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Cyclic ordering of edges without long acyclic subsequences

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ABSTRACT

Let G be a connected graph with m edges. The density of a nontrivial subgraph H with $\omega(H)$ components is $d(H) = |E(H)|/(|V(H)|-\omega(H))$. A graph G is uniformly dense if for any nontrivial subgraph H of G, $d(H) \leq d(G)$. For each cyclic ordering $o = (e_1, e_2, \cdots, e_m)$ of $E(G)$, let $h(o)$ be the largest integer k such that every k cyclically consecutive elements in o induce a forest in G; and the largest $h(o)$, taken among all cyclic orderings of G, is denoted by $h(G)$. A cyclic ordering o of G is a cyclic base ordering if $h(o) = |V(G)|-\omega(G)$. In [15], Kajitani et al proved that every connected nontrivial graph with a cyclic base ordering is uniformly dense, and conjectured that every uniformly dense graph has a cyclic base ordering. This motivates the study of $h(G)$. In this paper, we investigate the value of h for some families of graphs and determine all connected graphs G with $h(G) \leq 2$.

Keywords: uniformly dense graphs, cyclic orderings, cyclic base ordering

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1. Introduction

In this paper, graphs considered are finite and loopless. We follow [1] for undefined terms and notation. A graph is nontrivial if its edge set is not empty. For a nontrivial graph

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G, let $\omega(G)$ be the number of connected components of G. Following the notations in [3], define the density $d(G)$ and the fractional arboricity $\gamma(G)$ as follows:

$$d(G) = \frac{|E(G)|}{|V(G)| - \omega(G)} \text{ and } \gamma(G) = \max\{d(H) : \emptyset \neq E(H) \subseteq E(G)\}.$$

A graph G is uniformly dense if $\gamma(G) = d(G)$. Uniformly dense graphs have been considered as models of certain secured networks, as seen in [12]. There have been quite a few studies on uniformly dense graphs and matroids, as can be seen in [3, 2, 4, 5, 6, 8, 9, 10, 11, 12, 7, 14, 15, 13, 16, 17, 18, 19], and the references therein, among others.

Let m denote the number of edges of G. A sequence to list all the edges in G is a cyclic ordering of G if the subscripts of the sequence are taken modulo m. Let $\mathcal{O}(G)$ be the collection of all cyclic ordering of G. For each $o = (e_1, e_2, \cdots, e_m) \in \mathcal{O}(G)$, let $h(o)$ be the largest integer k such that every k cyclically consecutive elements in o induces a forest in G, where a forest is a graph that contains no cycles; and define $h(G) = \max\{h(o) : o \in \mathcal{O}(G)\}$. By definition, $1 \leq h(G) \leq |V(G)| - \omega(G)$ for any loopless nontrivial graph G. It is therefore of particular interests to investigate the conditions when equality holds in either the lower bound or the upper bound. A cyclic order $o \in \mathcal{O}(G)$ is a cyclic base ordering if $h(o) = |V(G)| - \omega(G)$. The following has been proved by Kajitani et al, relating cyclic orderings with the property of being uniformly dense in graphs.

Theorem 1.1. [15] *Let G be a connected nontrivial graph on n vertices. If $h(G) = n - 1$, then G is uniformly dense.*

Kajitani et al. in [15] proposed the conjecture that if G is uniformly dense, then $h(G) = n - 1$. Thus investigating the value of $h(G)$ is of interests. The purpose of this study is to investigate the relationship between the value of $h(G)$ and the structural properties of G which may measure the closeness for a graph G to being uniformly dense. In the next section, we present some preliminaries, and the main results are stated and proved in Section 3.

2. Preliminaries

Proposition 2.1. *Let G be a connected nontrivial graph with n vertices. Then every cyclic ordering of G is a cyclic base ordering if and only if G is either a tree or a cycle.*

Proof. We prove the sufficiency first. If G is a tree, then $|E(G)| = n - 1$. Clearly, for any cyclic ordering o of G, $h(o) = h(G) = n - 1$ and so it is a cyclic base ordering. Now we suppose G is a cycle. We know $|E(G)| = n$ and $h(G) < n$. Notice that every set of $n - 1$ edges induces a path. Therefore, for any cyclic ordering o of G, $h(o) = n - 1$ and so $h(G) = n - 1$. Every cyclic ordering of G is a cyclic base ordering.

We prove the necessity by contradiction. Suppose every cyclic ordering of G is a cyclic base ordering but G is neither a tree nor a cycle. Since G is connected, $|E(G)| \geq n$ and G contains a cycle C but $G \neq C$. Without loss of generality, suppose $E(G) = \{e_1, e_2, \cdots, e_m\}$ and $C = e_1 e_2 \cdots e_i e_1$ is a cycle of G. Consider the cyclic ordering $o =$

$(e_1, e_2, \cdots, e_i, e_{i+1}, \cdots, e_m)$. We have $h(o) < i \leq n-1$ because $e_1 e_2 \cdots e_i e_1$ is a cycle and $G \neq C$. However, by the assumption, it is a cyclic base ordering of G, so $h(o) = n-1$, a contradiction. This completes the proof. □

Suppose $u, v \in V(G)$. Let $E_{uv} = \{e \in E(G) | \ e$ is incident with both u and $v\}$, and $m_{uv} = |E_{uv}|$. If there is no edge between u and v, then $E_{uv} = \emptyset$ and $m_{uv} = 0$. Let $\Delta_m(G) = \max\{m_{uv} | u, v \in V(G)\}$. We prove the following.

Lemma 2.2. *For every graph G, if $|E(G)| < (i+1)\Delta_m(G)$, then $h(G) \leq i$ where $i \geq 1$.*

Proof. Assume that $h(G) \geq i+1$. Suppose $|E_{uv}| = \Delta_m(G)$ where $u, v \in V(G)$. Then for any cyclic ordering, every $i+1$ cyclically consecutive elements induce a forest in G, and contain at most one edge in E_{uv}. Then $m \geq (i+1)\Delta_m(G)$. It contradicts to $m < (i+1)\Delta_m(G)$. Hence $h(G) \leq i$. □

Corollary 2.3. *For a nontrivial graph G, $m < 2\Delta_m(G)$ if and only if $h(G) = 1$.*

Proof. We prove the necessity first. It is known that $h(G) \geq 1$. By Lemma 2.2, $h(G) \leq 1$. Therefore, $h(G) = 1$. It remains to prove the sufficiency. Assume that $m \geq 2\Delta_m(G)$. Suppose $E_{uv} = \{e_1, \cdots, e_{\Delta_m(G)}\}$, where $u, v \in V(G)$. Since $m \geq 2\Delta_m(G)$ and $|E_{uv}| = \Delta_m(G)$, we can put at least one edge after e_i and the edges between e_i and e_{i+1} are different for $1 \leq i \leq \Delta_m(G)$ and $e_{\Delta_m(G)+1} = e_1$. Then we have a cyclic ordering o such that $h(o) \geq 2$. It is a contradiction to $h(G) = 1$. Thus $m < 2\Delta_m(G)$. □

3. Results

In this section, we study $h(G)$ of the complete graph on three vertices, denoted by K_3, where multiple edges are allowed. Also, we prove a formula for obtaining $h(G)$ under a particular set of constraints. Finally, we propose the structure for all graphs with $h(G) \leq 2$ and prove the necessity of the conjecture.

Theorem 3.1. *Let G be the graph $(m_1, m_2, m_3)K_3$ where m_1, m_2, m_3 are the number of edges between two vertices and $m_1 \leq m_2 \leq m_3$. Then*

$$h((m_1, m_2, m_3)K_3) = \begin{cases} 1, & \text{if } m_3 > m_1 + m_2 \\ 2, & \text{if } m_3 \leq m_1 + m_2 \end{cases} \tag{1}$$

Proof. Let $V(G) = \{v_1, v_2, v_3\}$, E_i be the set of all edges with endpoints v_i and v_{i+1} for $1 \leq i \leq 2$, E_3 be the set of all edges with endpoints v_3 and v_1, and $m_i = |E_i|$ for $i \in \{1, 2, 3\}$. Then $m_3 = \Delta_m(G)$.

If $m_3 > m_1 + m_2$, then $m = m_1 + m_2 + m_3 < 2m_3 = 2\Delta_m(G)$. By Corollary 2.3, $h(G) = 1$. Now suppose $m_3 \leq m_1 + m_2$. Let $E_i = \{e_1^i, e_2^i, ..., e_{m_i}^i\}$. Then there exists a cyclic ordering o such that the ordering alternates between an edge from E_3 and an edge from E_1 or E_2. Since $m_3 \leq m_1 + m_2$, there are at least as many edges in $E_1 \cup E_2$ as in E_3. So $h(G) \geq 2$. Notice that $h(G) \leq |V(G)|-1 = 2$. Therefore, $h(G) = 2$. □

Let S_G be the graph obtained from G by combining all multiple edges between two vertices to only one edge. Let $g(S_G)$ be the girth of S_G, which is the length of the shortest cycle in S_G. See Figure 1 for an example. If S_G is a tree, we define $g(S_G) = \infty$.

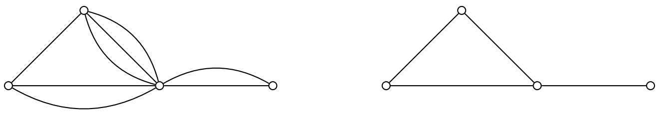

Fig. 1. G and S_G where $g(S_G) = 3$

Theorem 3.2. *For a graph G, suppose $g(S_G) \geq i+1$. Then each of the following holds.*
(i) *If $i\Delta_m(G) \leq m$, then $h(G) \geq i$.*
(ii) *If $m \leq (i+1)\Delta_m(G) - 1$, then $h(G) \leq i$.*

Proof. Let $E(S_G) = \{e_1, e_2, \cdots, e_t\}$, E_i be the set of all edges in G that were combined to e_i in S_G and $E_i = \{e_1^i, e_2^i, ..., e_{m_i}^i\}$ where $m_i = |E_i|$ for $1 \leq i \leq t$. Without loss of generality, we suppose $\Delta_m(G) = m_1 \geq m_2 \geq \cdots \geq m_t$. For each cyclic ordering of G, $(e_{j_1}^1, \cdots, e_{j_2}^1, \cdots, e_{j_{\Delta_m(G)}}^1, \cdots)$ where $\{j_1, j_2, \cdots, j_{\Delta_m(G)}\} = \{1, 2, \cdots, \Delta_m(G)\}$, the edges $\{e_{j_1}^1, e_{j_2}^1, \cdots, e_{j_{\Delta_m(G)}}^1\}$ in E_1 divide the ordering into $\Delta_m(G)$ intervals with the first edges from E_1.

If $m \geq i\Delta_m(G)$, since $g(S_G) \geq i+1$, there is a cyclic ordering o such that each interval has at least i edges and no multiple edges from the same E_i are in the same interval for $1 \leq i \leq t$. So $h(G) \geq h(o) \geq i$. This proves part (i).

Suppose $m \leq (i+1)\Delta_m(G) - 1$. To prove by contradiction, assume that G has a cyclic order o with $h(o) \geq i+1$. The edges in E_1 divide the ordering into $\Delta_m(G)$ intervals. If one such interval has at most i edges, then length of this interval together with the edges from E_1 on both sides of the interval is at most $1 + (i-1) + 1 = i+1$ and it contains a 2-cycle, contrary to $h(o) \geq i+1$. Hence, every interval must have at least $i+1$ edges. So $m \geq \Delta_m(G)(i+1)$. □

Suppose $g(S_G) = 3$. Let $\Delta_\Delta = \max\{\lceil \frac{m_1+m_2+m_3}{2} \rceil \mid (m_1, m_2, m_3)K_3 \text{ is a subgraph of } G\}$. If $g(S_G) > 3$, we define $\Delta_\Delta = 0$. Let $\ell = m - m_1 - m_2 - m_3$.

We propose a structure for all graphs that satisfy $h(G) = 2$ and prove the necessity. Below, we use Δ_m for $\Delta_m(G)$.

Conjecture 3.3. *Let G be a connected graph. Then*

$$\begin{cases} 2\Delta_m \leq m \leq 3\Delta_m - 1, & if\ \Delta_\Delta \leq \Delta_m \\ \ell < \Delta_\Delta, & if\ \Delta_\Delta > \Delta_m \end{cases}$$

if and only if $h(G) = 2$.

Theorem 3.4. *Let G be a connected graph. If*

$$\begin{cases} 2\Delta_m \leq m \leq 3\Delta_m - 1, & if\ \Delta_\Delta \leq \Delta_m \\ \ell < \Delta_\Delta, & if\ \Delta_\Delta > \Delta_m \end{cases}$$

then $h(G) = 2$.

Proof. If $g(S_G) > 3$, then G has no 3-cycle and $\Delta_\Delta = 0 \leq \Delta_m(G)$. If $2\Delta_m \leq m \leq 3\Delta_m - 1$, by Theorem 3.2, $h(G) = 2$.

It is sufficient to prove the case when $g(S_G) = 3$. If $\Delta_\Delta \leq \Delta_m$, then $2\Delta_m \leq m \leq 3\Delta_m - 1$. By Lemma 2.2, $h(G) \leq 2$. And by Corollary 2.3, $h(G) \neq 1$. Therefore, $h(G) = 2$. If $\Delta_\Delta > \Delta_m$, suppose $(m_1, m_2, m_3)K_3$ satisfies Δ_Δ. Since $\Delta_\Delta > \Delta_m$, $m \geq m_1 + m_2 + m_3 \geq 2\Delta_\Delta - 1$ by the definition of Δ_Δ. Therefore, $m > 2\Delta_m - 1$, i.e. $m \geq 2\Delta_m$. By Corollary 2.3, $h(G) \geq 2$. Assume that $h(G) \geq 3$. Let o be the cyclic ordering such that $h(o) \geq 3$. Then every 3 cyclically consecutive elements in o has at most two edges in $(m_1, m_2, m_3)K_3$. Then $\ell \geq \frac{m_1+m_2+m_3}{2}$, so $\ell \geq \Delta_\Delta$. It contradicts to $\ell < \Delta_\Delta$. Thus $h(G) = 2$. □

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Declarations

The authors declare no conflict of interest.

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