Independence Number of Maximal Planar Graphs

Allan Bickle

Penn State Altoona

Contact at allanbickle.wordpress.com

Abstract

It is known that for a maximal planar graph G with order $n \geq 4$, the independence number satisfies $\frac{n}{4} \leq \alpha(G) \leq \frac{2n-4}{3}$. We show the lower bound is sharp and characterize the extremal graphs for $n \leq 12$. For the upper bound, we characterize the extremal graphs of all orders.

The independence number $\alpha(G)$ of a graph G is the size of the largest independent set. This parameter is difficult to determine in general, but can be bounded on various graph classes. This paper considers planar and maximal planar graphs.

1 The Lower Bound

The lower bound was known to follow immediately from the Four Color Theorem.

Proposition 1. *If G is a planar graph with order n, then $\alpha(G) \geq \frac{n}{4}$.*

Proof. The Four Color Theorem says that any planar graph is 4-colorable. Each color class is an independent set, so one has size at least $\frac{n}{4}$. □

Prior to the proof of the Four Color Theorem, Michael Albertson showed that any planar graph has $\alpha(G) \geq \frac{2}{9}n$ [2] and found an algorithm to obtain an independent set of size at least $\frac{2}{9}n$ [1].

Proposition 1 is sharp. The only extremal graph of order 4 is K_4. Any number of disjoint copies of K_4 is also an extremal graph, and so is any graph formed by adding edges while preserving planarity. One example of this is the cube of P_n, which is maximal planar.

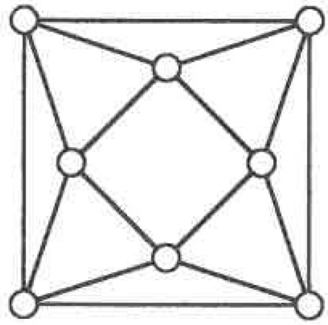
Definition 2. A critical extremal graph is a planar graph with $\alpha(G) = \frac{n(G)}{4}$ that contains no proper subgraph H with $\alpha(H) = \frac{n(H)}{4}$.

Proposition 3. *Any critical extremal graph G other than K_4 has $\delta(G) \geq 4$.*

CONGRESSUS NUMERANTIUM 234 (2019), pp.61-68

Proof. Let G be a critical extremal graph and suppose $v \in G$ has $d(v) \leq 3$. Let $S = N(v) \cup \{v\}$. Then any maximal independent set of G contains a vertex in S. Then $\alpha(G-S) \leq \frac{n(G)}{4} - 1 = \frac{n(G)-4}{4} \leq \frac{n(G-S)}{4}$, which is impossible unless $G - S$ is a critical extremal graph, a contradiction. □

The square of the cycle C_8 is a critical extremal graph of order 8. It is mentioned in [3]. Note that it is not maximal planar.

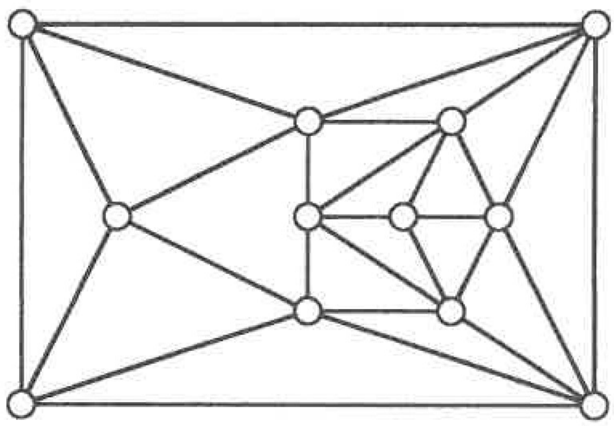


Proposition 4. *The square of C_8 is the unique critical extremal graph of order 8.*

Proof. We consider a maximal planar graph G with order 8 and independence number 2, then check whether any edges can be removed. Now $\Delta(G) \in \{5, 6, 7\}$. If $\Delta(G) = 7$, G would have a vertex of degree 3. If $\Delta(G) = 6$ and G has no degree 3 vertex, $G = C_6 + \overline{K}_2$, and $\alpha(G) = 3$.

If $\Delta(G) = 5$, G contains $C_5 + K_1$, and the other two vertices must be adjacent. One of them must be adjacent to four vertices of C_5, and the other adjacent to three vertices of C_5. Thus we find a maximal planar graph in which the four vertices of degree 5 induce a 4-cycle. Checking the edges of this cycle, we find that two nonadjacent edges may be removed while keeping the independence number at 2. This produces the square of C_8. □

There are two critical extremal graphs of order 12–the icosahedron (mentioned in [7]), and the graph shown below. It is not regular and not maximal.



Theorem 5. *There are exactly two critical extremal graphs of order 12, the icosahedron and the graph shown above.*

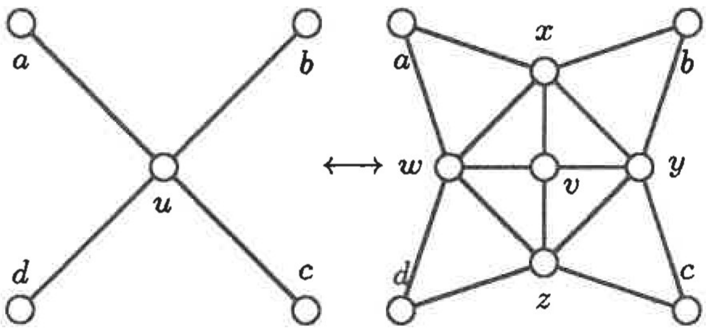
Proof. We consider a maximal planar graph G with order 12 and independence number 3, then check whether any edges can be removed. Now $\Delta(G) \in \{5, 6, 7, ..., 11\}$. If $\Delta(G) = 5$, G is the icosahedron, which is a critical extremal graph. If $\Delta(G) \geq 8$, clearly $\alpha(G) \geq 4$.

If $\Delta(G) = 7$, G contains $C_7 + K_1$, and the other four vertices induce $K_4 - e$. Since 11 edges remain unassigned, one of these four vertices is incident with at most two of these edges, and they must be incident with consecutive vertices of the C_7. But this produces an independent set of size four.

If $\Delta(G) = 6$, G contains $C_6 + K_1$, and the other five vertices induce $P_4 + K_1$. Now 11 edges remain unassigned and each of these five vertices is incident with at least two of the 11 edges, so one of them is incident with exactly three of the 11 edges. Checking three cases, we find that only one choice (making the three edges incident with a degree two vertex in $P_4 + K_1$) avoids an independent set of size four. Checking each edge of the graph shows that four edges can be (separately) deleted without creating an independent set of size four, and up to symmetry, only one pair can be so deleted. This produces the graph shown above. □

The graph shown above can be constructed from the square of C_8 using the following operation.

Let u be a degree 4 vertex of a graph, and $\{a, b, c, d\}$ be its neighborhood. Replace u and its incident edges with vertices $\{v, w, x, y, z\}$ and edges $\{aw, ax, bx, by, cy, cz, dz, dw, vw, vx, vy, vz, wx, wy, wz, zw\}$. This increases the order of a graph by 4 and preserves planarity.



Proposition 6. *Let G be a graph, and H be a graph that results from applying the operation described above. Then $\alpha(H) = \alpha(G) + 1$.*

Proof. Let S be an independent set of G. If $u \notin S$, let $S' = S \cup \{v\}$. If $u \in S$, let $S' = S - u \cup \{w, y\}$. Either way, S' is an independent set of H, and $|S'| = |S| + 1$, so $\alpha(H) \geq \alpha(G) + 1$.

Let S be a maximum independent set of H. At most two vertices of $\{v, w, x, y, z\}$ are in S. If exactly two are in S, none of $\{a, b, c, d\}$ are, so let $S' = S - \{v, w, x, y, z\} \cup \{u\}$. If only one of $\{v, w, x, y, z\}$ is in S, let $S' = S - \{v, w, x, y, z\}$. Either way, S' is an independent set of G, and $|S'| = |S| - 1$, so $\alpha(H) \leq \alpha(G) + 1$. □

If G is a critical extremal graph, then applying the operation produces an extremal graph H, that may be critical if $\{u, a, b, c, d\}$ does not induce a wheel.

2 The Upper Bound

Maximizing $\alpha(G)$ over planar graphs is not an interesting problem, since empty graphs are planar. The problem is interesting if we restrict ourselves to maximal planar graphs. Caro and Roditty found an upper bound for this case. We present a somewhat streamlined proof.

Theorem 7. *[5] If G is a maximal planar graph with order $n \geq 4$ and minimum degree $\delta = \delta(G)$, then $\alpha(G) \leq \frac{2n-4}{\delta}$.*

Proof. Let G be a maximal planar graph with order $n \geq 4$. Then G has $2n - 4$ triangular regions. Each triangle bounding a region contains at most one vertex of an independent set S. Now each vertex in S is on the boundary of at least $\delta(G)$ regions. Thus $\delta|S| \leq 2n - 4$, so $|S| \leq \frac{2n-4}{\delta}$. □

When $\delta(G) = 3$, Caro and Roditty identified the extremal graphs, but did not prove that they were the only extremal graphs.

Theorem 8. *If G is a maximal planar graph with order $n \geq 4$ and minimum degree $\delta = \delta(G)$, the equality $\alpha(G) = \frac{2n-4}{\delta}$ holds if and only if G can be formed from a planar graph H, all of whose regions have length δ by adding a vertex of degree δ inside each region.*

Proof. ($\Rightarrow$) Equality requires every vertex in S have degree δ, and every region of G have a vertex of S on its boundary. Deleting each vertex in S produces a planar graph H, all of whose regions have length δ, and all of which contained a vertex of S.

($\Leftarrow$) Let H be a planar graph, all of whose regions have length δ. Form G by adding a set S of vertices of degree δ inside each region. As in Theorem 7, $\delta|S| = 2n - 4$, so $|S| = \frac{2n-4}{\delta}$. □

When $\delta = 3$, the graph H is maximal planar. Caro and Roditty identified the extremal graphs, but did not prove that they were the only extremal graphs.

When $\delta = 4$, Caro and Roditty gave the example of $C_4 \square P_k$, a planar graph with all regions of length 4, to show that the bound is sharp. However, these are not the only extremal graphs. Any planar graph with all regions of length 4 is the dual of a 3-connected 4-regular planar graph. All 3-connected 4-regular planar graphs can be constructed from the octahedron graph using three operations [4].

When $\delta = 5$, Caro and Roditty provided a construction that begins with the dodecahedron, and iteratively identifies 5-cycles in a dodecahedron and the existing graph. This graph is planar with all regions of length 5, which shows that the bound is sharp. However, these are not the only extremal graphs. Any planar graph with all regions of length 5 is the dual of a 3-connected 5-regular planar graph. There is an operation characterization for such graphs [6].

When $\delta = 3$, the maximum independent set in the extremal graphs is unique. The extremal graphs in the previous theorem exist only when $n \equiv 2 \bmod 3$. Characterizations of the extremal graphs also exist for all other orders. When $n \equiv 1 \bmod 3$, a slight modification of Theorem 8 works.

Corollary 9. *A maximal planar graph G with order $n = 3k+1$, $k \geq 1$, has $\alpha(G) = \frac{2n-5}{3}$ if and only if it can be constructed from a maximal planar graph H of order $\frac{1}{3}n + \frac{5}{3}$ by either*

1. *adding vertices of degree 3 in all but one region, or*
2. *deleting an edge, adding a degree 4 vertex in the resulting region, and adding degree 3 vertices in all other regions.*

Proof. ($\Rightarrow$) Let S be an independent set of size $\frac{2}{3}n - \frac{5}{3}$. We have $3|S| = 3\left(\frac{2}{3}n - \frac{5}{3}\right) = (2n-4) - 1$. Then either S has one vertex of degree 4 and all others of degree 3, or one region contains no vertex of S, and all vertices of S have degree 3. Delete all vertices of S, and in the first case, add one edge to the region of length 4. This produces a maximal planar graph H from which G can be constructed.

($\Leftarrow$) Let H have order $\frac{1}{3}n + \frac{5}{3}$. Then H has $2\left(\frac{1}{3}n + \frac{5}{3}\right) - 4 = \frac{2}{3}n - \frac{2}{3}$ regions, so G has order $\left(\frac{1}{3}n + \frac{5}{3}\right) + \left(\frac{2}{3}n - \frac{2}{3} - 1\right) = n$, and the set of vertices added is independent, with $|S| = \frac{2}{3}n - \frac{5}{3}$. □

When $n > 4$, the maximum independent set in the extremal graphs is unique.

When $n \equiv 0 \bmod 3$, the characterization of the extremal graphs is somewhat more complicated.

Corollary 10. *A maximal planar graph G with order $n = 3k$, $k \geq 2$, has $\alpha(G) = \frac{2n-6}{3}$ if and only if it can be constructed from a maximal planar graph H of order $\frac{1}{3}n + 2$ by either*

1. adding vertices of degree 3 in all but two regions

2. deleting an edge, adding a degree 4 vertex in the resulting region, and adding degree 3 vertices in all but one other region

3. deleting two non-incident edges, adding degree 4 vertices in the resulting regions, and adding degree 3 vertices in all other regions

4. deleting two incident edges so that the resulting region is bounded by a 5-cycle, adding a degree 5 vertex in it, and adding degree 3 vertices in all other regions.

Proof. ($\Rightarrow$) Let S be an independent set of size $\frac{2}{3}n - 2$. We have $3|S| = 3\left(\frac{2}{3}n - 2\right) = (2n-4) - 2$. Then the number of regions containing no vertex of S plus $\sum_{v \in S}(d(v) - 3)$ must be two. There are four possibilities–two missed regions, one missed region and a degree 4 vertex in S, two degree 4 vertices in S, or one degree 5 vertex in S. In each case, deleting the vertices of S and if necessary, adding edges in any nontriangular regions, produces a maximal planar graph H from which G can be constructed.

($\Leftarrow$) Let H have order $\frac{1}{3}n + 2$. Then H has $2\left(\frac{1}{3}n + 2\right) - 4 = \frac{2}{3}n$ regions, so G has order $\left(\frac{1}{3}n + 2\right) + \left(\frac{2}{3}n - 2\right) = n$, and the set of vertices added is independent, with $|S| = \frac{2}{3}n - 2$. □

A maximum independent set in this construction need not be unique. For any order $n = 3k$, $k \geq 2$, there is a maximal planar graph with order n with more than one independent set of size $\frac{2}{3}n - 2$. However, when $n = 3k$, $k \geq 3$, there is also a maximal planar graph with order n with a unique independent set.

References

[1] M. Albertson, *Finding an independent set in a planar graph.* Graphs and Combinatorics. Lecture Notes in Mathematics, R. A. Bari and F. Harary (eds) vol 406, 173-179. Springer, Berlin, Heidelberg (1974).

[2] M. Albertson, *A lower bound for the independence number of a planar graph*, J Comb Theory, Series B. 20 1, 84-93 (1976).

[3] M. Albertson, B. Bollobas, and S. Tucker, *The independence ratio and maximum degree of a graph*, Congr. Numer. 17 43-50 (1976).

[4] H.J. Broersma, A.J.W. Duijvestijn, F. Gobel, Generating all 3-connected 4-regular planar graphs from the octahedron graph J. Graph Theory, 17 5 (1993), 613-620

[5] Y. Caro and Y. Roditty, On the vertex-independence number and star decomposition of graphs, Ars Combin. 20 (1985), 167–180.

[6] G. Ding, J. Kanno, J. Su, Generating 5-regular planar graphs, J. Graph Theory 61 3 (2009) 219-240.

[7] R. Steinberg, C. Tovey, *Planar Ramsey Numbers*, J. Comb Theory, Series B. 59, 288-296 (1993).