PERFECT MATCHINGS, CHANNELS, AND 2-DIVISIBILITY

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ABSTRACT. A *perfect matching* of a graph is a subset of edges in the graph such that each vertex is contained in exactly one edge. We study the number of perfect matchings of a given graph. In particular, we are interested in the power of two that divides this number. A new type of vertex set called a channel is considered, the presence of which is associated to powers of two in the perfect matching count. This gives a method for determining lower bounds on such powers. Algebraic and involutive proofs are given for these results, and methods for channel identification are given. We specialize to perfect matchings on subgraphs of the square lattice, which are identified with domino tilings of the plane, and apply channels to some conjectures by Pachter.

1. INTRODUCTION

A *perfect matching* of a graph $G = (V, E)$ is a set of edges from E such that each vertex in V is contained in exactly one edge. Let m_G denote the number of perfect matchings of G. In general, determining m_G for an arbitrary graph is $\#P$-hard [16]. However, it was first shown by Kasteleyn [7] that for planar graphs we may construct a signed version of the adjacency matrix with the property that its determinant is m_G^2. Such a matrix is called a *Kasteleyn matrix* for G.

Using the Kasteleyn matrix gives a method for computing the number of perfect matchings in polynomial time. For some families of graphs we may even find explicit formulas for m_G using this matrix. For instance, a classical result of Temperley and Fisher [15] and Kasteleyn [6] gives the following for the number of perfect matchings of the $m \times n$ grid graph:

$$m_G = \prod_{k=1}^{\frac{m}{2}} \prod_{l=1}^{n} 2\left[\cos^2 \frac{k\pi}{m+1} + \cos^2 \frac{l\pi}{n+1}\right]^{1/2}.$$

Much attention has been given to the fact that, for $m = n = 2r$, this formula produces a number of the form $2^r b_r^2$ with b_r an odd integer (see, e.g., [1], [3], [4], [5], [8], [12]). This can be shown directly from Kasteleyn's formula by extracting the factors where $k = l$. However, this method does not readily extend to non-rectangular regions since few exact formulas such as (1) are known. Other proofs have been given using combinatorial methods that rely on the reflective symmetry of the square grid graph. The key result in this vein is the following:

Proposition 1.1 (Ciucu's Factorization Theorem [1]). *If a bipartite graph has a line of symmetry containing $2r$ vertices, and no edges connect two vertices on opposite sides of the line, then the number of perfect matchings of the graph is divisible by 2^r.*

Many more power of 2 patterns for non-symmetric graphs have been proposed [12][14]. For example, Pachter gives the following conjecture:

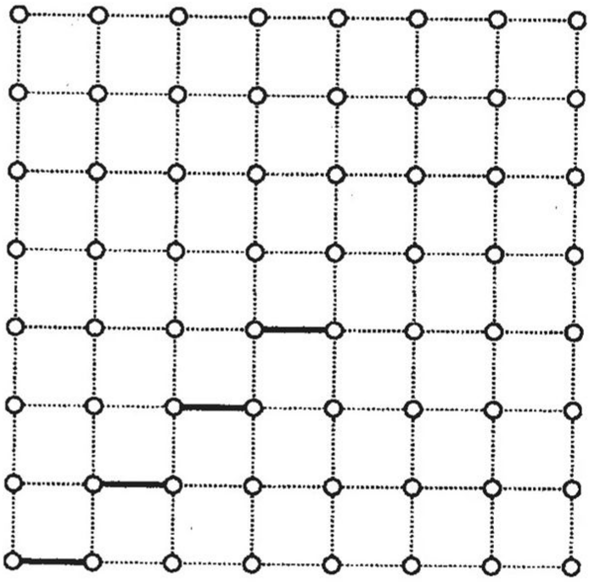

FIGURE 1

Conjecture 1.2 (Deleting from step-diagonals). *Let G be the $2r \times 2r$ grid graph shown in Figure 1. If any k of the highlighted edges are removed, then*

$$m_G = 2^{r-k}b$$

for some odd integer b.

The goal of this paper is to introduce a unified method for tackling these 2-divisibility problems. We do so via the introduction of special vertex sets, called *channels*. Channels build upon the following result, known to Lovász, who gave an algebraic proof.

Proposition 1.3 ([11], Problem 5.18). *Let G be any graph. Then m_G is even if and only if there is a non-empty set $C \subseteq V$ such that every vertex is adjacent to an even number of vertices in C.*

We will strengthen this for a planar graph G to show that in fact the number of channels divides m_G^2. Since the number of channels will always be a power of 2, this gives us a lower bound on the power of 2 dividing m_G. We will give an algebraic proof of this result and a combinatorial proof for Proposition 1.3.

We will provide a number of applications of channels to subgraphs of the square lattice. Perfect matchings of such graphs may identified with domino tilings of planar regions. In particular, our methods address the problems by Propp and Pachter mentioned above.

The paper is organized as follows. In Section 2, we remind the reader of the properties of Kasteleyn matrices and Smith normal forms. In Section 3 we use properties of the adjacency matrix mod 2 to show a divisibility result. In Section 4 we introduce channels and use them with the result of Section 3 to show a 2-divisibility property of rectangles and give evidence for Conjecture 1.2. In Section 5 we give a combinatorial proof of Proposition 1.3 for bipartite matrices. In Section 6 we apply the concepts from Section 5 to determine which rectangle grid graphs have an odd perfect matching count. We conclude in Section 7 with some directions for future work.

2. Preliminaries

All graphs in this paper are simple, undirected, and finite. If a graph is bipartite, we will consider its vertices to be colored black and white. Additionally, all matchings discussed will be perfect matchings, and thus the word "perfect" will be omitted in the future for brevity. For a graph $G = (V, E)$, V denotes the vertex set, E denotes the edge set, and A denotes the adjacency matrix. For an edge e, we use the notation $G - e$ to denote the subgraph $(V, E - e)$. For a vertex set S we use the notation $G - S$ to denote the subgraph of G induced by $V - S$. Recall that m_G is the number of matchings of G. As an exercise in this notation, we have the following proposition.

Proposition 2.1. *Let G be any graph, and let $e = (v_1, v_2)$ be any edge in the graph. Then*

$$m_G = m_{G-e} + m_{G-\{v_1,v_2\}}.$$

Also, fix any vertex v. Then

$$m_G = \sum_{(v,v')\in E} m_{G-\{v,v'\}}$$

where the sum is over edges containing v.

Proof. For the first relation, notice that matchings of $G - e$ are just matchings of G that do not use the edge e. The rest of the matchings of G do use e, and therefore for these matchings the vertices v_1 and v_2 are never in an edge with any vertex other than each other. Thus such matchings are equivalent to matchings of $G - \{v, w\}$, plus the edge e, and the first equation is shown.

For the second equation, partition the set of matchings of G based on the vertex that v shares an edge with in the matching. By the same reasoning as the last paragraph, the number of matchings in which v shares an edge with v' is $m_{G-\{v,v'\}}$. Summing over the possible edges shows the claim. □

Many of our examples will come from subgraphs of the square lattice $\mathbf{Z}^2$. In this case there is a canonical identification between matchings of the graph and domino tilings of a corresponding planar region, as shown in Figure 2. We will typically show such examples as planar regions for geometric intuition.

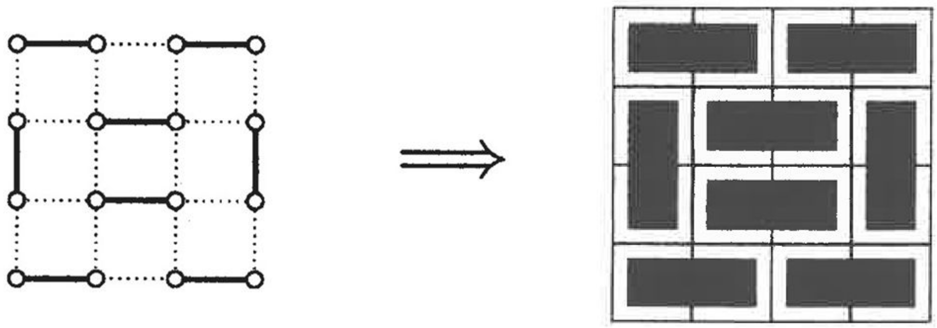

FIGURE 2

Our main tools for Section 3 are the Kasteleyn matrix and the Smith decomposition of a matrix.

Proposition 2.2 ([7]). *Let G be a planar graph with adjacency matrix $A = (a_{ij})$. Then there exists a matrix K such that $K = (\pm a_{ij})$, and*

$$\det K = m_G^2.$$

The matrix K is called a *Kasteleyn matrix*, and if a (not necessarily planar) graph G admits such a K, then G is said to have a *Kasteleyn signing*. For bipartite graphs we may be more specific.

Definition 2.3. Let G be a bipartite graph with adjacency matrix A. The *bipartite adjacency matrix* of G is the minor of A formed by selecting rows from A associated to white vertices and columns from A associated to black vertices.

The bipartite adjacency matrix has its own Kasteleyn matrix, called the *bipartite Kasteleyn matrix.*

Proposition 2.4 ([13]). *Let G be a bipartite planar graph with bipartite adjacency matrix $B = (b_{ij})$. Then there exists a matrix H, the bipartite Kasteleyn matrix, such that $H = (\pm b_{ij})$, and*

$$\det H = m_G.$$

We will want to diagonalize these matrices over the integers. The canonical tool for doing so is called *Smith normal form.*

Proposition 2.5. *Let A be a matrix over a principal ideal domain (PID) $\mathcal{R}$. Then there exist matrices S, D, T over $\mathcal{R}$ with the following properties:*

(i) $A = SDT$.

(ii) S and T are invertible over $\mathcal{R}$. For $\mathcal{R} = \mathbf{Z}$, this means $\det S, \det T = \pm 1$.

(iii) D is diagonal, with diagonal entries $\alpha_1, \ldots, \alpha_n$ satisfying

$$\alpha_1 \text{ divides } \alpha_2 \text{ divides } \ldots \text{ divides } \alpha_n.$$

The matrices S, D, T are called a Smith decomposition for A.

For our purposes, $\mathcal{R}$ will be the integers or a finite field. Smith decompositions have many useful properties. We will want the following result in particular.

Proposition 2.6. *Let $A = SDT$ be a Smith decomposition for a matrix A over a PID. Then*

$$\ker A \cong \ker D \text{ as abelian groups and } \det A = \det D.$$

We are now prepared to study 2-divisibility of m_G.

3. Kasteleyn 2-kernels

In this section we shall study graphs G with a Kasteleyn matrix K. As described above, planarity is a sufficient condition for G to have a Kasteleyn signing. Using the Smith normal form of K, we will find 2-divisibility results for such graphs that we develop further in the next section.

Definition 3.1. For an integral matrix A, define the *reduction of A modulo 2* to be the matrix A_2 over $\mathbf{Z}/2\mathbf{Z}$ given by reducing the entries of A mod 2 and considering them as elements of $\mathbf{Z}/2\mathbf{Z}$.

Define $\ker_2 A$, the *2-kernel* of A, to be the kernel of A_2 as a vector space over $\mathbf{Z}/2\mathbf{Z}$.

Notice that for an adjacency matrix A and corresponding Kasteleyn matrix K, we have $A_2 = K_2$. This follows from the definition of K as a signed version of A. This is a key observation that will allow us to translate our algebraic results in this section into geometric results in Section 4. Before that, let us see what we can learn from reducing the Kasteleyn matrix mod 2. Let the *2-nullity* of a matrix be the dimension of its 2-kernel:

$$\operatorname{null}_2 A := \dim \ker_2 A.$$

Lemma 3.2. *Let A be a square matrix with integer entries. Then*

$$2^{\operatorname{null}_2 A} \mid \det A.$$

Proof. Let $A = SDT$ be a Smith decomposition of A. Reducing modulo 2 gives $A_2 = S_2D_2T_2$. One may check this is a Smith normal form of A_2. Then by Proposition 2.6, the kernel of A_2 is isomorphic to the kernel of D_2. Set $k := \mathrm{null}_2 A = \mathrm{null}_2 D$. Since D_2 is diagonal, its kernel has a basis consisting of standard basis vectors that indicate columns where the diagonal entry is 0. Therefore there are exactly k such entries. Since 0 entries in D_2 correspond to even integral entries in D, there are exactly k even entries on the diagonal of D. Thus the determinant of D contains at least k factors of 2, and the result follows by Proposition 2.6. □

Applying this lemma to the Kasteleyn matrix or the bipartite Kasteleyn matrix will let us use the 2-kernel to find powers of two in the number of matchings of a graph.

Theorem 3.3. *Let G be a graph with a Kasteleyn signing (e.g. a planar graph). If A is the adjacency matrix of G, then*

$$2^{\mathrm{null}_2 A} \text{ divides } m_G^2.$$

Proof. As remarked above, $A_2 = K_2$, so in particular $\mathrm{null}_2 A = \mathrm{null}_2 K$. Additionally, by Proposition 2.2, the determinant of K is m_G^2. Thus by Lemma 3.2,

$$2^{\mathrm{null}_2 A} = 2^{\mathrm{null}_2 K} \text{ divides } \det K = m_G^2. \qquad \square$$

Theorem 3.4. *Let G be a bipartite graph with a Kasteleyn signing. If B is the bipartite adjacency matrix of G, then*

$$2^{\mathrm{null}_2 B} \text{ divides } m_G.$$

Proof. Same as the previous theorem, using the bipartite Kasteleyn matrix. □

We therefore may deduce powers of 2 dividing m_G by finding elements of the 2-kernel of A. The next section details how this may be done.

4. Channels

Let G be a graph with adjacency matrix A. Then a vector x in $\ker_2 A$ has entries in $\mathbf{Z}/2\mathbf{Z}$, and can be lifted to a vector $\tilde{x}$ with entries $0, 1 \in \mathbf{Z}$. The condition $A_2 x = 0$ then becomes $A\tilde{x} = 2y$ for some integral vector y. Because each row of x corresponds to a vertex in G, we may interpret x as the indicator function for a vertex set C, where a row with a 1 indicates the vertex is in C and a row with a 0 indicates the vertex is not in C. This leads to the following interpretation of 2-kernel elements.

Definition 4.1. Let $G = (V, E)$ be any graph. A *channel* is a set C of vertices such that every vertex in G is adjacent to an even number of vertices in C. In other words, letting $N(v)$ denote the neighborhood of v, a channel satisfies

$$|N(v) \cap C| \text{ is even for all } v \in V.$$

Let the set of channels in G be denoted $\mathcal{C}(G)$. If G is bipartite, let $\mathcal{C}_B(G)$ be the subspace of $\mathcal{C}(G)$ consisting of channels that use only black vertices from G.

The 2-kernel also has an additive structure as a $\mathbf{Z}/2\mathbf{Z}$ vector space. This transfers to $\mathcal{C}(G)$ by defining the sum of $C_1, C_2 \in \mathcal{C}(G)$ to be

$C_1 \oplus C_2 := (C_1 \cup C_2) - (C_1 \cap C_2)$, the symmetric difference of C_1 and C_2.

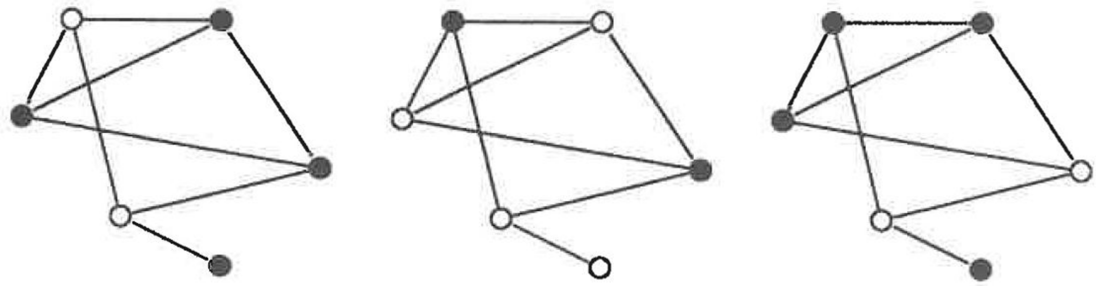

FIGURE 3. A graph G with its three nonempty channels indicated by shading. Any two of these form a basis for the space $\mathcal{C}(G)$.

With these definitions, we have the following result.

Lemma 4.2. *The spaces $\mathcal{C}(G)$ and $\ker_2 A$ are isomorphic vector spaces over $\mathbf{Z}/2\mathbf{Z}$.*

If G is additionally bipartite, with bipartite adjacency matrix B, then the results of the previous section indicate the following lemma.

Lemma 4.3. *The spaces $\mathcal{C}_B(G)$ and $\ker_2 B$ are isomorphic vector spaces over $\mathbf{Z}/2\mathbf{Z}$. Additionally, if B is square, then $\dim \mathcal{C}_B(G) = \frac{1}{2} \dim \mathcal{C}(G)$.*

Combining these observations with the 2-divisibility results from the last section, we have the following key theorem.

Theorem 4.4 (Channeling 2s). *If $\{C_1, ..., C_n\}$ is a linearly independent set of channels in a graph G with a Kasteleyn signing, then*

$$2^n \textit{ divides } m_G^2.$$

If additionally G is bipartite, and $\{C_1, ..., C_n\} \subseteq \mathcal{C}_B(G)$, then

$$2^n \textit{ divides } m_G.$$

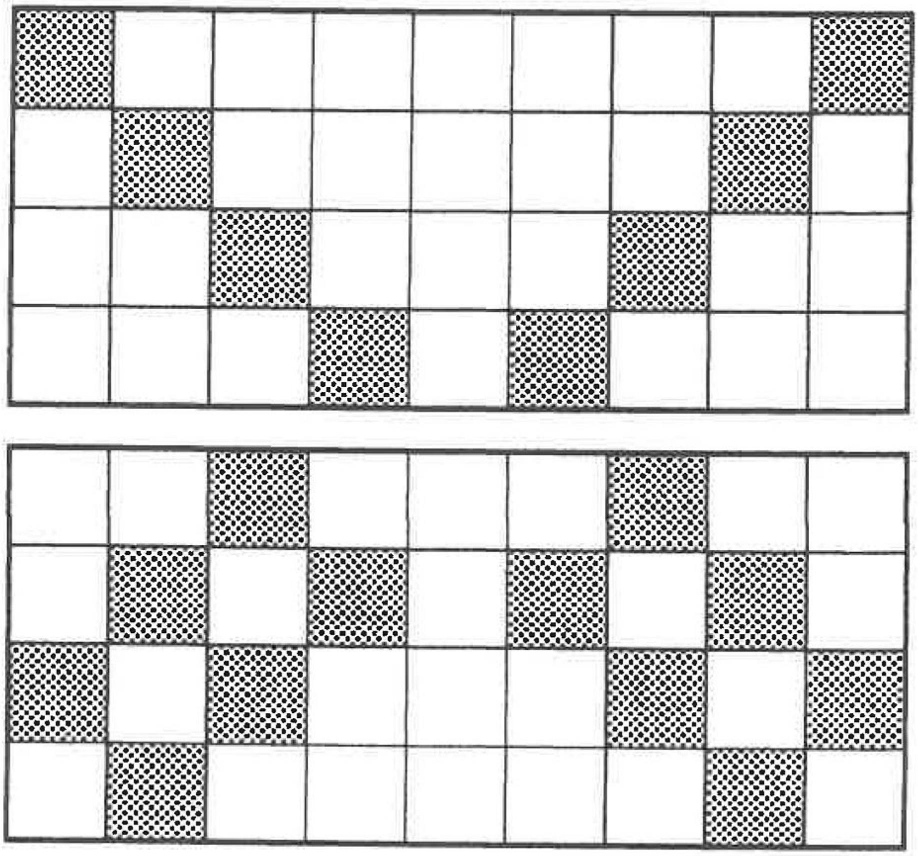

FIGURE 4

Example 4.5. We now have the tools we need to begin a study of the 2-divisibility of rectangle grid graphs. Let $R_{m\times n}$ denote the $m\times n$ rectangular grid graph. The shading in Figure 4 shows a basis for $\mathcal{C}_B(R_{4\times 9})$. By channeling 2s, we have that 2^2 divides $m_{R_{4\times 9}}$. And indeed, $m_{R_{4\times 9}} = 6336$ is divisible by 4.

Our results from Example 4.5 help demonstrate why the step-diagonal conjecture of Pachter should hold.

Proposition 4.6. *Let G be the $2r \times 2r$ grid graph shown in Figure 1. If any k of the highlighted vertex pairs are removed, then*

$$2^{r-k} \textit{ divides } m_G.$$

Proof. For a $2r\times 2r$ square grid graph, we will find that the space $\mathcal{C}_B(R_{2r\times 2r})$ has r independent channels which each intersect exactly one of vertex pairs removed from the step diagonal. Thus removing k of the step diagonal vertex pairs interrupts just k of the channels. The other $r-k$ channels will still be present. The result then follows from channeling 2s once we construct these channels.

Consider Figure 5. To construct each channel, pick a black vertex b on the step diagonal. In Figure 5, these are the vertices along the diagonal

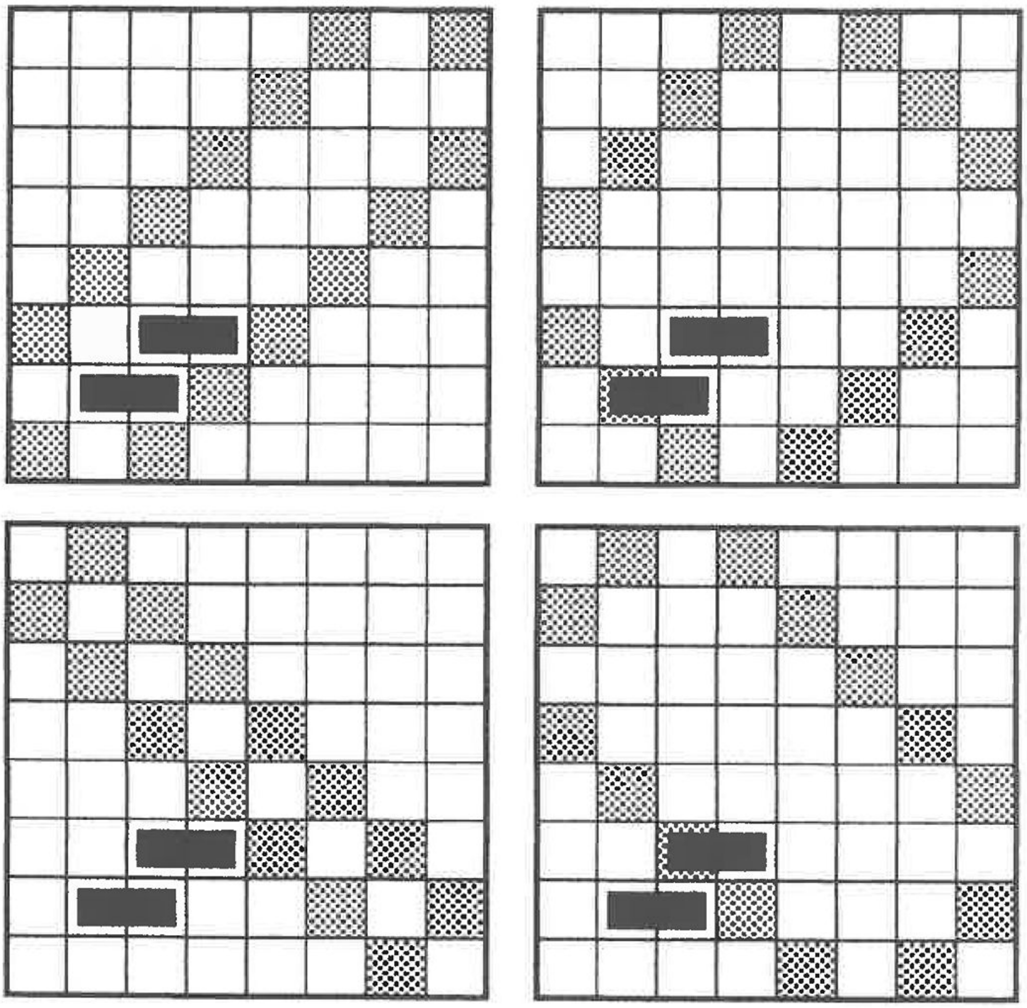

FIGURE 5. An illustration of Proposition 4.6 for $r = 4$ and $k = 2$. When the bolded vertex pairs are removed, the two shaded channels on the left remain valid.

from the bottom left to the top right. Our channel will consist of four diagonal segments of vertices. Two segments will intersect the step diagonal transversally, one at b in the bottom left and one at b's mirror image in the top right. The other two segments will be parallel to the step diagonal and placed so that the channel vertices along the sides of the grid graph each have one vertex between them. It is straightforward to check that each vertex of the grid graph is adjacent to an even number of elements of this vertex set, so it is indeed a channel. By following this construction for each b lying along the lower r black vertices of the step diagonal, we create r channels that each contain a different vertex from the step diagonal, and therefore must be independent. □

In this section we give a combinatorial proof that existence of a nonempty channel in G is necessary and sufficient for m_G to be even. Lovász originally proved this for completely general graphs in [11], problem 5.18. His argument studies the permanent of the adjacency matrix modulo 2, and is of a surprisingly different nature from our methods in the previous section that rely on a Kasteleyn signing. Our results in this section also do not need a Kasteleyn signing but do require the graph be bipartite. We prove the forward and backward directions separately.

5.1. Existence of a channel implies m_G even. Our approach for this direction will be to construct an involution on the set of matchings $\mathcal{M}(G)$. If we can construct an involution with no fixed points, then we can pair off elements of G that map to each other under the involution. This would imply that $|\mathcal{M}(G)| = m_G$ is even.

To construct such an involution, we shall employ a technique called cycle flipping. This is a commonly used method to build involutions on perfect matchings and show 2-divisibility results. See for example [1] or [12] to see this applied to graphs with reflective symmetry, or [9] for disjoint unions of two graphs. Given a perfect matching $\mu \in \mathcal{M}(G)$, the idea is to find a cycle Y of edges in the graph such that every second edge in the cycle is in μ. We may then construct a new edge set μ' by replacing the edges of $\mu \cap Y$ with the edges of $Y - \mu$. Since each vertex in Y is contained in exactly one edge in either case, μ' is also a perfect matching.

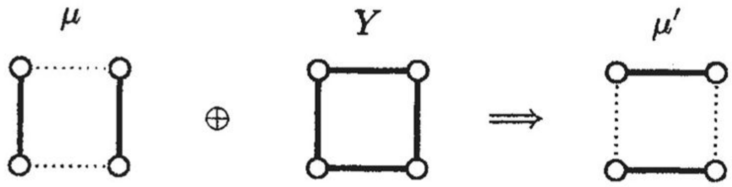

Figure 6

For the map this produces to be an involution, the same cycle has to be identified for both μ and μ'. The following results will show we can do so, given a nonempty channel. First we set up a tool we will need to find our cycle.

Definition 5.1. A *pairing function* across a vertex set $C \subseteq V$ is a collection of involutions f_v associated to each $v \in V$ which act on the edge set

$$\{(v, v') \in E \mid v' \in C\}$$

such that $f_v \circ f_v = \text{id}$ and f_v has no fixed points.

The existence of a pairing function across C is a combinatorial realization of the statement that the neighborhood $N(v)$ contains an even number of points from C for each vertex v. Applying this to the definition of a channel gives the following lemma.

Lemma 5.2. *Let G be a graph and let $C \subseteq V$ be any vertex set. Then C is a channel if and only if there exists a pairing function across C.*

Now we may use the pairing function from this lemma to trace out a path along edges of a matching. Finiteness of our graph will force this path to eventually become a cycle with the properties we require.

Theorem 5.3. *Let G be a bipartite graph with a nonzero channel $C \in \mathcal{C}(G)$. Then for each matching μ of G there is a nonempty set of edges $C(\mu) \subseteq E$ satisfying the following properties.*

(i) $C(\mu)$ is a simple cycle of even length.

(ii) Every second edge in $C(\mu)$ is in μ.

(iii) $C(\mu)$ depends only on the edges of μ containing a vertex from C.

(iv) If μ' is a matching satisfying $C(\mu) = \mu \oplus \mu'$ then $C(\mu') = C(\mu)$.

Proof. Fix a $v_0 \in C$ and pairing function $f_\bullet$ across C.

We construct $C(\mu)$ as follows. For even $n \in \mathbf{N}$, set e_n to be the edge of the matching μ that contains v_n, and set v_{n+1} to be the other vertex contained in that edge. For odd $n \in \mathbf{N}$, set $e_n = f_{v_n}(e_{n-1})$ and v_{n+1} to be the other vertex in e_n.

Because G is finite, we must have $v_{n+p} = v_n$ for some $p > 0$ and some $n \geq 0$. Let n_0 be minimal among such n and let p be the smallest value such that $v_{n_0+p} = v_{n_0}$.

We claim n_0 is even and p is even. That p is even follows from the bipartiteness of G, since the sequence of vertices $\{v_n\}$ must alternate in color. If n_0 were odd, then $v_{n_0+p} \in e_{n_0+p-1} \in \mu$ and $v_{n_0} = v_{n_0+p} \in e_{n_0} \in \mu$ so $e_{n_0} = e_{n_0+p-1}$ since vertices have degree one in μ. But then the other vertices in both edges are equal, giving $v_{n_0-1} = v_{n_0+p-1}$ and contradicting the minimality of n_0. Thus n_0 is even.

We claim

$$C(\mu) = \{e_n \mid n_0 \leq n < n_0 + p\}$$

satisfies the above properties.

(i) $C(\mu)$ is a simple cycle of even length:

$C(\mu)$ is indeed a cycle, since by definition e_n and e_{n+1} share vertex v_{n+1} for all $n \in \mathbf{N}$ and $v_{n_0+p} = v_{n_0}$. It is simple by minimality of p, and it has even length since p is even.

(ii) Every second edge in $C(\mu)$ is in μ:

This follows from the definition of e_n for even n.

(iii) $C(\mu)$ depends only on the edges of μ containing a vertex from C:

The construction of $\{v_n\}$ and $\{e_n\}$ depended only on the choice of the initial vertex v_0, the pairing function $f_\bullet$, and edges containing vertices from C. Thus $C(\mu) = C(\mu')$ for matchings μ, μ' that differ away from C.

(iv) If μ' is a matching satisfying $C(\mu) = \mu \oplus \mu'$ then $C(\mu') = C(\mu)$:

Denote the vertex and edge sequence associated to $C(\mu')$ by $\{v'_n\}$ and $\{e'_n\}$ respectively. If $n < n_0$, then e_n is not contained in $C(\mu)$ by minimality of n_0. Additionally, if $e_n \in \mu$ then necessarily $e_n \in \mu'$ because $\mu \oplus \mu' = C(\mu)$. Therefore for n even if $v_n = v'_n$ we find $v_{n+1} = v'_{n+1}$. The pairing function f_{v_n} is independent of μ so $e_{n+2} = e'_{n+2}$ and $v_{n+2} = v'_{n+2}$. Since $v'_0 = v_0$, induction gives $v_n = v'_n$ for $n \le n_0$ and $e_n = e'_n$ for $n < n_0$.

Now, since e'_{n_0} must contain $v'_{n_0} = v_{n_0}$ and v_{n_0} is in $C(\mu) = \mu \oplus \mu'$, it follows that e'_{n_0} must be in exactly one of μ and $C(\mu)$. Since $v'_{n_0} = v_{n_0}$ is a vertex in $C(\mu)$, by item (ii) the edge containing it in μ is in $C(\mu)$. Thus if e'_{n_0} were in μ, then it would also be in $C(\mu)$, a contradiction. Therefore e'_{n_0} is in $C(\mu)$ and not in μ. Since $C(\mu)$ is a simple cycle, v_{n_0} is only contained in the edges e_{n_0} and e_{n_0+p} within $C(\mu)$, so e'_{n_0} must be one of those edges. It cannot be e_{n_0} since this is contained in μ, so we must have $e'_{n_0} = e_{n_0+p-1}$ and therefore $v'_{n_0+1} = v_{n_0+p-1}$ as well. Then since

$$f_{v_{n_0+p-1}}(e_{n_0+p-2}) = e'_{n_0}$$

we get

$$f_{v_{n_0+p-1}}(e'_{n_0}) = e_{n_0+p-2}$$

and therefore $e'_{n_0+1} = e_{n_0+p-2}$ and $v'_{n_0+2} = v_{n_0+p-2}$. Proceeding by induction shows that for $k \le p$, we have $v'_{n_0+k} = v_{n_0+p-k}$ and $e'_{n_0+k-1} = e_{n_0+p-k}$. Thus in particular $n'_0 = n_0$ and $p' = p$, since the first repetition in the $\{v'_n\}$ vertex sequence is $v'_{n_0} = v'_{n_0+p}$.

Therefore

$$C(\mu') = \{e'_n \mid n_0 \le n < n_0 + p\},$$

which is just the elements of $C(\mu)$ in reverse order. Hence $C(\mu) = C(\mu')$ as claimed.

□

Corollary 5.3.1. *Let G be a bipartite graph. If $\mathcal{C}(G)$ contains a nonempty channel, then m_G is even.*

Proof. If $C \in \mathcal{C}(G)$ is nonzero, then we claim the map on matchings of G given by

$$\mu \longmapsto \mu' := \mu \oplus C(\mu)$$

is an involution with no fixed points. By our discussion at the start of the section, we just need to show that $C(\mu') = C(\mu)$. This follows directly from Theorem 5.3(iv). □

Remark 5.4. It is interesting (and rather inconvenient) to note that the action of channels on matchings we define above can not in general be extended to a group action of $\mathcal{C}(G)$ or $\mathcal{C}_B(G)$ since, for instance, the action of two distinct channels need not commute. Such a group action would be a very useful combinatorial tool. We give some thoughts on this at the end of the paper.

5.2. **Even m_G implies existence of a channel.** Proving the converse statement will take some different machinery, which will turn out to have more general applications. First, a useful notation.

Definition 5.5. Let G be a graph and v be a vertex. Define $\mathcal{C}[v](G)$ to be the subspace of channels in $\mathcal{C}(G)$ that do not contain v.

The following lemma describes how channels are affected by removal of an edge. We restrict to bipartite G to clean up the statement of the result, but the arguments will generalize if care is taken about how vertices can appear in channels. Recall that $\mathcal{C}_B(G)$ denotes the channels in a bipartite graph G containing only black vertices.

Lemma 5.6 (Channel Digging Lemma). *Let $G = (V, E)$ be any bipartite graph, and fix an edge $e = (b, w)$ (with vertices the corresponding colors). Define the subgraphs*

$$G^e = G - e$$

and

$$G' = G - \{b, w\}.$$

Then the following statements hold:

(i) $\mathcal{C}_B[b](G) = \mathcal{C}_B[b](G^e) = \mathcal{C}_B(G^e) \cap \mathcal{C}_B(G')$.

(ii) *If $\mathcal{C}_B[b](G^e) \neq \mathcal{C}_B(G^e)$, then there is a bijection $\mathcal{C}_B(G) \leftrightarrow \mathcal{C}_B(G')$ preserving $\mathcal{C}_B[b](G)$.*

Proof.

(i) First, notice that G and G^e have the same vertex set, and $G - \{b, w\} = G^e - \{b, w\}$. Thus for any channel C of G or G^e that does not contain b, the intersection $N(v) \cap C$ will be the same for all vertices in both graphs. Therefore $\mathcal{C}_B[b](G) = \mathcal{C}_B[b](G^e)$. It also follows that channels of G not containing b will be channels of $G' = G - \{b, w\}$ since the only neighborhoods of white points that differ between these graphs are again those containing b.

Conversely, let C be a channel in both G^e and G'. Then C does not contain b or w, since G' does not have those vertices. And since $C \in \mathcal{C}_B(G^e)$ it is also in $\mathcal{C}_B[b](G^e)$ as desired.

(ii) Let $B = \{C_1, ..., C_n\}$ be a basis for $\mathcal{C}_B(G^e)$. If any of the basis elements contain b, we may reorder them so that C_1 contains b. We may then eliminate b from the rest of the basis elements by replacing a channel C_i containing b with $C_i \oplus C_1$. We then have two cases. If b is contained in any element of $\mathcal{C}_B(G^e)$, then

$$\mathcal{C}_B[b](G^e) = \text{span}\{C_2, ..., C_n\}.$$

Otherwise, if b is not contained in any element of $\mathcal{C}_B(G^e)$,

$$\mathcal{C}_B[b](G^e) = \mathcal{C}_B(G^e).$$

Our hypothesis states that we are in the first case. By part (i) we may write $\mathcal{C}_B[b](G) = \text{span}\{C_2, ..., C_n\}$. Extend this to a full basis of $\mathcal{C}_B(G)$ given by $\{C_2, ..., C_n, C_{n+1}, ..., C_{n+m}\}$. In particular all of $\{C_{n+1}, ..., C_{n+m}\}$ contain b. We construct a bijection

$$\mathcal{C}_B(G) \leftrightarrow \mathcal{C}_B(G')$$

by sending channels in $\mathcal{C}_B(G)$ according to

$$f: C \longmapsto \begin{cases} C & \text{if } b \notin C \\ C_1 \oplus C & \text{if } b \in C \end{cases}.$$

We check the image of f is indeed in $\mathcal{C}_B(G')$. That C is in $\mathcal{C}_B(G')$ for $b \notin C$ follows from part (i). Otherwise, $b \in C$. Then the symmetric difference $C_1 \oplus C$ does not contain b or w and thus is a vertex set of G'. Any white vertex $v \in G'$ is adjacent to an even number of elements in both C_1 and C by the evenness constraint of channels. Therefore v is also adjacent to an even number of elements in $C_1 \oplus C$ by properties of the symmetric difference. This implies $C_1 \oplus C$ is indeed a channel in $\mathcal{C}_B(G')$.

We define the inverse map similarly. For a channel C in $\mathcal{C}_B(G')$,

$$g: C \longmapsto \begin{cases} C & \text{if } |N(w) \cap C| \text{ is even} \\ C_1 \oplus C & \text{if } |N(w) \cap C| \text{ is odd} \end{cases}.$$

We check the image of g is indeed in $\mathcal{C}_B(G)$. Notice that the only way a channel C in $\mathcal{C}_B(G')$ may fail to be a channel in $\mathcal{C}_B(G)$ is if w is adjacent to an odd number of points in C, since the evenness constraint is imposed on every other white vertex in G. Thus if $|N(w) \cap C|$ is even, all evenness constraints are satisfied and C is also a channel in $\mathcal{C}_B(G)$. Otherwise, $|N(w) \cap C|$ is odd. We also have that $|N(w) \cap C_1|$ is odd, since the evenness constraint is satisfied at w before including b in $N(w)$. Therefore $|N(w) \cap C_1 \oplus C|$ is even and the evenness constraint holds everywhere. Thus $C_1 \oplus C_2$ is a channel in $\mathcal{C}_B(G)$.

Both maps are the identity on $\mathcal{C}_B[b](G)$: all channels in this set do not contain b and do have $|N(w) \cap C|$ even, so the first condition for is satisfied for both f and g. For channels in $\mathcal{C}_B(G)$ that do contain b, the set $C_1 \oplus C$ has odd intersection with the neighborhood of w and therefore $g \circ f$ is the identity. Channels in $\mathcal{C}_B(G')$ do not contain b, so those that have odd intersection with the neighborhood of w map to a channel $C_1 \oplus C$ which contains b. Thus $f \circ g$ is the identity as well, and the claim follows.

□

Channel digging is a versatile tool. To begin with, let us use it to prove constructively the claim titling this section.

Theorem 5.7. *If a bipartite graph G has an even number of matchings, then it has a nonempty channel.*

Proof. First note that G is nonempty, since the empty graph has one matching. We proceed by induction on $|V| + |E|$. If every white vertex of G has even degree, then we can take the black vertices of V to be our channel. Otherwise, some white vertex w has odd degree. Then for some edge $e = (b, w)$, the subgraph $G - \{b, w\}$ has an even number of matchings, since otherwise by Proposition 2.1

$$m_G = \sum_{(b,w)\in E} m_{G-\{b,w\}}$$

would be the sum of an odd number of odd numbers, and thus odd, a contradiction. Define $G^e := G - e$ and $G' := G - \{b, w\}$. Now, because $m_{G'}$ is even, so is

$$m_G - m_{G'} = m_{G^e}.$$

Thus by the inductive hypothesis both G' and G^e have nonempty channels (G' cannot be the empty graph since that would imply G is a single edge, which would have an odd number of matchings). By channel digging (i) and (ii), either $|\mathcal{C}_B(G)| \geq |\mathcal{C}_B(G^e)|$ or $|\mathcal{C}_B(G)| = |\mathcal{C}_B(G')|$, and in both cases we have a channel for G. □

6. Digging channels

Channel digging allows us to remove one pair of vertices at a time from our graph while keeping track of the available channels. In some cases, repeated application of this process can reduce our graph to one with known properties. The following theorem describes one use of this process.

Theorem 6.1. *Let G be a bipartite graph with n vertex disjoint edges $e_1 = (b_1, w_1), \ldots, e_n = (b_n, w_n)$ selected. Set*

$$G^e = G - \{e_1, \ldots, e_n\}$$

$$G' = G - \{b_1, w_1, \ldots, b_n, w_n\}.$$

If $\dim \mathcal{C}_B(G^e) \geq n$, then m_G and $m_{G'}$ have the same parity.

Proof. If any nonempty channel in $\mathcal{C}_B(G^e)$ does not use any of $b_1, ..., b_n$, then it is also a channel in G' and so both m_G and $m_{G'}$ are even. Otherwise there exists a basis $B = \{C_1, ..., C_n\}$ for $\mathcal{C}_B(G^e)$ such that each C_i contains b_i and does not contain b_j with $j \neq i$. This follows since if any two distinct basis vectors C_i and C_j contained the same b_i, then we could replace C_j with $C_i \oplus C_j$ to get a basis with one less repeated b. If the basis had more elements than n we would be able to remove all bs from one of the basis elements and reduce to the previous case. Thus $\dim \mathcal{C}_B(G^e) = n$ and we have a basis as described.

As a result of this, we may apply channel digging (ii) to remove each $\{b, w\}$ pair. Removing $\{b, w\}$ preserves all channels that do not contain b. Thus condition (ii) of channel digging continues to apply, and we may remove all n pairs of vertices while preserving $|\mathcal{C}_B(G)|$. Thus

$$|\mathcal{C}_B(G)| = |\mathcal{C}_B(G')|$$

and the result follows. □

Proposition 6.2. *If $gcd(m + 1, n + 1) = 1$, then the $m \times n$ rectangle grid graph has an odd number of matchings.*

Proof. Refer to Figure 7. Without loss of generality, $m \leq n$. Set $r = \lceil m/2 \rceil$. Let $e_1, ..., e_r$ be the edges between the vertices indicated in bold in Figure 7 and set $G^e = G - \{e_1, ..., e_r\}$. The channels constructed in Proposition 4.6 give r independent channels in $\mathcal{C}_B(R_{m \times m})$. These are also valid channels of G^e. Thus by Theorem 6.1 we may remove black vertices in column m and white vertices in column $m+1$ while preserving the parity of m_G. Any matching of the resulting graph G' must use all remaining edges between vertices in columns m and $m + 1$. Thus we may remove the rest of the vertices in those columns without changing the number of tilings. The resulting graph will be the disjoint union of a $m \times (m - 1)$ rectangle and a $m \times (n - m - 1)$ rectangle. The result then follows by induction on the size of the rectangle, since

$$\gcd(m+1, n-m-1+1) = \gcd(m+1, n+1) = 1 \text{ and } \gcd(m+1, m-1+1) = 1.$$

□

7. Conclusion

In addition to the results we have shown, channels may be used to give lower bounds supporting Conjectures 1, 2, 4, and 5 in [12] and likely for Problems 15 and 30 in [14] as well. As we have seen, channels provide an effective lower bound on the power of two dividing a matching count. In addition, when there are no nonzero channels they tell us that power is exactly 0. It would be nice to find exact powers of two more generally. This prompts a natural question.

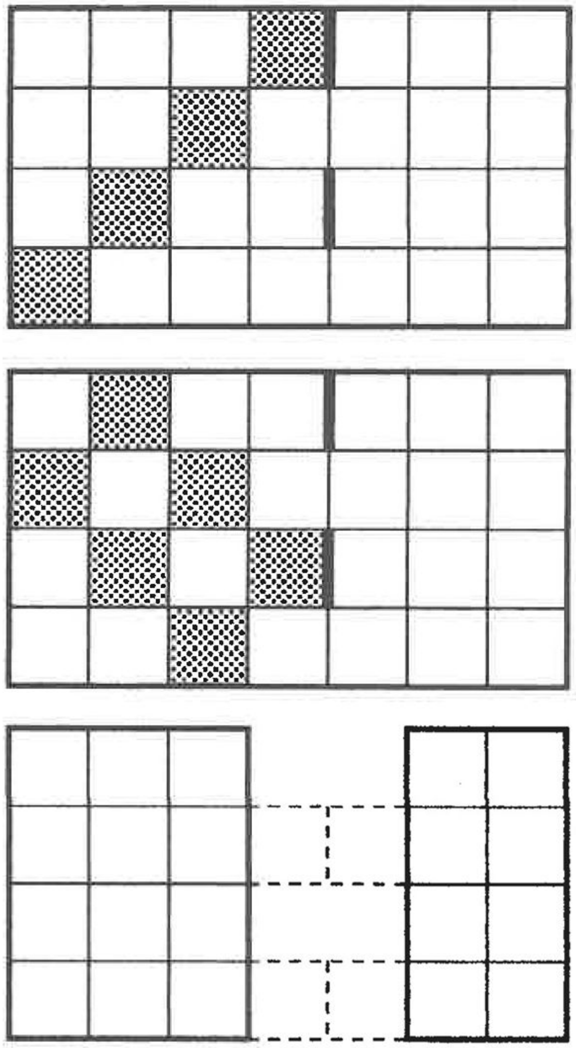

FIGURE 7. The shaded sets are channels of G^e, where e_1 and e_2 are the edges that cross the bold lines. Thus, using Theorem 6.1, the number of matchings of G' (the third figure) has the same parity as that of the original graph. Any matching of G' will use the edges between the dashed vertices, so this result continues to hold if we remove them.

Problem 7.1. *Determine when the number of channels is the exact power of two dividing m_G^2. Even better, determine how many additional powers of two are carried by each channel more generally. Alternatively, provide a method for determining an upper bound on powers of two dividing m_G.*

For more on how additional powers of two are distributed in the Smith normal form of the Kasteleyn matrix (and therefore among channels), see [10]. Additional powers of two may be associated to a result such as Ciucu's Factorization Theorem (Proposition 1.1). Indeed, graphs where this theorem applies tend to have additional powers of two beyond what channels would predict (see for example the Aztec Diamond [2]). One possible route for approaching Problem 7.1 is to consider factoring out the action of channels in some manner, and examining the remaining structure. Another route may arise by solving the following problem.

Problem 7.2. *Find a combinatorial proof of Theorem 4.4.*

Since Theorem 4.4 requires the Kasteleyn signing of G, such a proof would likely invoke planarity. As mentioned in Remark 5.4, one possible approach to this is constructing a free action of $\mathcal{C}_B(G)$ on the set of matchings of G. For the general, non-bipartite case we would want an action of $\mathcal{C}(G)$ on pairs of matchings. Because the definition of channels involves neighborhoods of even size, searching for an action that uses properties of Eulerian circuits may yield productive results.

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References

[1] Mihai Ciucu, *Enumeration of Perfect Matchings in Graphs With Reflective Symmetry*, 1996.

[2] ———, *Perfect Matchings and Perfect Powers*, Journal of Algebraic Combinatorics **17** (2003jan), no. 3, 335–375, available at 0501521.

[3] Henry Cohn, *2-Adic behavior of numbers of domino tilings*, Electronic Journal of Combinatorics **6** (1999aug), no. 1, available at 0008222.

[4] William Jockusch, *Perfect matchings and perfect squares*, Journal of Combinatorial Theory, Series A **67** (1994), no. 1, 100–115.

[5] Peter E. John and Horst Sachs, *On a strange observation in the theory of the dimer problem*, Discrete Mathematics **216** (2000), no. 1-3, 211–219.

[6] P. W. Kasteleyn, *The statistics of dimers on a lattice. I. The number of dimer arrangements on a quadratic lattice*, Physica **27** (1961dec), no. 12, 1209–1225, available at 0901.2686.

[7] ———, *Dimer statistics and phase transitions*, Journal of Mathematical Physics **4** (1963feb), no. 2, 287–293.

[8] K. P. Kokhas, *Domino tilings of Aztec diamonds and squares*, Journal of Mathematical Sciences **158** (2009may), no. 6, 868–894.

[9] Eric H. Kuo, *Applications of graphical condensation for enumerating matchings and tilings*, Theoretical Computer Science **319** (2004), no. 1-3, 29–57, available at 0304090.

[10] Greg Kuperberg, *Kasteleyn cokernels*, Electronic Journal of Combinatorics **9** (2002aug), no. 1 R, 1–30, available at 0108150.

[11] L. Lovász, *Combinatorial Problems and Exercises, Second Edition*, Elsevier Science, 1993.

[12] Lior Pachter, *Combinatorial approaches and conjectures for 2-divisibility problems concerning domino tilings of polyominoes*, Electronic Journal of Combinatorics **4** (1997), #R29.

[13] Jerome K. Percus, *One more technique for the dimer problem*, Journal of Mathematical Physics **10** (1969oct), no. 10, 1881–1884.

[14] James Propp, *Enumeration of Matchings: Problems and Progress*, New Perspectives in Geometric Combinatorics **38** (1999apr), 255–291, available at 9904150.

[15] H. N.V. V. Temperley and Michael E. Fisher, *Dimer problem in statistical mechanics-an exact result*, Philosophical Magazine **6** (1961aug), no. 68, 1061–1063.

[16] L. G. Valiant, *The complexity of computing the permanent*, Theoretical Computer Science **8** (1979jan), no. 2, 189–201.

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