A characterization of 2-neighborhood degree list of diameter 2 graphs

N. Benakli
Department of Mathematics
NYCCT, CUNY
Brooklyn, NY 11201
nbenakli@citytech.cuny.edu

E. Halleck
Department of Mathematics
NYCCT, CUNY
Brooklyn, NY 11201
ehalleck@citytech.cuny.edu

S. R. Kingan
Department of Mathematics
Brooklyn College, CUNY
Brooklyn, NY 11210
skingan@brooklyn.cuny.edu

Abstract

Let $N_2DL(v)$ denote the set of degrees of vertices at distance 2 from v. The 2-neighborhood degree list of a graph is a listing of $N_2DL(v)$ for every vertex v. A degree restricted 2-switch on edges v_1v_2 and w_1w_2, where $deg(v_1) = deg(w_1)$ and $deg(v_2) = deg(w_2)$, is the replacement of a pair of edges v_1v_2 and w_1w_2 by the edges v_1w_2 and v_2w_1 given that v_1w_2 and v_2w_1 did not appear in the graph originally. Let G and H be two graphs of diameter 2 on the same vertex set. We prove that G and H have the same 2-neighborhood degree list if and only if G can be transformed into H by a sequence of degree restricted 2-switches.

1 Introduction

Two graphs G_1 and G_2 are *isomorphic* if there is a bijection from $V(G_1)$ to $V(G_2)$ that preserves adjacencies. Two graphs G_1 and G_2 are *label isomorphic* or *identical* if there is a bijection from $V(G_1)$ to $V(G_2)$ that preserves labeled adjacencies. The *degree* of a vertex v, denoted by $deg(v)$,

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is the number of edges incident to v. The *distance* between a pair of vertices u and v in a graph G, denoted by $d(u,v)$, is the length of the shortest path between u and v. The *diameter*, denoted by diam(G), is the maximum value of $d(u,v)$, where the maximum is taken over all pairs of vertices u and v in G.

Let G be a graph with vertices $v_1, \ldots, v_n$. The *degree sequence* of a graph is a listing of the degrees of its vertices: $deg(v_1), deg(v_2), \ldots, deg(v_n)$. The set of all vertices adjacent to a vertex v is denoted by $N(v)$ and called the *neighborhood* of v. Note that the neighborhood of v does not contain v itself. For each vertex v, the *neighborhood degree list* of v, denoted by $NDL(v)$, is the list of degrees of vertices in $N(v)$. The *neighborhood degree list* of G, denoted by $NDL(G)$, is the list of lists

$$\{NDL(v_1), NDL(v_2), \ldots NDL(v_n)\}$$

By convention the degree sequence and the neighborhood degree list of a vertex are written in descending order. The concept of neighborhood degree list was introduced independently by Barrus and Donavan [1] and Bassler *et al* [2].

In this paper we generalize the notion of neighborhood degree list. Let $N_k(v)$ be the set of vertices of distance $k \geq 1$ from v. In this notation $N(v) = N_1(v)$. Observe that $k \leq diam(G)$. The *k-neighborhood degree list* of v, denoted by $N_kDL(v)$, is the list of degrees of vertices in $N_k(v)$. The *k-neighborhood degree list* of G, denoted by $N_kDL(G)$, is the list of lists

$$\{N_kDL(v_1), N_kDL(v_2), \ldots N_kDL(v_n)\}.$$

Essentially we are considering concentric balls of vertices of increasing distance centered around a vertex.

The motivation for the definition of k-neighborhood degree list comes from the problem of identifying fake followers on Instagram and Twitter. A New York Times article titled "The follower factory"[1] explains how celebrities purchase followers from companies that create millions of such accounts and sell them as followers. Such companies are called follower factories. An Instagram influencer's follower count (i.e. degree in the social network) may be high but the followers may be mostly vertices of low degree. Such fake influencers would be flagged by computing their NDL. In case the fake accounts have a degree greater than 1 in an effort to hide that they are fake

[1] **https://www.nytimes.com/interactive/2018/01/27/technology/social-media-bots.html**

accounts, then computing N_kDL, for $k \geq 1$, would reveal an anomaly in the pattern of N_kDL lists. Measures of centrality like betweenness centrality, eigenvalue centrality, PageRank, etc. can also be used to compare vertices, but they are global measures designed for specialized applications. On the other hand N_kDL is a local measure. In many cases the entire graph is unknown and a local measure of influence is needed.

The main result of this paper is a characterization of diameter 2 graphs with the same N_2DL. A *2-switch* in a graph is the replacement of a pair of edges v_1v_2 and w_1w_2 by the edges v_1w_2 and v_2w_1 given that v_1w_2 and v_2w_1 did not appear in the graph originally. There may or may not be edges between pairs of vertices v_1, w_1 and v_2, w_2. The 2-switch operation is illustrated in Figure 1. Observe that a 2-switch on a pair of edges does not alter the degrees of the four vertices involved. Therefore a 2-switch does not alter the degree sequence of the resulting graph. If some 2-switch turns G into G', then a 2-switch on the same four vertices turns G' into G.

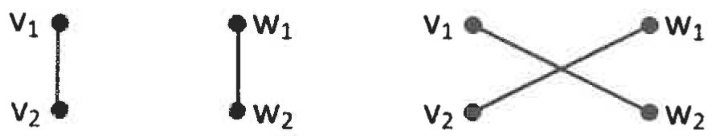

Figure 1: The 2-switch operation

A *degree restricted 2-switch* on edges v_1v_2 and w_1w_2 is a 2-switch performed when $deg(v_1) = deg(w_1)$ and $deg(v_2) = deg(w_2)$. (Barrus and Donavan call this an n-switch.) The next theorem is the main result in this paper.

Theorem 1 *Let G and H be two graphs on n vertices with diameter 2. Then G and H have the same 2-neighborhood degree list if and only if G can be transformed into H by a sequence of degree restricted 2-switches.*

2 The proof of Theorem 1.1

When generalizing NDL to N_2DL one problem that comes up is that the 2-switch operation can alter the diameter. Consider for example a 2-switch performed on the the cube graph (circular 4-ladder) that converts it to the Mobius 4-ladder as shown in Figure 2. Observe that NDL is preserved in both graphs. However, N_2DL is not preserved. The diameter of the cube is 3, but when a 2-switch operation is done to obtain the Mobius 4-ladder, the diameter is reduced to 2. Thus the 2 graphs have different N_2DL.

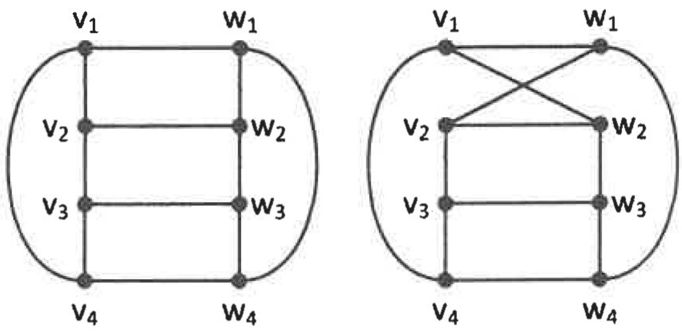

Figure 2: Circular 4-ladder and Mobius 4-ladder

Let G and H be two graphs on n vertices. Berge proved that G and H have the same degree sequence if and only if G can be transformed into H by a sequence of 2-switches. See **[4, p. 47]**. We do not use this result, rather we use the technique that Berge uses. The next result appears in **[1, Theorem 3.3]**. We give a different proof based on the lexicographic ordering of the neighborhood degree list.

Let a and b be two vertices in a graph such that $deg(a) = deg(b) = t$. Let $N(a) = \{a_1, \ldots, a_t\}$ and $N(b) = \{b_1, \ldots, b_t\}$, where the vertices are listed in descending order based on their degrees. We say $NDL(a) = NDL(b)$ if the ordered lists of degrees are the same. In other words

$$(deg(a_1), \ldots, deg(a_t)) = (deg(b_1), \ldots, deg(b_t))$$

We say $NDL(a) < NDL(b)$ if using the lexicographic ordering

$$(deg(a_1), \ldots, deg(a_t)) < (deg(b_1), \ldots, deg(b_t)).$$

Lexicographic ordering is defined recursively. If $deg(a_1) < deg(b_1)$, then $NDL(a) < NDL(b)$. If $deg(a_1) = deg(b_1)$, then the order is determined by the lexicographic order of $(deg(a_2), \ldots, deg(a_t))$ and $(deg(b_2), \ldots, deg(b_t))$. If $deg(a_2) < deg(b_2)$, then $NDL(a) < NDL(b)$. If $deg(a_2) = deg(b_2)$, then check the sequences $(deg(a_3), \ldots deg(a_t))$ and $(deg(b_3), \ldots deg(b_t))$, and so on.

Lemma 2 (Barrus and Donavan 2018) *Let G and H be two graphs on n vertices. Then G and H have the same neighborhood degree list if and only if G can be transformed into H by a sequence of degree restricted 2-switches.*

Proof. One direction is straightforward. If G can be transformed into H by a sequence of degree-restricted 2-switches, then clearly G and H have the same neighborhood degree list.

Conversely, suppose G and H have the same neighborhood degree list. The proof is by induction on $n \geq 4$. The result holds for graphs on 4 vertices trivially. Assume that the result holds for all graphs with $n-1$ vertices.

Let w be a vertex of maximum degree Δ in G. Let z be a neighbor of w and let S be the set of all vertices that are not neighbors of w, but have the same degree as z. If $S = \phi$, then proceed to the next neighbor. Otherwise suppose $S \neq \phi$. Choose $x \in S$ so that $NDL(x)$ is highest among vertices of S. If $NDL(x) \leq NDL(z)$, then again proceed to the next neighbor of S.

Suppose $NDL(x) > NDL(z)$. Note that w is a vertex of maximum degree. Since w is incident to z, but not to x, there exists y incident to x, but not to z, such that $deg(y) = deg(w)$. (See Figure 3.) Thus we can perform a degree restricted 2-switch operation on wz and yx. Delete wz and yx and add wx and yz. Observe that the resulting graph has the same NDL as G (and consequently the same degree sequence).

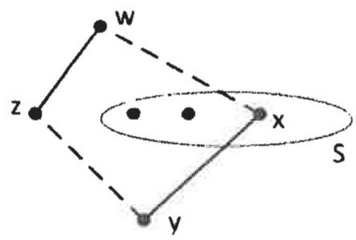

Figure 3: Degree-restricted 2-switch

Repeat the above process for every neighbor of w to obtain a graph G^* where $NDL(G^*) = NDL(G)$ and neighbors of w in G^* are adjacent to vertices with the same degrees as neighbors of w in G, but with highest NDL.

Similarly, choose a vertex w_H in H of highest degree such that $NDL_H(w_H) = NDL_G(w)$. Such a vertex exists since G and H have the same NDL. There exists a sequence of degree restricted 2-switches that transforms H into H^*, where $NDL(H^*) = NDL(H)$ and neighbors of w_H in H^* are adjacent to vertices with the same degrees as neighbors of w_H in H, but with highest NDL.

Observe that the degrees of the neighbors of w and w_H in G^* and H^*, respectively, are the same. In addition,

$$NDL_{H^*}(w_H) = NDL_H(w_H) = NDL_G(w) = NDL_{G^*}(w)$$

Consider $G' = G^* - w$ and $H' = H^* - w_H$. Then $NDL(G') = NDL(H')$. By the induction hypothesis applied to G' and H', there exists a sequence of degree restricted 2-switches that transforms G' to H'. These degree restricted 2-switches do not involve w and w_H, which have the same NDL in G^* and H^*, respectively. So applying this sequence of degree restricted 2-switches transforms G^* to H^*. Finally, we can transform G to H by transforming G to G^*, then G^* to H^*, and then (in reverse order) H to H^*. ■

Lemma 3 *Let G be a graph on n vertices with diameter 2. Then*

$$deg(v) = n - 1 - |N_2(v)|.$$

Proof. Since G has diameter 2, every vertex is of distance 1 or 2 from every other vertex. So for each $v \in V(G)$,

$$V(G) = \{v\} \cup N(v) \cup N_2(v),$$

where $|N(v)| \cap |N_2(v)| = \phi$. Since $deg(v) = |N(v)|$,

$$n = 1 + deg(v) + |N_2(v)|.$$

Therefore

$$deg(v) = n - 1 - |N_2(v)|.$$

■

The main idea in the proof of Theorem 1.1 is that if the graph has diameter 2, then we can recover NDL from N_2DL and vice versa. Moreover, we can recover the degree sequence from NDL in any graph. Let us look at an example to illustrate this point. Consider the graph G with diameter 2 shown in Figure 4. It has degree sequence 5, 5, 4, 4, 4, 4, 3, 3 and NDL and N_2DL

	NDL	N_2DL
v_1	5, 5, 4, 3	4, 4, 3
v_2	4, 4, 4, 4, 3	5, 3
v_3	5, 5, 4, 3	4, 4, 3
v_4	4, 4, 4, 4, 3	5, 3
v_5	5, 4, 3	5, 4, 4, 4
v_6	5, 4, 3	5, 4, 4, 4
v_7	5, 5, 4, 4	4, 3, 3
v_8	5, 5, 4, 4	4, 3, 3

Observe that $N_1(v_1) = \{v_2, v_4, v_6, v_8\}$. So the members of $N_2(v_1)$ are the rest of the vertices (except v_1 itself). Thus $N_2(v_1) = \{v_3, v_5, v_7\}$

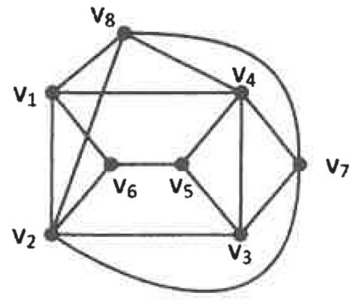

Figure 4: A diameter 2 graph

Proof of Theorem 1. Suppose G and H have the same N_2DL. Then for every vertex v_G in G, there is a vertex v_H in H with the same N_2DL and vice-versa. Thus there is a one-to-one correspondence between the vertices of G and H such that for every pair of corresponding vertices v_G and v_H,

$$N_2DL(v_G) = \{deg_G(u) \mid u \in N_2(v_G)\},$$

$$N_2DL(v_H) = \{deg_H(u) \mid u \in N_2(v_H)\},$$

and $N_2DL(v_G) = N_2DL(v_H)$. Observe that $|N_2(v_G)|$ is the number of entries in $N_2DL(v_G)$ and $|N_2(v_H)|$ is the number of entries in $N_2DL(v_H)$. Therefore $|N_2(v_G)| = |N_2(v_H)|$. By Lemma 3

$$deg_G(v_G) = n - 1 - |N_2(v_G)| = n - 1 - |N_2(v_H)| = deg_H(v_H).$$

Thus the degree sequence of G and H can be obtained from N_2DL. Moreover the degree sequence of G and H are the same.

Next observe that if $N_2DL(v) = \{deg(u) \mid u \in N_2(v)\}$, then since the degree sequence is known, $NDL(v) = \{deg(u) \mid u \notin \{v\} \cup N_2(v)\}$. Since G and H have the same N_2DL and the same degree sequence, G and H must have the same NDL. Lemma 2 implies that G can be transformed into H by a sequence of degree restricted 2-switches.

Conversely, suppose G can be transformed into H by a sequence of degree-restricted 2-switches. Lemma 2 implies that NDL is maintained at each stage (even if the diameter changes) so at the end of the sequence of degree-resticted 2-switches G and H have the same NDL. Thus there is a one-to-one correspondence between the vertices of G and H such that for every pair of corresponding vertices v_G and v_H,

$$NDL(v_G) = \{deg_G(u) | u \in N(v_G)\},$$

$$NDL(v_H) = \{deg_H(u) | u \in N(v_H)\}$$

and $NDL(v_G) = NDL(v_H)$. Observe that $|N(v_G)|$ is the number of entries in $NDL(v_G)$ and $|N(v_H)|$ is the number of entries in $NDL(v_H)$. Therefore $|N(v_G)| = |N(v_H)|$ and $deg(v_G) = deg(v_H)$. Thus the degree sequence of G and H can be obtained from N_2DL and they are the same.

Next, observe that if $NDL(v) = \{deg(u) \mid u \in N(v)\}$, then since the degree sequence is known, $N_2DL(v) = \{deg(u) \mid u \notin \{v\} \cup N(v)\}$. In conclusion, if G and H have the same NDL and the same degree sequence, then since G and H are diameter 2 graphs they must have the same N_2DL. ■

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