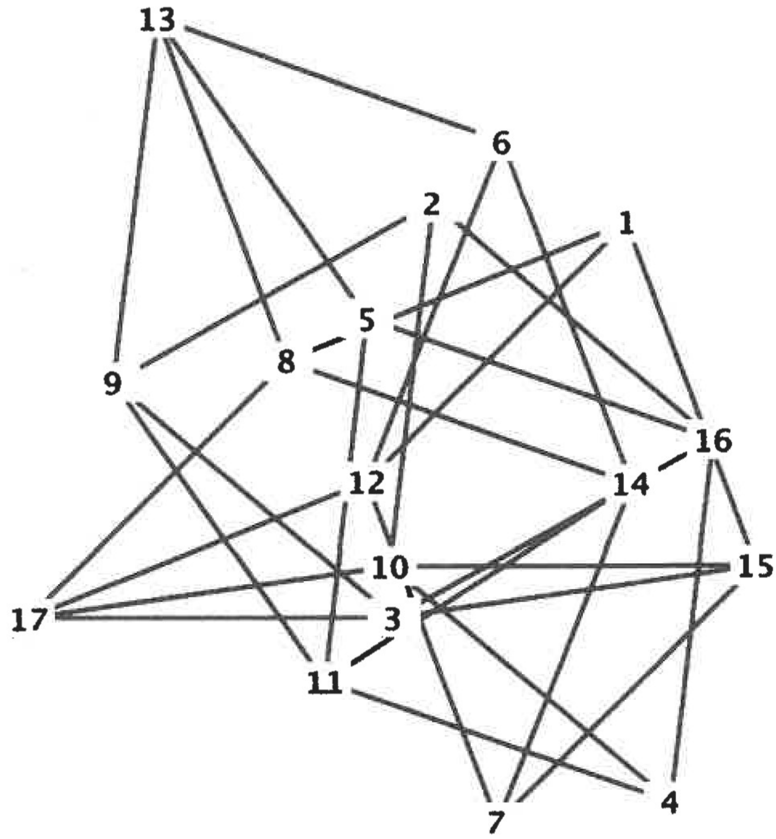
This record stood for two decades. *Twenty Years After*, as Alexandre Dumas named his sequel to *The Three Musketeers*, using the approach of Hochberg and O'Donnell, Geoffrey Exoo and Dan Ismailescu constructed consecutively triangle-free 4-chromatic unit distance graphs on 21, 19, and 17 vertices, respectively. They then proved that the value 17 cannot be reduced, thus, completely solving my problem 15.4. Let me introduce you to their three graphs.



The Exoo-Ismailescu triangle-free 4-chromatic unit distance graph
of order 21



The Exoo-Ismailescu triangle-free 4-chromatic unit distance graph of order 19



The Exoo-Ismailescu World Record Graph:
A triangle-free 4-chromatic unit distance graph of order 17

I had the honor to be a member of Paul O'Donnell's Doctoral Defense Committee at Rutgers University, together with Endre Szemerédi, Michael Saks, and János Komlós. The crux of Paul O'Donnell's thesis (see it in [O'D]) was the existence of four-chromatic unit-distance graphs of arbitrary girth, which constituted a complete solution Paul Erdős's problem 5.6, and delivered to Paul Erdős an ultimate surprise by negatively answering his general conjecture:

O'Donnell's Theorem 5.8. ([O'D], [Soi1]]). There exist 4–chromatic unit distance graphs of arbitrary finite girth.

Ron Graham's assessment of this important result was natural [Gra7], [Gra8]): he cites O'Donnell's Theorem 45.4 as "perhaps, the evidence that the chromatic number of the plane is at least 5."

Chromatic Number of the plane is my favorite open problem in mathematics. Everyone I know likes this problem very much. Paul Erdos called it "a very nice problem." There is only one exception known to me. After struggling for 5 years with trying to cook up a negative review of *The Mathematical Coloring Book*, Günter Ziegler wrote in *Jahresbericht der Deutschen Mathematiker-Vereinigung*:

> The chromatic number of the plane: Is this a good problem? Again this is a question of taste. In my view the fact that there is so little progress on the original problem in so many years, and progress only on variations, and that the answer might depend on set theory all indicate that it is not a productive, helpful problem.

Ziegler's logic is of course absurd. The indication that the problem is hard and consequently takes a long time to be conquered, he uses as a 'proof' that it is bad! If "the fact that there is so little progress on the original problem in so many years" were to imply a bad problem, then all the great classic problems of mathematics would be bad, from Fermat's Last Theorem, which required ca. 360 years, to the Poincare Conjecture, the Goldbach Conjecture, the Riemann Hypothesis, the P versus NP Problem, the Hadwiger Conjecture, etc.

Ziegler's taste puts him perhaps in the minority of one, when he suggests that CNP problem is bad. Paul Erdős and Ronald L. Graham liked CNP problem and included it in their numerous problem papers and talks. CNP problem was selected for the inclusion in the well-known problem books "Unsolved Problems in Geometry" by Croft–Falconer–Guy, Springer, 1991, and "Old and New Unsolved Problems in Plane Geometry and Number Theory" by Klee–Wagon, Mathematical Association of America, 1991. I was invited to write Chapter 8 on CNP and related problems for the book "Topics in Chromatic Graph Theory", Cambridge University Press, 2015, by Lowell W. Beineke and Robin J. Wilson (editors). The Nobel Prize (1994) and Abel Prize (2015) winner John F. Nash, Jr. invited me to write a chapter on CNP for the 2016 Springer book edited by Nash and Michael Rassias on famous "Open Problems in Mathematics," where other chapters were dedicated to such celebrated classic unsolved problems as the Riemann Hypothesis, the Goldbach Conjecture, the P versus NP Problem, the Hadwiger Conjecture, etc.

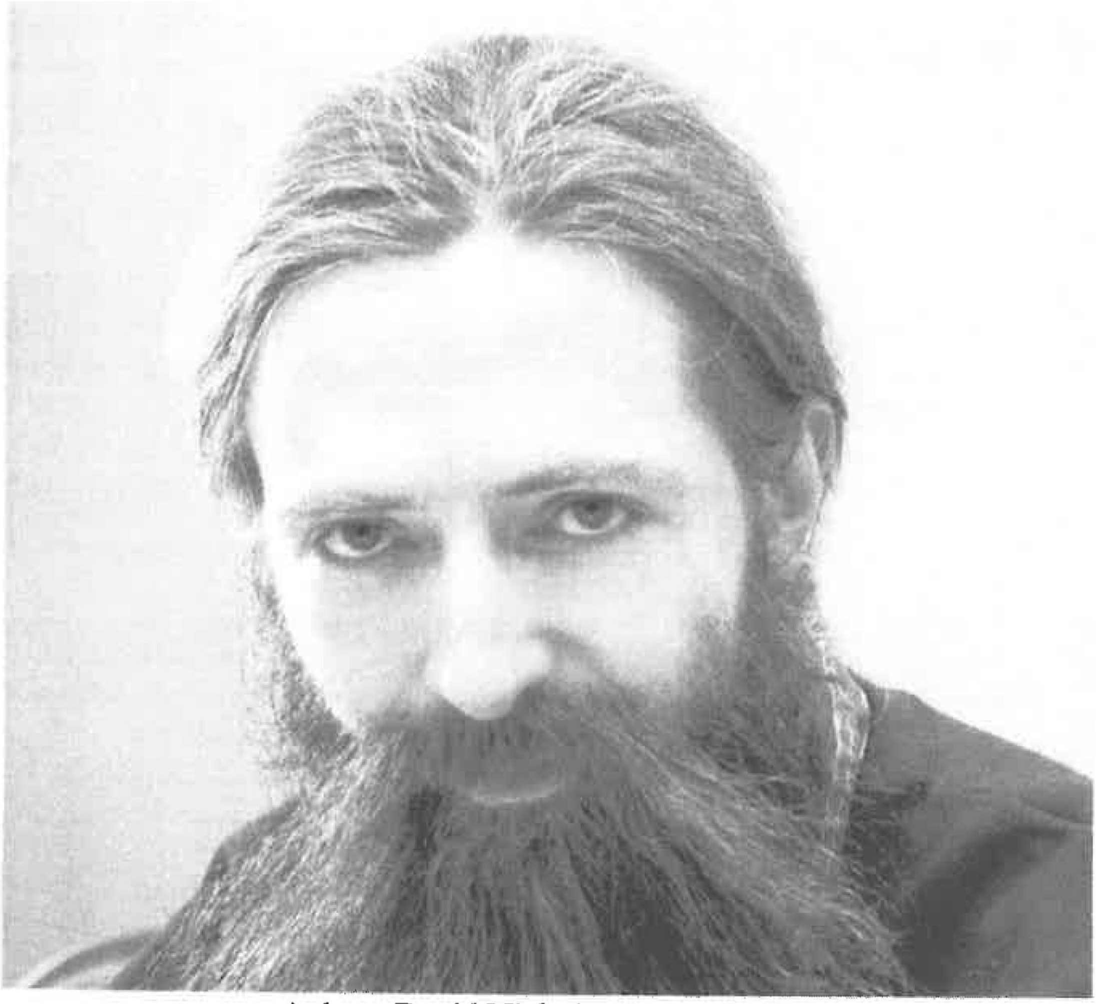
To further disappoint Ziegler, the year 2018 saw breathtaking breakthroughs in the general case of CNP problem.

The train of thought I presented above, paid the main attention to our ability to embed a unit-distant graph in the plane. This in turn prompted elimination of rigidity of an equilateral triangle and thus Mosers' Spindle. One man from the outside of mathematics, who read *The Mathematical Coloring Book*, was not sold on the necessity of girth 4. Free from preconceived notions, he achieved a major breakthrough.

3. Aubrey D.N.J. de Grey's Breakthrough

In the 68 years of the problem's life, many fine mathematicians obtained many beautiful results in special circumstances [Soi1]. However, no progress has occurred in the general case until Aubrey de Grey, a Ph.D. in Biology from the University of Cambridge, succeeded in a joint effort of his imagination and a clever computer program he wrote for this purpose. On January 16, 2018, de Grey sent me the first version, followed by a corrected one on April 7, 2018 with the following introductory note:

> I append a copy of an email I sent you in January, which you may have overlooked. It's good that you did, because the graph that I told you about is in fact 4-colorable after all, and my failure to discover this was due to a bug in my code. However, I'm pleased to report that after fixing the bug I was rapidly able, using the same basic approach, to find somewhat larger unit-distance graphs that do indeed have no 4-colourings. My confidence that my code is not still lying to me arises largely from the fact that the 4-colouring of the earlier graph was found by Dr. Robert Hochberg, to whom I wrote at the same time as you; he became interested enough to spend time writing code that could test quite large graphs, and he has not found a 4-colouring of the (progressively smaller, but still four-digit) graphs I have been discovering since January even though his code can 4-colour all his previous attempts at 5-chromaticity in seconds. We appreciate that this is not a proof... but it makes us feel good enough about the graphs that I have now written up the discovery as a paper. I have just submitted it to the arXiv and it is scheduled to go live on Monday. I still very much hope that you will be inclined to consider it for publication in *Geombinatorics*; I am in absolute awe of your 2008 book and I hope that this might serve as some sort of mark of my gratitude.



Aubrey David Nicholas Jasper de Grey

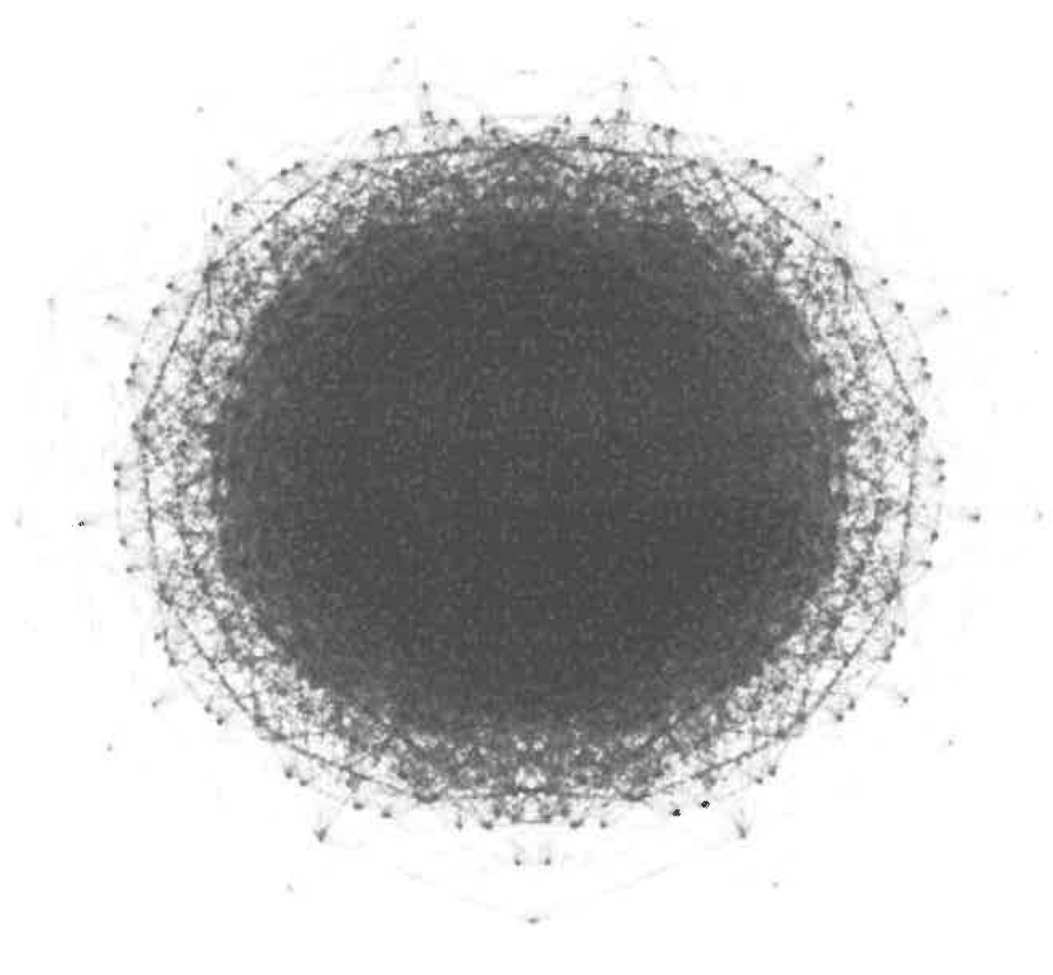
As I mentioned, previous pursuits of a 5-chromatic unit-distance graph (see O'Donnell [O'D], for example) were first of all concerned with embeddability of unit-distance graphs in the plane, and thus put emphasis on unit-distance graphs without 3-cycles. No triangles, of course, means no Mosers' Spindles. De Grey goes the opposite way: he starts with a piece of a regular triangular lattice and superimposes its images under various rotations. As a result, he floods his construction with high density of Mosers' Spindles and of edges. His goal is to force a certain coloring of small number of vertices and then create contradictions to those forced colorations.

He constructs a unit-distance graph on 20,425 vertices [G]. Can one check whether it is 4-colorable? Normally one would not even try. But de Grey bravely goes for it, and with a clever use of specific properties of his graph succeeds in verifying that this giant graph is indeed not 4-colorable – all on his MacBook Air using his custom program he wrote for Mathematica 11. Hence, we arrive at

De Grey's Theorem [G]. $\chi \geq 5$.

De Grey then dramatically reduces the size of his 5-chromatic unit-distance graph; see it in *Geombinatorics* [G]. It also decorates the cover of this issue. In short, he identifies vertices of his large graph, removal of which does not affect the required properties and thus the chromatic number of the graph. He also finds new vertices whose addition allows the removal of more than one preexisting vertex. This permits him to shrink the size of his 5-chromatic unit distance graph by a factor of almost 13, to the 1581-vertex graph G. De Grey then presents his new graph in an artistically beautiful embedding.

De Grey's Example. There is a 5-chromatic unit-distance graph on 1581 vertices.

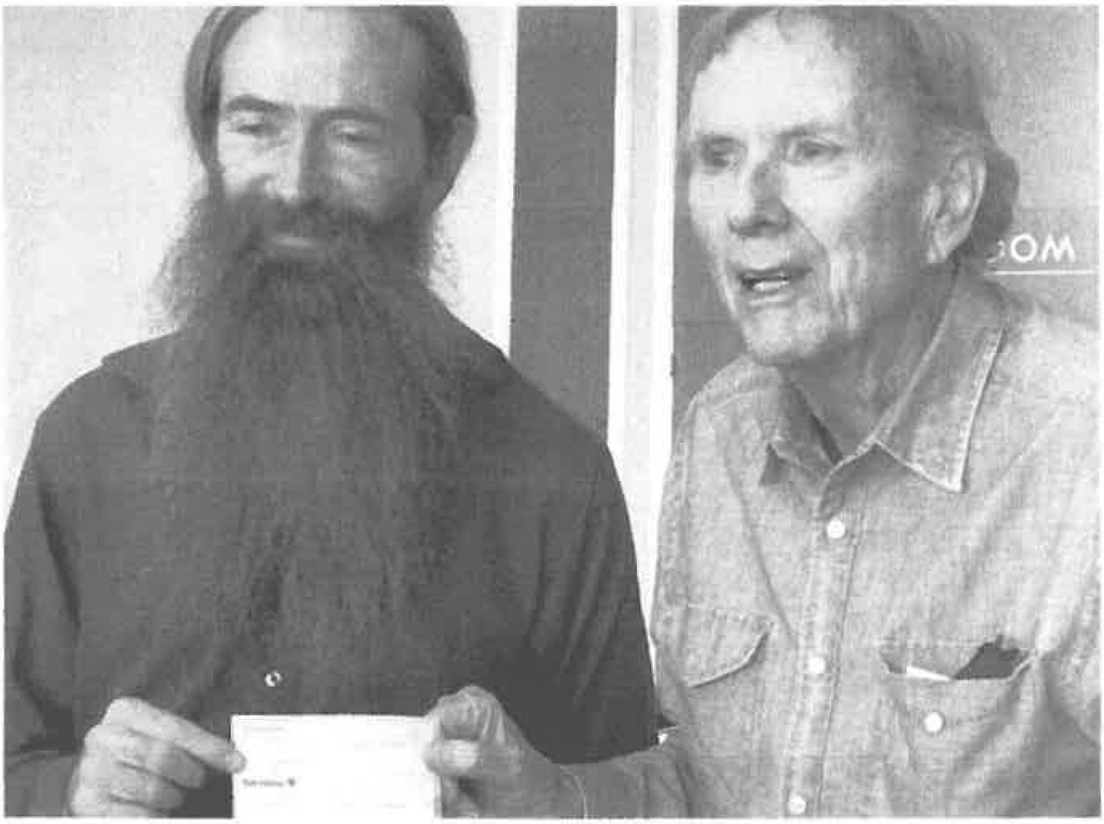


The de Grey Graph: A 1581-vertex 5-chromatic unit-distance graph

With this size, the verification becomes within the reach of his computer program. Heule observes [H] that the de Grey Graph is almost minimal: at most 4 vertices can be removed without introducing a 4-coloring of the remaining graph.

What does Aubrey do next? I knew many colleagues, who would keep their approach in secret, or worse, would publish hardly comprehensible description – in order to position themselves ahead of the competition. Aubrey de Grey is a true scholar. He does not wish to compete with others, but rather invites them to join in to conquer mathematics herself. He succeeds in commencing a Polymath project where new blood is attracted to try their twist on the problem. And try they do.

As to Aubrey de Grey, he receives the $1,000 prize from the hands of Ronald L. Graham by solving Ron's $1,000 Problem posed on May 4, 2002:



Ronald L. Graham presents Aubrey de Grey the Prize: $1,000
September 22, 2018

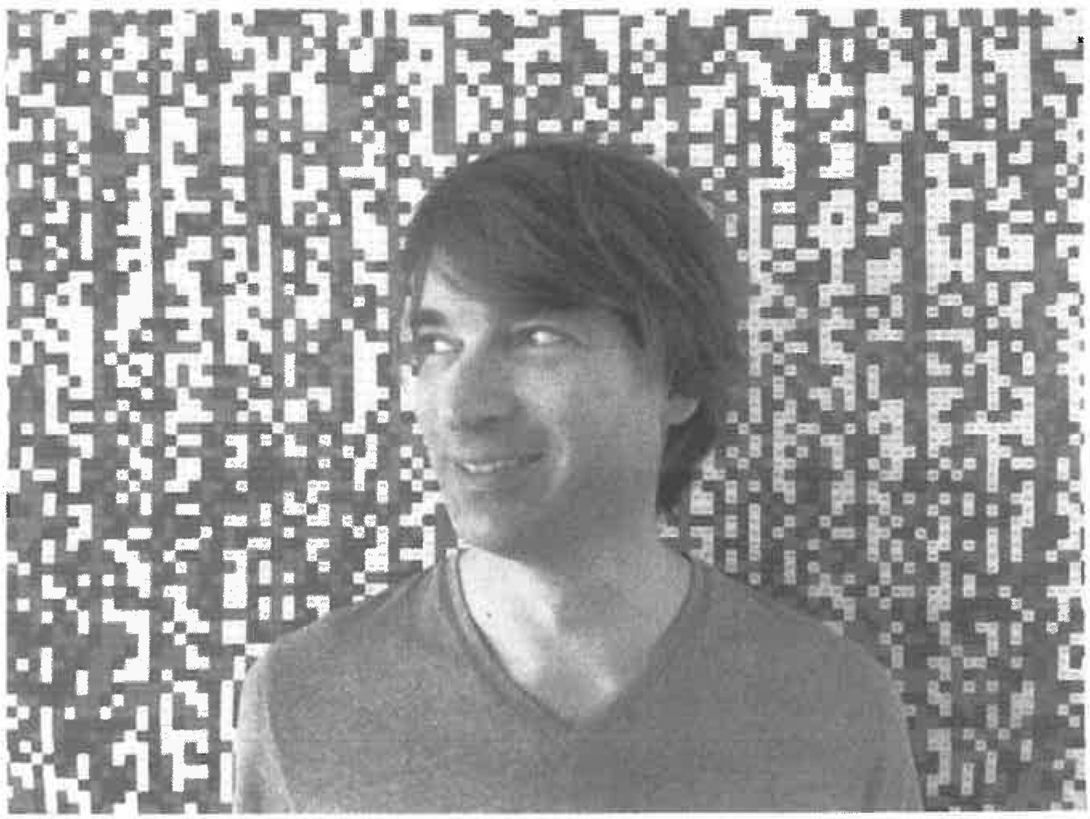
"I will certainly be adhering to the convention of framing the check rather than cashing it," wrote Aubrey.

I would like to pose here Ron's next $1,000 problem, subject to his approval, of course (breaking news: Ron has approved!):

The Ronald L. Graham New $1,000 Problem. Prove or disprove the existence of a 6-chromatic unit-distance graph.

In 1991, I started *Geombinatorics*, jointly with the great geometer Branko Grünbaum, in the style of Paul Erdős' problem-posing talks and essays. I found existing mathematical journals to be like cemeteries for honorable burials of finished research. My goal was to publish research in progress, so that people can join in the efforts. Clearly, *Geombinatorics* has been a precursor to Polymath-type blogs. With your contributions and enthusiasm, it will continue to be our meeting place, the melting pot of ideas.

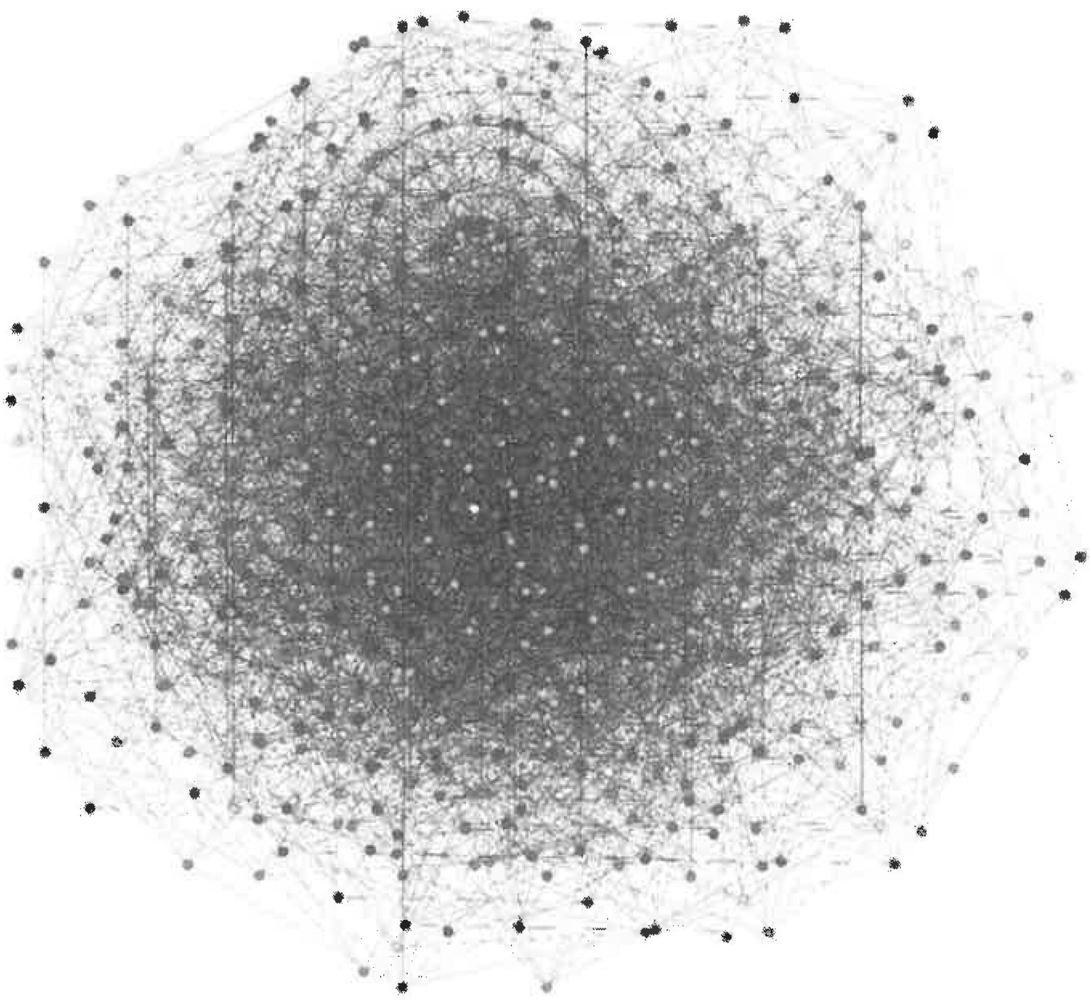
4. Marijn J.H. Heule



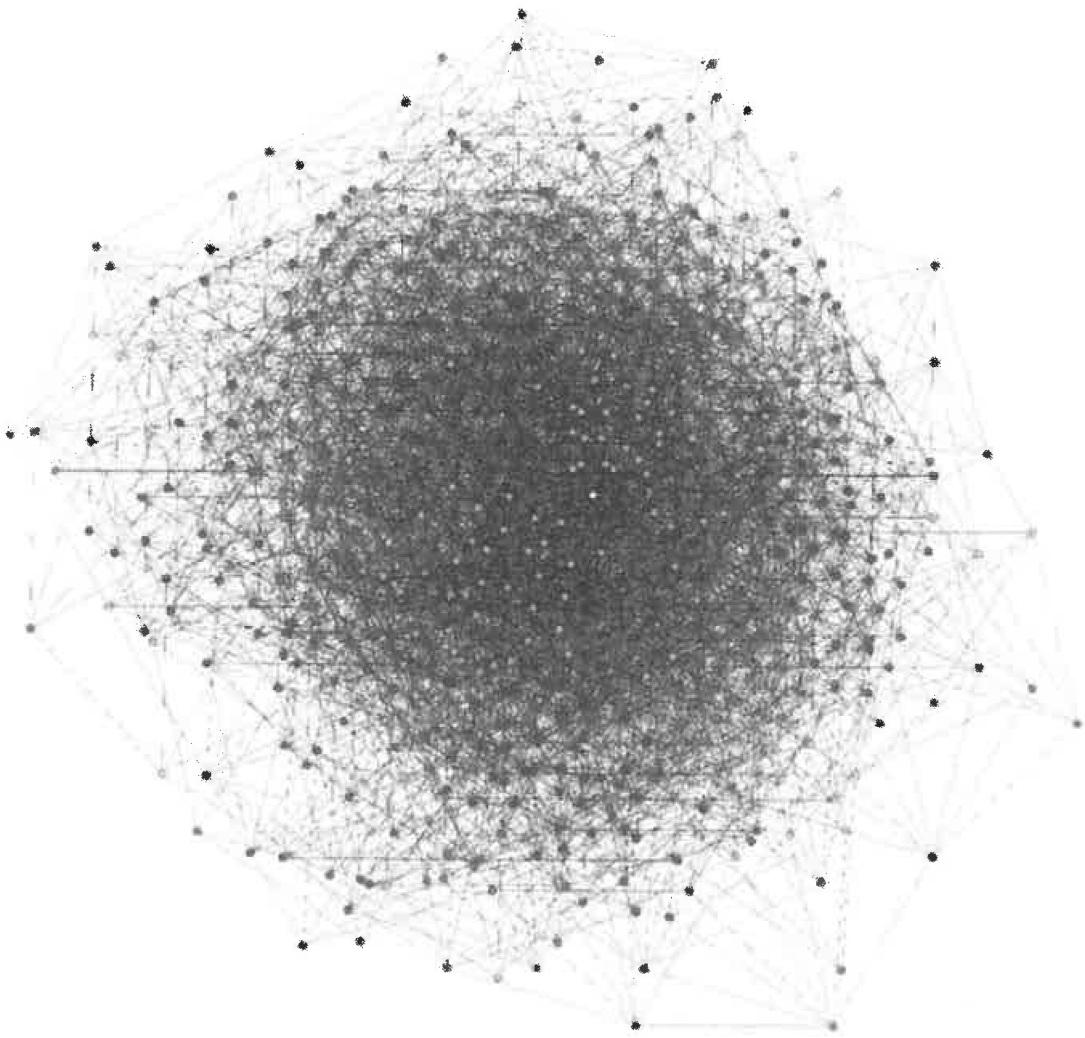
Marienus Johannes Hendrikus Heule

Marijn Heule, a virtuoso computer scientist, who does not always rely on existing software but rather creates his own. We are fortunate that he became excited about CNP problem, for he has produced a series of world records for the smallest known 5-chromatic unit-distance graphs, with apparently no one able to compete with him in this endeavor. Let us document for posterity the succession of his 5-chromatic unit-distance graph records [Soi2].

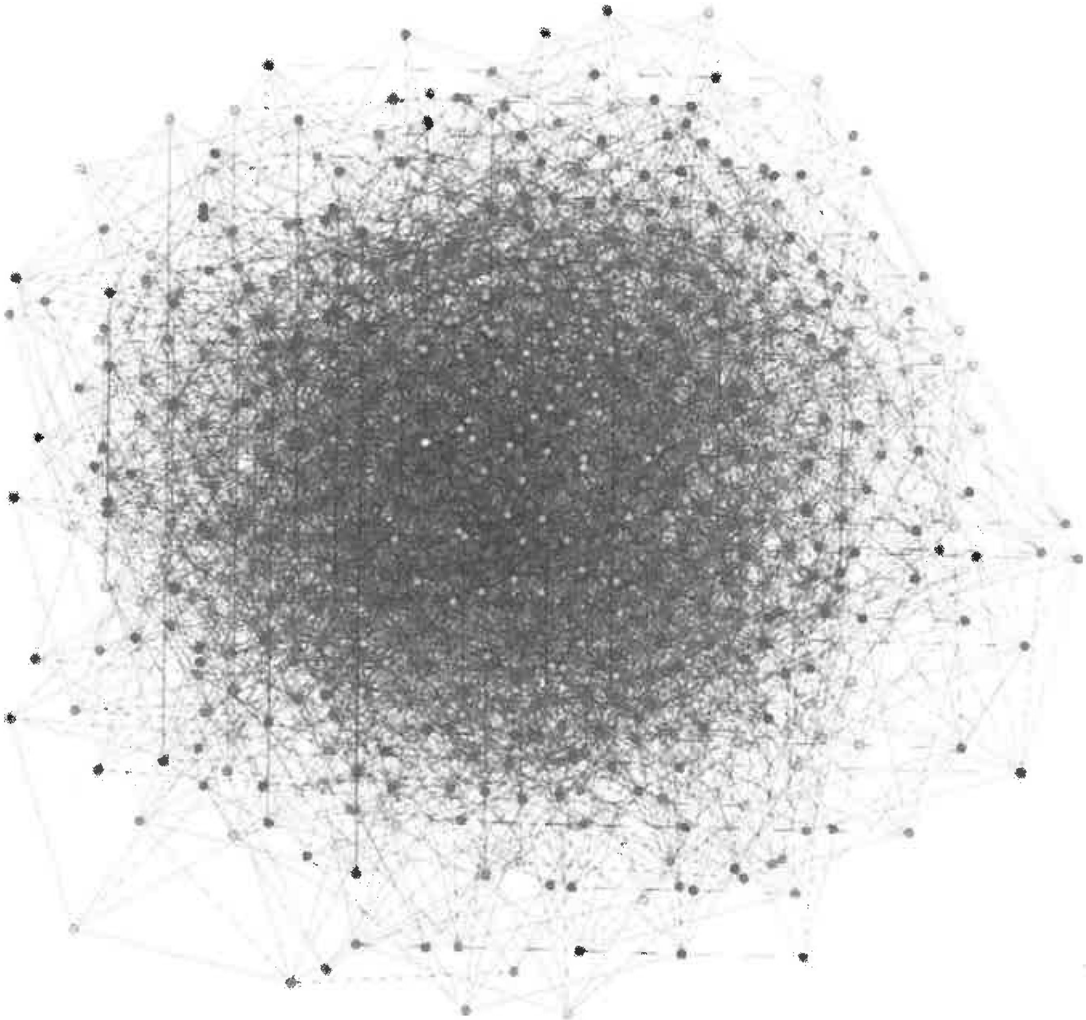
Heule's Example 1. 874 vertices and 4461 edges. Created on April 14, 2018.



Heule's Example 2. 826 vertices and 4273 edges. Created on April 16, 2018

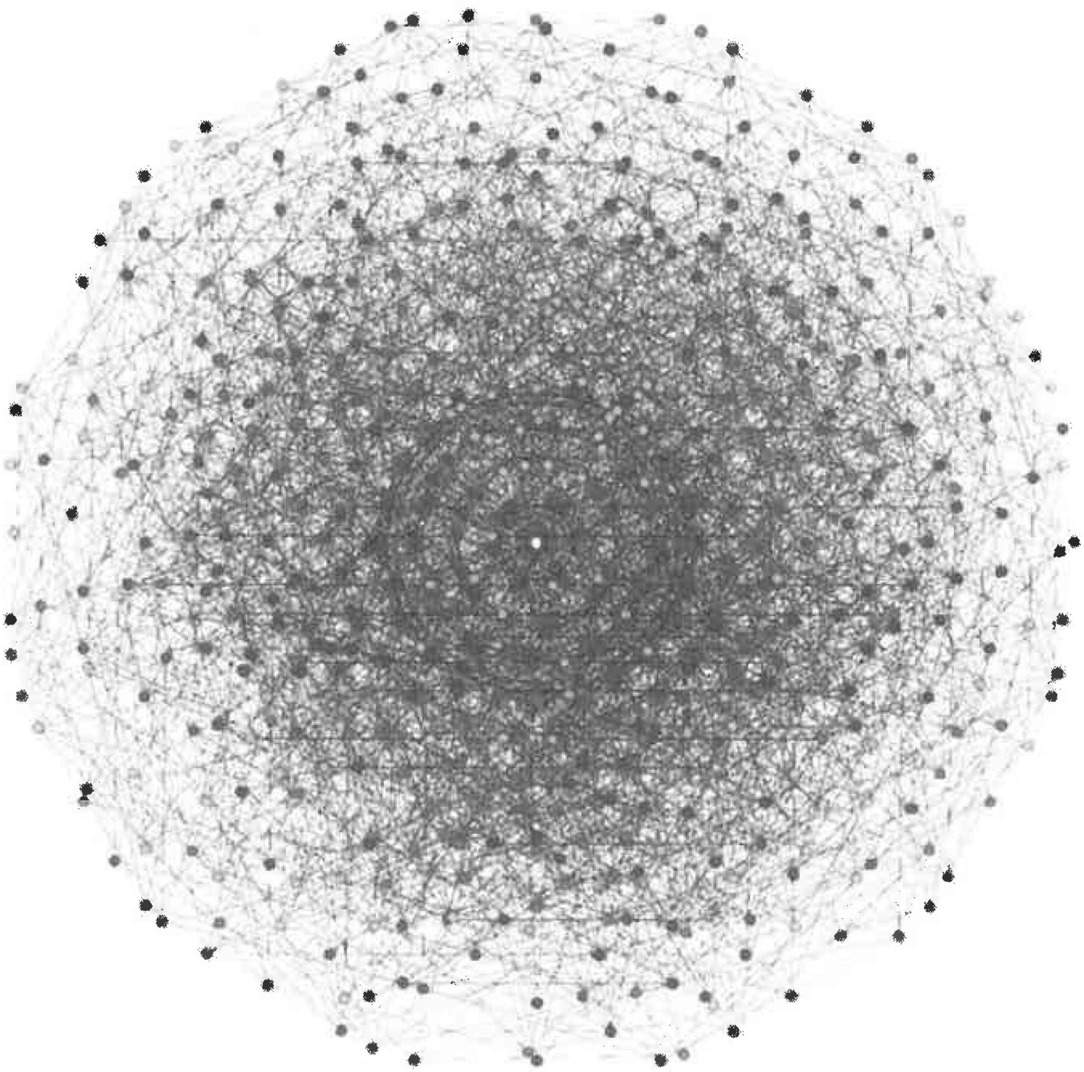


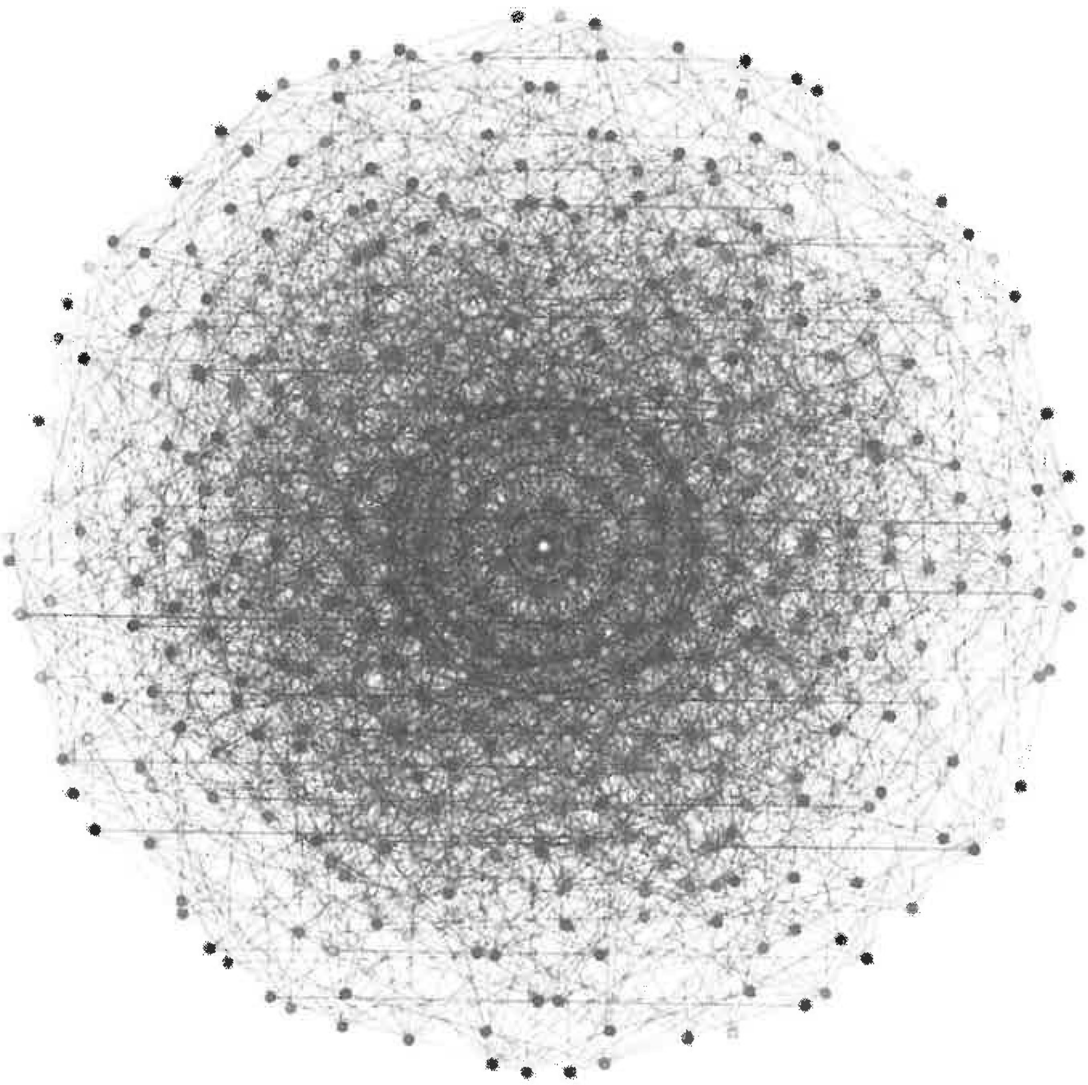
Heule's Example 3. 803 vertices and 4144 edges. Created on April 30, 2018.



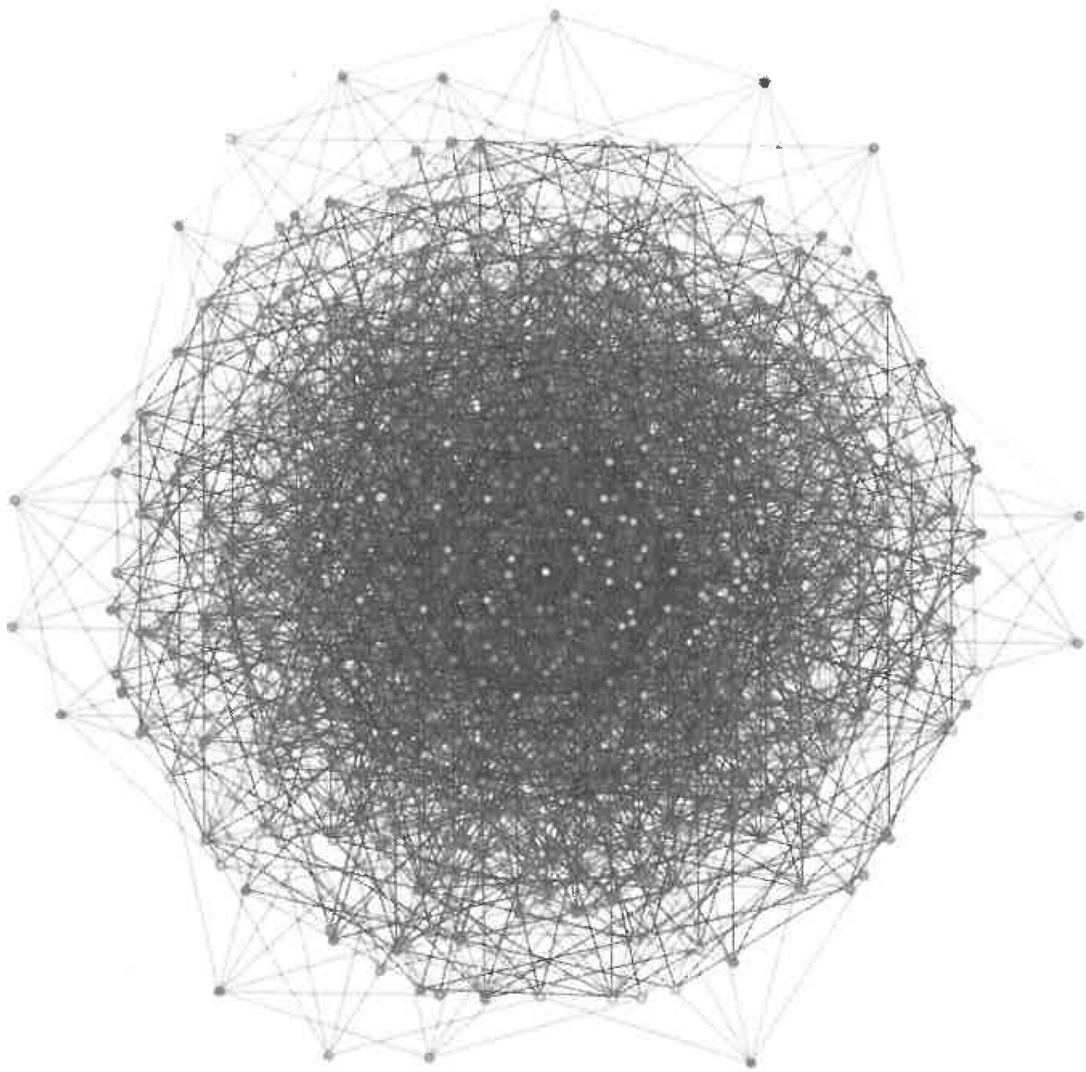
Heule's Example 4. 633 vertices and 3166 edges. Created on May 6, 2018.

Heule's Example 5. 610 vertices and 3000 edges. Created on May 14, 2018





Example 6 [H1]: a 5-chromatic unit-distance graph on 553 vertices with 2722 edges; May 18, 2018.



A visualization of a 553-vertex unit-distance graph with chromatic number 5.

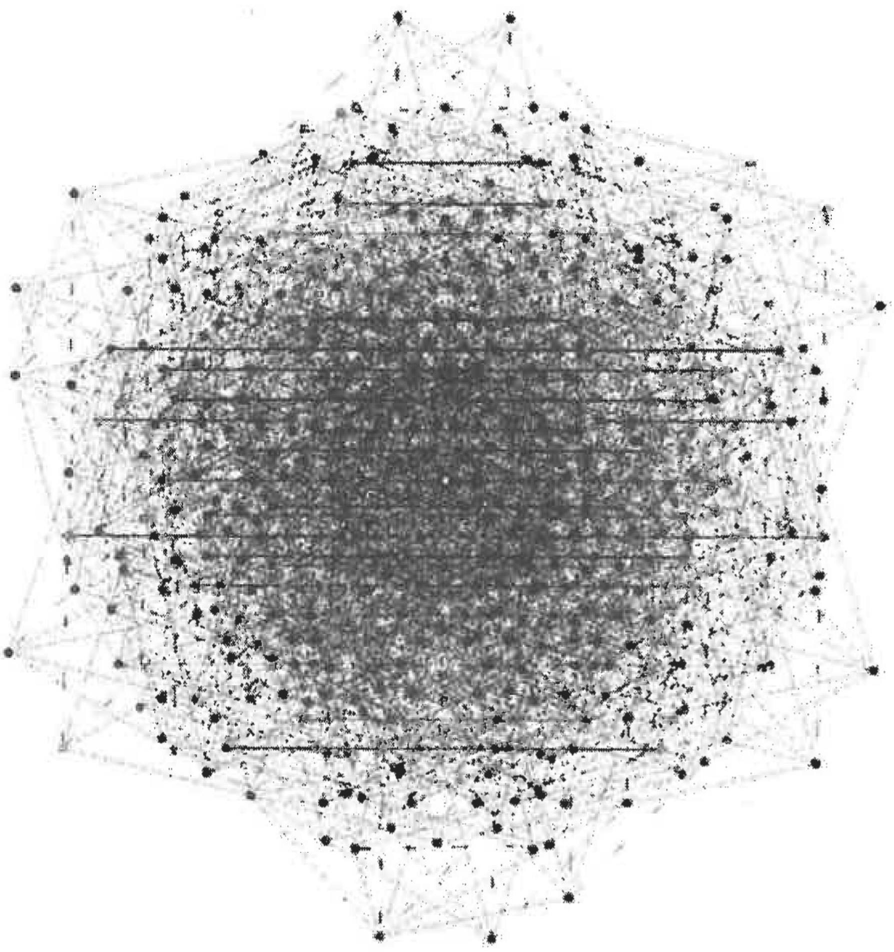
Recently Marijn informed me of an even smaller unit-distance 5-chromatic graph. Let me quote his February 15, 2019 e-mail, so that we could appreciate the difficulty and the effort that went into discovering this graph:

> I was able to reduce smallest UD graph with chromatic number 5 to 529 vertices? Attached is an illustration of that graph. I spend a few thousand CPU hours to find it. This seems the best what I can do with the current methods.

Example 7: The New World Record [H2]. A 5-chromatic unit-distance graph on 529 vertices with 2670 edges; September 19, 2018.

To my question about this graph's symmetries, he replied as follows:

The graph has almost a rotation by 120 degrees. Enforcing this symmetry adds only a few vertices.



A visualization of a 529-vertex unit-distance graph with chromatic number 5. Five colors are used for the vertices. Only the center uses the 5th color (white). In fact, all Heule's graphs are vertex critical. This implies that there exists a coloring in which every vertex can be the only one with the fifth color. On Aubrey's suggestion, Marijn let the central vertex to be the only one of the 5th color.

"I spent a few thousand CPU hours to find it [the 529-vertex record]. This seems the best what I can do with the current methods," writes Marijn to me on February 15, 2019.

Heule shares his approach [H]:

> The techniques were originally designed for verification purposes and applying them to graph minimization is novel and unexpected...
>
> Our method exploits two formal-methods technologies: the ability of *satisfiability (SAT) solvers* to find a short refutation for unsatisfiable formulas (if they exist) and *proof checkers* that can minimize refutations and unsatisfiable formulas.

Heule ends his essay [H] with the following conclusions:

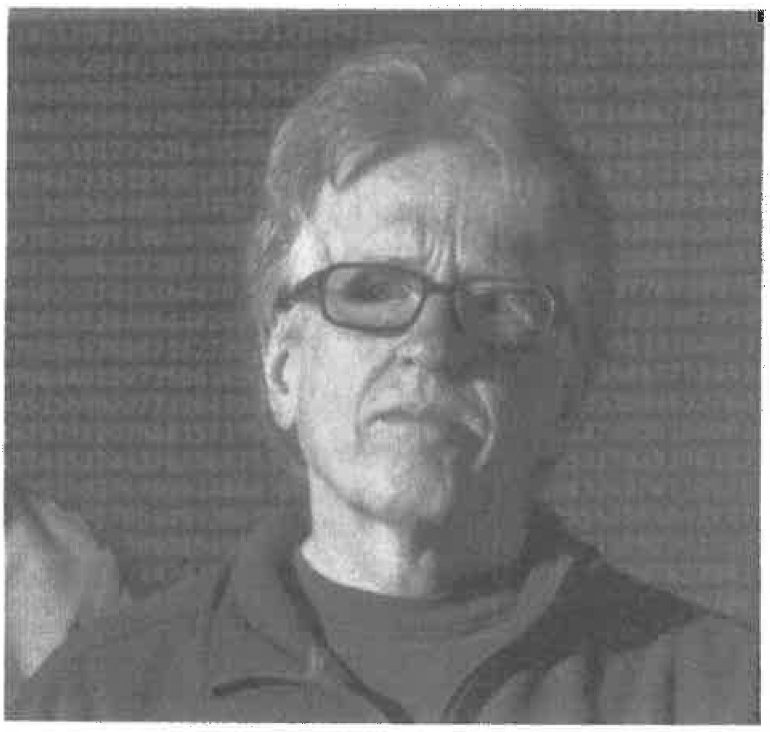
> Applying clausal-proof techniques to provide mathematical insights is an interesting twist in the discussion about the usefulness of mechanized mathematics.
> Finally, all graphs used in our experiments could be easily colored with 5 colors, even the ones with many thousands of vertices. However, we observed that this does not hold for the graph $(S_{199} \oplus S_{199}) \cup \theta_4 (S_{199} \oplus S_{199})$. This graph is 5-colorable, even when requiring two colors for the central vertex, but computing such a coloring is expensive. Consequently, such colorings may be rare and thus may contain certain patterns. This could point to the existence of unit-distance graphs with chromatic number 6 with thousands of vertices.

Paul Erdős' reply to me regarding Appel-Haken proof of the Four-Color Conjecture, is applicable here too:

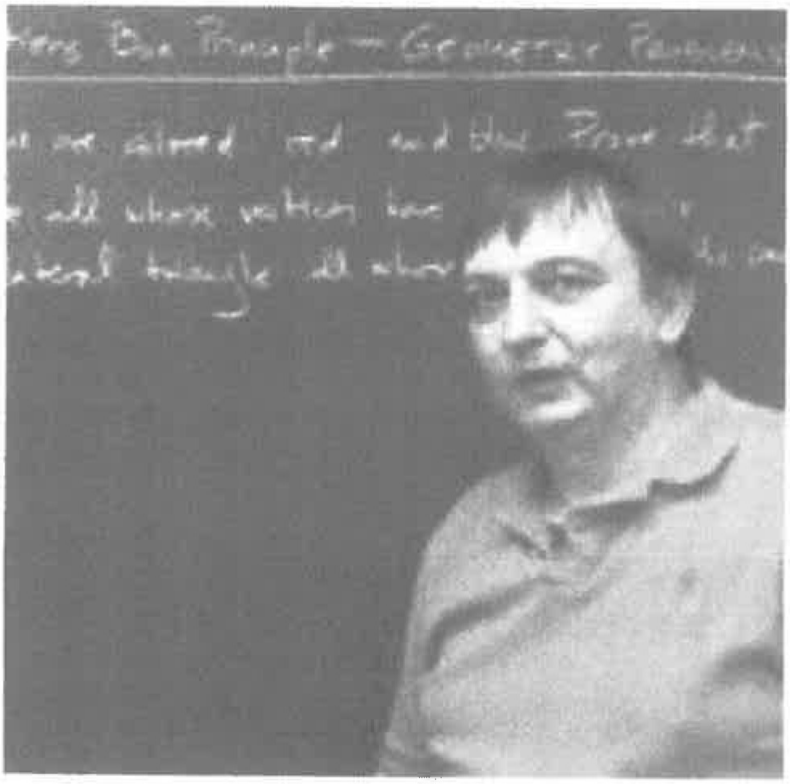
> *I prefer a computer-free proof of the Four-Color Conjecture, but I am willing to accept the Appel-Haken solution. Beggars cannot be choosers.*

A computer-free proof is preferred only because we are a curious bunch and a human-checkable proof *may* shed light on why things are the way they are.

5. Geoffrey Exoo and Dan Ismailescu



Geoffrey Exoo



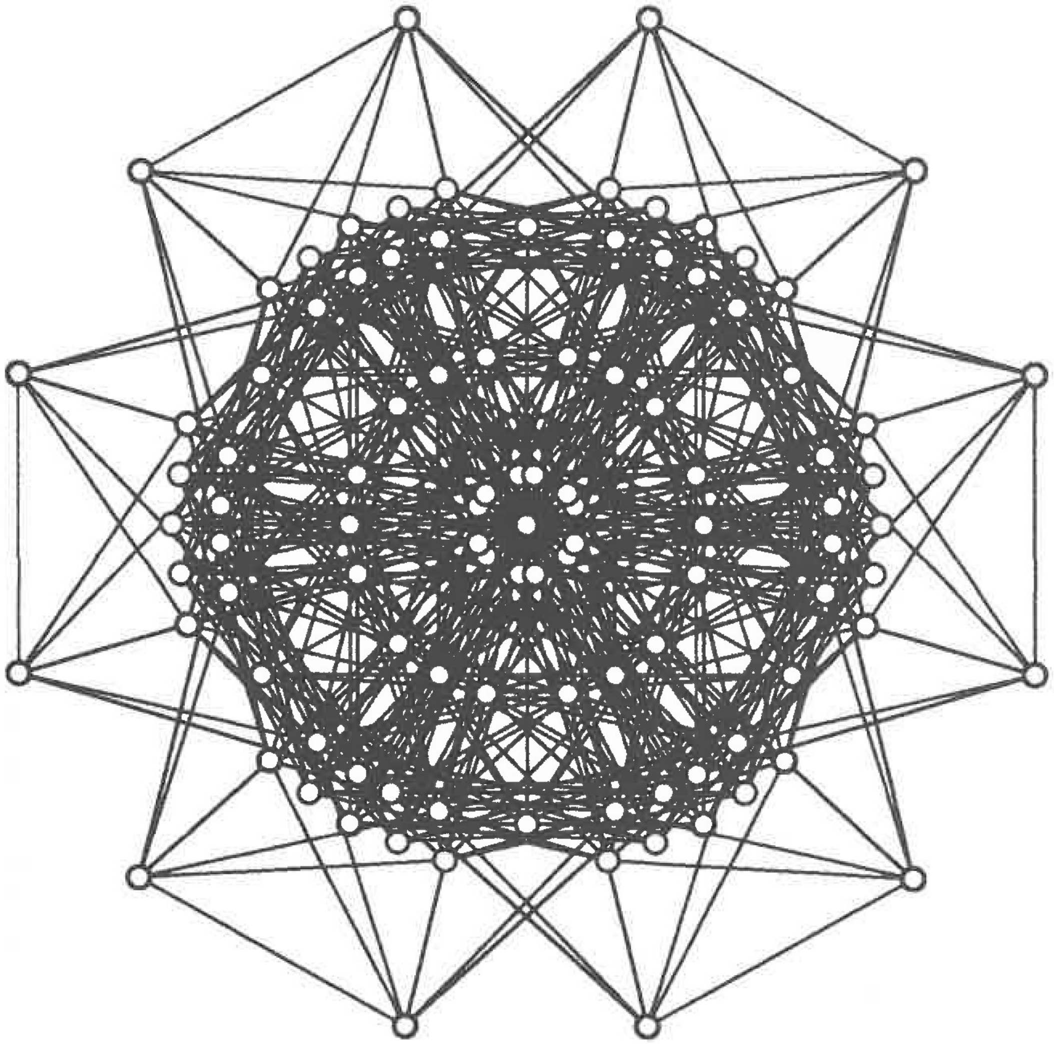
Dan Ismailescu

We have a team of a *Geombinatorics Editor* Geoffrey Exoo and Dan

Ismailescu. As you already know, this duo – which never met in person! – not only created a 4-chromatic unit-distance graph of girth 4 on 17 vertices, but also proved that 17 is best possible, thus settling [EI] an important open problem 15.4 I posed in [Soi1].

Geoffrey tells me – and I certainly trust him – that they had their 5-chromatic unit-distance graph at about the same time as Aubrey de Grey, in January 2018. They then took their time to improve construction, reduce its size, and … missed the boat. Having acknowledged – as they ought – de Grey's priority, they published their graph elsewhere.

In [EI2], the duo is building tools clearly aimed at constructing a 6-chromatic unit-distance graph. They construct a good number of 5-chromatic dual-distance graphs, i.e., two distances are declared that may not appear monochromatically when the graph's chromatic number is determined. The examples are large and small. Let me show you a large one, for its aesthetic beauty and symmetries:



A 103-vertex $\{1,\ 2/\sqrt{3}\}$-graph with chromatic number 5

The authors explain their reason of studying dual-distance graphs and formulate the following conjecture:

The Exoo-Ismailescu Conjecture [EI2]. There exist values $d \neq 1$ such that

$$\text{If } \chi(E^2, \{1\}) = 5, \text{ then } \chi(E^2, \{1, d\}) = 5.$$

The authors conclude:

> If one can then find a 6-chromatic {1, d}-graph, the above conjecture would immediately imply that χ (E2) ≥ 6. At this time however, no such graphs are known.

6. The Road Ahead

Of course, we all interested in further reducing the size of the smallest 5-chromatic unit distance graph. We have a World Series of mathematical kind.

5-Chromatics World Series. Find the smallest known 5-chromatic unit-distance graph.

Marijn Heule has already achieved the order of 529. The title of the next problem rhymes:

CNP – Triangle-Free. Construct a triangle-free 5-chromatic unit-distance graph.

Book-Prize 5-Chromatic Triangle-Free Competition. Find the smallest 5-chromatic triangle-free unit-distance graph.

Yes, as a prize for the smallest graph I am offering a copy of the second expanded edition of *The Mathematical Coloring Book*, which should appear in Springer, New York, in 2020 or so.

Why do I pose this problem when the smallest size of a graph in the first problem (without a non-triangle condition) is lower or equal to the size of the graph with this condition?

The triangle-free condition makes the graph infinitely embeddable. Moreover, cited above Exoo-Ismailescu result allows for a relatively small building block: the smallest unit-distance 4-chromatic graph has only 17 vertices, whereas without a unit-distance requirement, the Grötzsch graph is not much smaller at 11 vertices. I therefore believe that in triangle-free unit-distance 5-chromatic graphs, we will succeed in lowering the order of the graph faster than in the general case – we could do it, I trust, 'in real time.'

There is a new energy in the air, promising a 6-chromatic unit-distance graph in not too distant future. Some scholars tell me that they commence working on reducing the upper bound of χ from 7 to 6, and the upper bound for the chromatic number of 3-space from 15 to 14. Nevertheless, I stand by my old conjectures:

CNP Conjecture for the Plane [Soi1]. $\chi = 7$.

CNP Conjecture for E^3 [Soi1]. $\chi(E^3) = 15$.

The genius things are often simple. In the end, simplicity is what we are after. Boris Pasternak (*The Waves*, 1931)[3] comes to mind:

Involved in kinship with all living,
Foreseeing future day by day,
As heretic, I choose amazing
Simplicity in every way.

We shall abandon faith in mercy
When one Simplicity reveals –
Though it's essential to peoples,
Complexity is yet their will.

And so, here is my old simple conjecture for the general case:

CNP Conjecture for the Euclidean *n*-space E^n [Soi1]**:**

$$\chi(E^n) = 2^{n+1} - 1.$$

When will we learn the exact value of CNP? Niels Bohr chuckled "It is hard to predict, especially the future." And yet in 1991, the late Victor Klee & Stan Wagon [KW] braved the prediction:

> *If Problem 8 [CNP] takes that long to settle [as the Four-Color Conjecture], we should know the answer by the year 2084.*

There is still time remaining before the year 2084, the century pass George Orwell's favorite year of 1984. And so, Ladies and Gentlemen, start your engines!

The exciting breakthroughs presented here (and 1000 hard copies sold and

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3 [Pas] Translated by Ilya Hoffman and Alexander Soifer for this essay. The original Russian text is this:

В родстве со всем, что есть, уверясь
И знаясь с будущим в быту,
Нельзя не впасть к концу, как в ересь,

В неслыханную простоту.
Но мы пощажены не будем,
Когда ее не утаим.
Она всего нужнее людям,
Но сложное понятней им.

64,000 downloads of the 2009 edition) inspired Springer to sign a contract for the new expanded edition [Soi3] of *The Mathematical Coloring Book.* I hope it will appear in 2020 with these and further breakthroughs on the theme of this colorful problem.

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