β-PACKING SETS IN GRAPHS

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ABSTRACT. A set $S \subseteq V$ is *α-dominating* if for all $v \in V - S$, $|N(v) \cap S| \geq \alpha|N(v)|$. The *$\alpha$-domination number* of G equals the minimum cardinality of an α-dominating set S in G. Since being introduced by Dunbar, et al. in 2000, α-domination has been studied for various graphs and a variety of bounds have been developed. In this paper, we propose a new parameter derived by flipping the inequality in the definition of α-domination. We say a set $S \subset V$ is a *β-packing set* of a graph G if S is a proper, maximal set having the property that for all vertices $v \in V - S$, $|N(v) \cap S| \leq \beta|N(v)|$ for some $0 < \beta \leq 1$. The *β-packing number* of G (β-pack(G)) equals the maximum cardinality of a β-packing set in G. In this research, we determine β-pack(G) for several classes of graphs, and we explore some properties of β-packing sets.

Keywords: β-packing, α-domination, graph theory, graph parameters

1. INTRODUCTION

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and *order* $n = |V|$. The *open neighborhood* of a vertex v is the set $N(v) := \{u \mid uv \in E\}$ of vertices u that are adjacent to v; the *closed neighborhood* of v, $N[v] := N(v) \cup \{v\}$.

A set $S \subseteq V$ is *α-dominating* if for all $v \in V - S$, $|N(v) \cap S| \geq \alpha|N(v)|$. The *$\alpha$-domination number* of G equals the minimum cardinality of an α-dominating set S in G. Since being introduced by Dunbar, Hoffman, Laskar, and Markus [4] in 2000, α-domination has been studied for various graphs and a variety of bounds have been developed, see [1, 8, 5, 7, 2]. In this paper, we present a new parameter that is motivated by flipping the inequality in α-domination, known as the β-packing set.

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Definition 1.1. For a graph $G = (V, E)$, a set $S \subset V$ is a *β-packing set* of a graph G if S is a proper, maximal set having the property (which we call the *β-packing property*) that for all vertices $v \in V - S$,

$$|N(v) \cap S| \leq \beta |N(v)|$$

for some $0 < \beta \leq 1$. The *β-packing number* of G, β-pack(G), equals the maximum cardinality of a β-packing set in G.

For example, we say that a set $S \subset V$ is a 1/2-beta packing set if $v \in V - S$, $\frac{|N(v) \cap S|}{|N(v)|} \leq 1/2$ and is maximal. The 1/2-beta packing number equals the maximum cardinality of a 1/2-beta packing set in G.

Example 1.2. In Figure 1 we show all of the 1/2-beta packing sets of the shown graph (up to symmetry). The β-packings sets are shown as the black filled vertices. Note that in each graph, no subset of $V - S$ can be added to S while preserving both the β-packing property and keeping the β-packing set a proper subset. The largest cardinality of these sets is 2, so $\frac{1}{2}\beta$-pack(G) $= 2$.

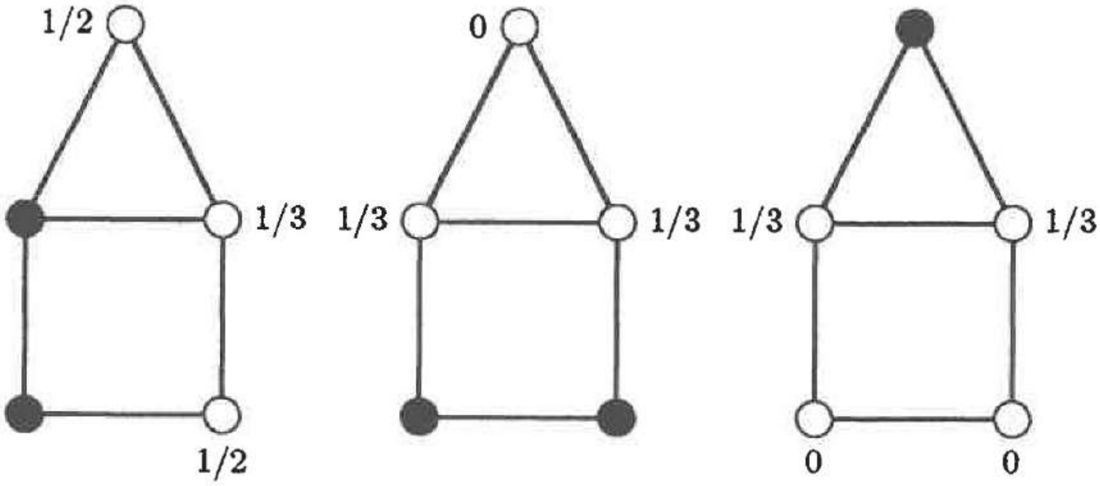

FIGURE 1. The 1/2-beta packing sets (up to symmetry), shown in black. $\frac{1}{2}\beta$-pack(G) $= 2$.

2. Examples and β-Packing Sets for Classes of Graphs

To begin we will consider some examples of different classes of graphs and try to determine some patterns about the β-packing number. We start by looking at the 1/2-beta packing sets for paths and then generalize these results to all paths and cycles. A $\frac{1}{2}\beta$-packing for P_6 is show in Figure 2.

FIGURE 2. The $\frac{1}{2}\beta$-packing set of a path, P_6.

Proposition 2.1. *Given a path P_n of length $n \geq 2$, $\frac{1}{2}\beta$-pack(P_n) $= n-2$ and $V - S$ is connected.*

Proof. Consider a path of length n, $P_n = (V, E)$. If $V - S$ is not connected, S is not maximal, see Proposition 3.2 where we show this in general. Suppose $S \subset V$ and $\{v_i, v_{i+1}\} = V - S$ for some $i \in [1, n-1]$. As S is proper, it suffices to show that the β-packing property is fulfilled and that S is maximal. To show the former, consider the following cases:

- If $\deg(v_i) = 1$, then $N(v_i) \cap S = \emptyset$ and
$$\frac{|N(v_i) \cap S|}{\deg(v_i)} = 0 \leq \frac{1}{2}.$$
- If $\deg(v_i) = 2$, then $N(v_i) \cap S = \{v_{i-1}\}$ and
$$\frac{|N(v_i) \cap S|}{\deg(v_i)} = \frac{1}{2} \leq \frac{1}{2}.$$
- If $\deg(v_{i+1}) = 1$ then $N(v_{i+1}) \cap S = \emptyset$ and
$$\frac{|N(v_i) \cap S|}{\deg(v_i)} = 0 \leq \frac{1}{2}.$$
- If $\deg(v_{i+1}) = 2$ then $N(v_{i+1}) \cap S = \{v_{i+2}\}$ and
$$\frac{|N(v_i) \cap S|}{\deg(v_i)} = \frac{1}{2} \leq \frac{1}{2}.$$

Thus the β-packing property holds in all cases. Now, we need to show that S is maximal. WLOG, suppose $V - S = \{v_i\}$. We will again consider cases:

- If $\deg(v_i) = 1$ then $N(v_i) \cap S = \{v_{i+1}\}$ and
$$\frac{|N(v_i) \cap S|}{\deg(v_i)} = 1 > \frac{1}{2}.$$

- If $\deg(v_i) = 2$ then $N(v_i) \cap S = \{v_{i-1}, v_{i+1}\}$

$$\frac{|N(v_i) \cap S|}{\deg(v_i)} = 1 > \frac{1}{2}.$$

□

The following three results cover all the possible values of β and show what the corresponding value of β-pack(P_n) is.

Proposition 2.2. *For $\frac{1}{2} \leq \beta < 1$ and $n \geq 2$, β-pack$(P_n) = n - 2$ and $V - S$ is connected.*

Proof. This follows the same proof as the $\frac{1}{2}\beta$-packing set. □

Proposition 2.3. *For $0 < \beta < \frac{1}{2}$, β-pack$(P_n) = 0$.*

Proof. For any $v_i \in V$, $\deg(v_i) = 1$ or 2. This implies $\frac{|N(v_i) \cap S|}{\deg(v_i)}$ is either 0, $\frac{1}{2}$ or 1. But $\frac{|N(v_i) \cap S|}{\deg(v_i)} \leq \beta < \frac{1}{2}$, which implies $S = \emptyset$. So, β-pack$(P_n) = 0$. □

Proposition 2.4. *For $\beta = 1$, β-pack$(P_n) = n - 1$.*

Proof. Letting $\beta = 1$ means that for any v_i, $\frac{|N(v_i) \cap S|}{\deg(v_i)} \leq \beta$. As S must be a proper subset, we have to leave one node out of S. Thus, β-pack$(P_n) = n - 1$. □

Corollary 2.5. *Given a cycle C_n of size $n \geq 3$,*

$$\beta\text{-pack}(P_n) = \begin{cases} 0 & 0 < \beta < \frac{1}{2} \\ n-2 & \frac{1}{2} \leq \beta < 1 \\ n-1 & \beta = 1 \end{cases}$$

and $V - S$ is connected.

Proof. Note that any path can be made into a cycle by adding an edge. Thus, the proof for a cycle is identical to that of a path except that we need only to consider the cases of degree 2. □

Next we will consider complete bipartite graphs and determine their β-packing numbers. An example of a β-packing set is shown in Figure 3 for $K_{4,5}$.

Proposition 2.6. *Let $K_{m,n} = (V_m, V_n, E)$ be a complete bipartite graph. Then for $\beta < 1$, all β-packing sets $S \cup S'$ have the same size, where $S \subset V_m \subset V$, $S' \subset V_n \subset V$, with $|S| = \lfloor \beta \cdot m \rfloor$ and $|S'| = \lfloor \beta \cdot n \rfloor$. Thus, β-pack$(K_{m,n}) = \lfloor \beta \cdot m \rfloor + \lfloor \beta \cdot n \rfloor$.*

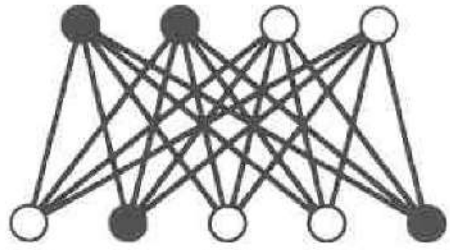

FIGURE 3. A possible $\frac{1}{2}\beta$-packing set of the complete bipartite graph, $K_{4,5}$.

Proof. Let $S \subset V_m$ be any subset of size $\lfloor \beta \cdot m \rfloor$ and $S' \subset V_n$ be any subset of size $\lfloor \beta \cdot n \rfloor$. Since $\beta < 1$, $\lfloor \beta \cdot m \rfloor < m$ and $\lfloor \beta \cdot n \rfloor < n$. Thus, $S \cup S'$ is proper. It suffices to show that the β-packing property is fulfilled and that $S \cup S'$ is maximal. Let $v \in V_m - S$. Then, $\deg(v) = n$ implies

$$|N(v) \cap S'| = |S'| = \lfloor \beta \cdot n \rfloor.$$

Thus,

$$\frac{|N(v) \cap (S \cup S')|}{|N(v)|} = \frac{|N(v) \cap (S')|}{|N(v)|} = \frac{\lfloor \beta \cdot n \rfloor}{n} \leq \frac{\beta \cdot n}{n} = \beta.$$

Now let $v' \in V_n - S'$. Then, following the same process, we see that

$$\frac{|N(v') \cap (S \cup S')|}{|N(v')|} \leq \beta.$$

Finally, we must show that $S \cup S'$ is maximal. Suppose for contradiction there was a proper subset $S \cup S' \cup U \subset V$ for which the β-packing property held with $\emptyset \neq U \subset V - (S \cup S')$. U must contain at least one vertex u. WLOG, let u be in the side $u \in V_m \cap U$. Then for $v \in V_n - (S' \cup U)$,

$$\frac{|N(v) \cap (S \cup S' \cup U)|}{|N(v)|} \geq \frac{|N(v) \cap (S \cup \{u\})|}{|N(v)|} = \frac{\lfloor \beta \cdot m \rfloor + 1}{m}.$$

Note that $\beta \cdot m < \lfloor \beta \cdot m \rfloor + 1$ which implies $\beta < \frac{\lfloor \beta \cdot m \rfloor + 1}{m}$. Thus

$$\frac{|N(v) \cap (S \cup S' \cup U)|}{|N(v)|} > \beta,$$

so $S \cup S'$ is maximal and $\beta\text{-pack}(K_{m,n}) = \lfloor \beta \cdot m \rfloor + \lfloor \beta \cdot n \rfloor$.

□

Proposition 2.7. *For $\beta = 1$, β-pack$(K_{m,n}) = m + n - 1$.*

Proof. As the β-packing set must be proper, we let all the nodes be in the β-packing set and then remove one. As $|K_{m,n}| = m + n$, β-pack$(K_{m,n}) < |K_{m,n}|$. Adding another node to this set would be all of $K_{m,n}$, so $S \cup S'$ is both proper and maximal.
Let $v \notin S \cup S'$. Then, $\frac{|N(v) \cap (S \cup S')|}{|N(v)|} = 1$, since every other node is in $S \cup S'$. Thus, β-pack$(K_{m,n}) = m + n - 1$. □

If we try to generalize these results to complete multipartite graphs, Proposition 2.6 does not generalize in the natural way, but Proposition 2.7 does.

Example 2.8. Consider the complete multipartite graph $K_{3,3,3,3}$ and let $\beta = 1/2$. A β-packing set is given by taking 1 vertex in each of three partitions and 2 vertices out of the forth partition, for a total of 5 vertices in S. One can check this gives

$$\frac{1}{2}\beta\text{-pack}(K_{3,3,3,3}) = 5 > \lfloor \beta \cdot 3 \rfloor + \lfloor \beta \cdot 3 \rfloor + \lfloor \beta \cdot 3 \rfloor + \lfloor \beta \cdot 3 \rfloor = 4.$$

Corollary 2.9. *For $\beta = 1$, β-pack$(K_{n_1,n_2,\ldots,n_m}) = n_1 + \cdots + n_m - 1$.*

Proof. The proof is similar to the bipartite case. □

3. General Properties of β-packing sets

In this section we present several general properties about β-packing sets and the β-packing number. Our first property shows how the β-packing numbers corresponding to different β's are related.

Proposition 3.1. *Let $0 < \beta_1 \leq \beta_2 \leq 1$. Then β_1-pack$(G) \leq \beta_2$-pack(G).*

Proof. Consider β_1-pack(G), for any β_1-packing set S, $\forall v \in V - S$,

$$\frac{|N(v) \cap S|}{|N(v)|} \leq \beta_1 \leq \beta_2.$$

So any such S is contained in a β_2-packing set and one could add vertices until S becomes maximal w.r.t β_2. □

It was already seen in Proposition 2.1 for paths that the complement a β-packing set is connected. This is in fact a general property that holds for all graphs.

Proposition 3.2. *For any β-packing set S, $V - S$ is connected.*

Proof. If $V - S$ is not connected, then S is not maximal since one of the components of $V - S$ could be added to S to form S' and for all other $v \in V - S'$ we still have the property

$$\frac{|N(v) \cap S'|}{|N(v)|} \leq \beta$$

and S' would still be proper. □

Proposition 3.3. *Let $\Delta(G)$ be the max degree of a vertex of a connected graph. If $\beta < \frac{1}{\Delta(G)}$, then β-pack$(G) = 0$.*

Proof. Suppose S is a nonempty β-packing set. For any vertex $v \in V - S$, if a neighbor is in a β-packing set S, then

$$\beta < \frac{1}{\Delta(G)} \leq \frac{1}{\deg(v)} \leq \frac{|N(v) \cap S|}{\deg(v)},$$

a contradiction. Thus no vertex has a neighbor in S. Therefore $S = \emptyset$. □

The next three properties investigate the question of which values for β in the interval $0 < \beta \leq 1$ are interesting to consider.

Proposition 3.4. *If $\beta = 1$, then β-pack$(G) = n - 1$.*

Proof. A β-packing set must be proper, but we can just leave out any one vertex. □

Proposition 3.5. *Let G be connected. If $\beta < 1$, then β-pack$(G) < n - 1$.*

Proof. Suppose $\{v\} = V - S$. Then

$$\frac{|N(v) \cap S|}{|N(v)|} = \frac{|N(v)|}{|N(v)|} = 1 > \beta.$$

□

Let us consider the following question a bit more.

Question 1. *Given a graph G how many "interesting" β's are there to consider? By interesting we mean that as β increases from 0 to 1 it is only at these values where the value of β-pack(G) could change.*

Let $\delta(G) = d_1, ..., d_t = \Delta(G)$ be the distinct degrees of vertices in the graph. Then we claim the possible interesting β's are a subset of the following ratios:

$$0, \frac{1}{d_1}, \frac{2}{d_1}, ..., \frac{d_1 - 1}{d_1}, 1$$

$$\frac{1}{d_2}, \frac{2}{d_2}, \ldots, \frac{d_2-1}{d_2}$$
$$\vdots \qquad \vdots$$
$$\frac{1}{d_t}, \frac{2}{d_t}, \ldots, \frac{d_t-1}{d_t}.$$

4. Related Parameters

The initial motivation for defining β-packing sets was from α-domination, and it is natural to ask what relationships the two parameters may have with each other. One might ask if $\beta = \alpha$ weather

$$\gamma_\alpha(G) \leq \beta\text{-pack}(G)? \quad \text{or} \quad \gamma_\alpha(G) \geq \beta\text{-pack}(G)?$$

The answer is neither one in general. We have by Proposition 2.3 that $\beta = \frac{1}{3}\text{-pack}(P_n) = 0$. But from [4, Prop. 1] that $\gamma_{\alpha=\frac{1}{3}}(P_n) = \lceil \frac{n}{3} \rceil$. So this is an example of $\frac{1}{3}\text{-pack}(G) < \gamma_{\frac{1}{3}}(G)$.

On the other hand, we have that by Proposition 2.6 that if when $\beta < 1$ $\beta\text{-pack}(K_{m,n}) = \lfloor \beta \cdot m \rfloor + \lfloor \beta \cdot n \rfloor$. In [4, Prop. 4] they have the result that for $1 \leq m \leq n$

$$\gamma_\alpha(K_{m,n}) = \min\{\lceil \alpha m \rceil + \lceil \alpha n \rceil, m\}.$$

Thus if we let for example $m = 1$, $n = 10$, $\beta = \alpha = 1/2$ we get than

$$\gamma_{\frac{1}{2}}(K_{1,10}) = 1 < \frac{1}{2}\text{-pack}(K_{1,10}) = 5.$$

We think it is an interesting open direction of study to consider if there are different relationship between α-domination and β-packing would be interesting to consider.

5. Conclusion

In conclusion, we have introduced the new graph parameter, the β-packing number, and studied some of its properties and given formulas for it for certain classes of graphs. Our motivation for defining β-packing sets comes for α-domination, but we leave it as an open direction to investigate what relationships these two parameters have with each other. Other interesting open directions would include determining the value of the β-packing number for other classes of graphs and determining the computational complexity of finding β-packing sets or the β-packing number. We hope that this introductory paper and promising future directions will promote further interest in considering β-packing.

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