On efficient total colorings of regular graphs

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ABSTRACT

Let $2 \leq k \in \mathbb{Z}$. We say that a total coloring of a k-regular simple graph via $k + 1$ colors is an *efficient total coloring* if each color yields an efficient dominating set, where the efficient domination condition applies to the restriction of each color class to the vertex set. We prove that Hamming shells of star transposition graphs and Hamming cubes have efficient total colorings. Also in this work, a focus is set upon the graphs of girths $2k$ and k. Efficient total colorings of finite connected simple cubic graphs of girth 6 are constructed. These are of two specific types, namely: **(a)** those whose 6-cycles use just 3 colors with antipodal monochromatic pairs of vertices or edges; **(b)** those whose 6-cycles do not respect item (a) so they use four colors. An orthogonality property holds for all graphs of type (a). Such orthogonality property allows further edge-half-girth colorings in the corresponding prism graphs.

Keywords: total coloring, efficient dominating set, regular graph, star transposition graph, generalized Petersen graph, Hamming shell

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1. Introduction

Given a simple connected graph Γ, a *total coloring* or *TC* of Γ is a color assignment to the vertices and edges of Γ such that no two incident or adjacent elements (vertices or edges) are assigned the same color. A recent survey [12] contains an updated bibliography on TCs, with the TC Conjecture, posed independently by Behzad [1, 2] and by Vizing [19], that asserts that the total chromatic number of Γ (namely, the least number of colors required by a TC of Γ) be either $\Delta(\Gamma) + 1$ or $\Delta(\Gamma) + 2$, where Δ is the largest degree of

any vertex of Γ.

The TC Conjecture was established for cubic graphs [4, 11, 16, 18], meaning that the total chromatic number of cubic graphs is either 4 or 5. To decide whether a cubic graph Γ has total chromatic number $\Delta(\Gamma) + 1$, even for bipartite cubic graphs, is NP-hard [17].

In [7], relations between total colorings of k-regular graphs of girth $k+1$ ($1 < k \in \mathbb{Z}$) and efficient dominating sets (EDSs) [6, 8, 9, 10, 13, 15] were established. In Section 2 below, we establish similar relations for girth $2k$ (and also k, but in Section 3) in place of $k+1$. We prove that Hamming shells of star transposition graphs (Remark 1.7, Subsection 2.3 and Theorem 2.6) and Hamming cubes (Remark 1.8, Section 3 and Theorem 3.2) have efficient total colorings. The following definition, given in [7], is employed.

Definition 1.1. [7] A coloring of a connected k-regular simple graph Γ ($2 \leq k \in \mathbb{Z}$) is said to be an *efficient TC*, or *ETC*, (and Γ said to be *ETCed*), if:

(a) (TC condition) each $v \in V(\Gamma)$ together with its neighbors are assigned all the colors in $[k+1] = \{0, 1, \ldots, k\}$ via a bijection $N[v] = N(v) \cup \{v\} \leftrightarrow [k+1]$, where $N[v]$ and $N(v)$ are the *closed neighborhood* of v and the *open neighborhood* of v, respectively [8];

(b) the TC in item (a) partitions $V(\Gamma)$ into $k+1$ *efficient dominating sets* (*EDSs*), also called *perfect codes*, namely independent (stable) subsets $S_i \subseteq V(\Gamma)$ such that for any $v \in V(\Gamma) \setminus S_i$, $|N[v] \cap S_i| = 1$, where i varies in $[k+1]$, [6, 8, 9, 10, 13, 15].

Under conditions (a)-(b), it is seen that the total chromatic number of Γ is $\Delta(\Gamma) + 1$.

Remark 1.2. For $1 < j \in \mathbb{Z}$, consider the $6j$-cycle graph $C_{6j} = (v_1, e_1, v_2, e_2, \ldots, e_{6j-1}, v_{6j}, e_{6j})$, where e_i is the edge with end-vertices v_i and v_{i+1}, for $1 \leq i \leq 6j-1$, and e_{6j} is the edge with end-vertices v_{6j} and v_1. Let the successive vertices and edges of C_{6j} be together colored in one fixed direction of C_{6j} in the order of the colors in the color cycle $(0, 1, 2, 0, 1, 2, \ldots, 0, 1, 2)$. This clearly yields an ETC of C_{6j}.

In Section 2, finite connected simple k-regular graphs Γ ($k \geq 2$) of girth $2k$ are dealt with, and in Section 3, girth k is considered instead of $2k$. The purpose is to determine ETCs of such graphs. Some of these ETCs yield *edge-half-girth colorings*, (an adaptation of the edge-girth colorings of [7], see Definition 1.3, below) on the prism $\Gamma \square K_2$, where $K_2 = P_2$ is the complete graph on two vertices, that is the path P_k consisting of $k = 2$ vertices.

Definition 1.3. Let Γ be a finite connected k-regular simple graph of girth $2k$. An *edge-girth coloring*, or *EGC*, of Γ is a proper edge coloring of Γ via $k+1$ colors, each girth cycle colored with k colors, each vertex color used precisely once.

To present most of our results, we need every girth cycle C of Γ to be colored with at least k colors, each such color used at most twice on $V(C)$. This leads to Definition 1.4, in which, in addition, the total coloring of the girth cycles is combined with the concept of ETC in Definition 1.1.

Definition 1.4. A *vertex partial-girth TC*, or *VpGTC*, of a connected simple k-regular graph Γ of girth $2k$ is a TC of Γ in which the vertices of each girth cycle C are colored with at least k colors, each such color used at most twice on $V(C)$. In addition, an ETC of Γ that is also VpGTC will be said to be an *efficient partial-girth total coloring*, or *EpGTC*, of Γ. If "at least" and "at most" are removed in the above definition of VpGTC, then "partial" is replaced by "half", so that we have *VhGTC* and *EhGTC* instead of VpGTC and EpGTC, where the lowercase "h" stands for "half".

The results of Section 2 involve the existence of EhGTCs and EpGTCs in finite connected simple k-regular graphs Γ of girth $2k$. They also involve the existence of k-regular graphs Γ of corresponding EGCs on the prisms $\Gamma \square K_2$, for which we need Definition 1.5.

Definition 1.5. Given a connected simple graph Γ and a TC C of Γ, if there exists a TC C' that coincides with C on $V(\Gamma)$ but differs with $E(\Gamma)$ edge by edge, i.e. $C(e) \neq C'(e)$, $\forall e \in E(\Gamma)$, then we say that C' is *orthogonal* to C, or that C and C' form a *pair of orthogonal* TCs, so that they enjoy an *orthogonality property*.

Definition 1.6. [7] An *ℓ-belt* in a plane graph Γ is a cycle bounding a face of Γ of length ℓ.

Below, efficient total colorings of finite connected simple cubic graphs of girth 6 are constructed. These are of two specific types, namely: (a) those whose 6-cycles use just 3 vertex colors with antipodal (distance 3) monochromatic vertex pairs; (b) those whose 6-cycles do not respect item (a), using in fact four vertex colors. Moreover, the orthogonality property of Definition 1.5 holds for all graphs of type (a). Such orthogonality property allows edge-half-girth colorings in the corresponding prism graphs.

In fact, Subsections 2.1 and 2.2 address EpGTCs in star transposition graphs [8] (type (a)), in some generalized Petersen graphs [4] (type b)) and in Hamming shells [5] of star transposition graphs [8] (type (a)), respectively. They also address EGCs when they do exist on the corresponding prisms $\Gamma \square K_2$. To refresh these concepts for the present purpose, we provide the following Remarks 1.7–1.8.

Remark 1.7. [8] Let $2 < n \in \mathbb{Z}$. The *star transposition graph* ST_n is the Cayley graph $Cay(Sym_n, C)$, where Sym_n is the group of n-permutations and C is composed by the n-transpositions given by the edges of the graph $K_{1,n-1}$, where its only vertex of degree $n-1$ is denoted y_0 and the remaining vertices are denoted $y_a, y_b, \ldots, y_x$, where x represents the $(n-1)$-letter of a possibly extended alphabet over z (for $n-1 = 26$) and $y_j \in [n]$, for $0 \leq j \leq x$ so a typical vertex of ST_n is expressible as $y_0 y_a y_b \cdots y_x$, so its neighbors are

$$y_a y_0 y_b \cdots y_x, \quad y_b y_a y_0 \cdots y_x, \quad \cdots \quad \cdots, \quad y_x y_a y_b \cdots y_0,$$

by means of the transpositions representing the edges of $K_{1,n-1}$.

Remark 1.8. [5] Let $1 < r \in \mathbb{Z}$ and let $n = 2^r - 1$. The Cayley graph

$$Q_n = Cay(\mathbb{Z}^n, \{10 \cdots 0, 010 \cdots 0, \ldots, 0 \cdots 01\}),$$

known as the *n-th Hamming cube*, has its vertex set admitting a partition formed by pairwise isomorphic EDSs (perfect codes)

$$\mathcal{S}_0, \qquad \mathcal{S}_1 = \mathcal{S}_0 + 10\cdots0, \qquad \mathcal{S}_2 = \mathcal{S}_0 + 010\cdots0, \qquad \ldots \qquad \ldots, \qquad \mathcal{S}_n = \mathcal{S}_0 + 0\cdots01.$$

Here $\mathcal{S}_0$, known as the *n-th Hamming code*, is a maximum stable set of $V(Q_n)$ with minimum distance 3 such that $d(v, \mathcal{S}_0) = 1$, for every vertex v in the *Hamming shell* $Q_n \backslash \mathcal{S}_0$. Moreover, $\mathcal{S}_0$ is formed from the vertex 0^n via the Steiner triple system $STS(n)$ whose triples are formed by the coordinate directions corresponding to the linearly dependent rows of the parity check $n \times r$-matrix H_r whose ℓ-row is the r-tuple of $\mathbb{Z}_2^r$ associated to the binary expression of the integer ℓ, with enough zeros to the left to attain length r, for $\ell = 1, \ldots, r$. Here, H_r has its kernel as $\mathcal{S}_0$, justifying our above denomination of *Hamming shell*. In Section 3, we modify Definition 1.4 by replacing girth $2k$ by girth k in order to present the cases of the Hamming shells.

2. Regular graphs of girth equal to twice the degree

Question 2.1. *Let Γ be a regular graph of degree $k \geq 3$ and girth $2k$. Does Γ possess an EpGTC C with $k + 1$ colors? Does there exists an EpGTC C' of Γ orthogonal to C?*

Question 2.1 is answered in the affirmative in Subsections 2.1 and 2.2.

2.1. Star transposition graphs

Definition 2.2. A *straight cutout* of a connected toroidal graph Γ is a closed rectangle Φ of $\mathbb{R}^2$ with $V(\Gamma) \subset \Phi \cap \mathbb{Z}^2$ and $E(\Gamma)$ given by an hexagonal tessellation formed by 6-belts (v, h, d, v', h', d') whose edges are two vertical and two horizontal unit-length segments v, v', h, h' and two main diagonals d, d' of 1×1-squares $v'' \times h'' \supset d, v''' \times h''' \supset d'$ in Φ inside the 2×2-square bordered by $(h, h'', v'', v', h', h''', v''', v)$ and such that identification of the left and right borders and of the top and bottom borders of Φ yield a representation of Γ, with the four corners of Φ representing a sole vertex of Γ in $\mathbb{Z}^2$.
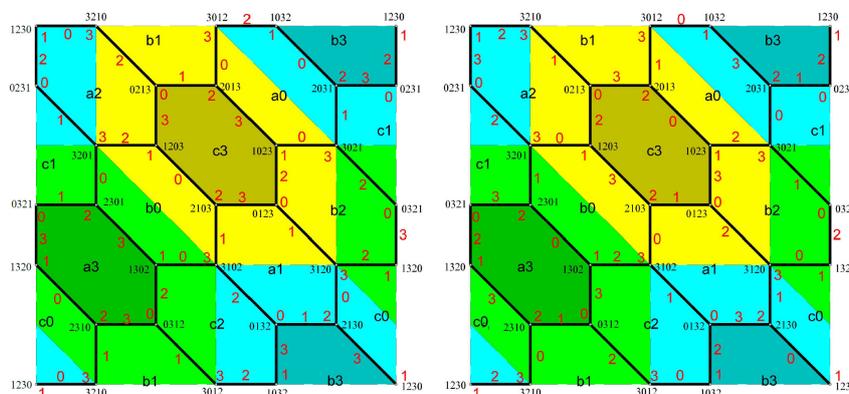

Fig. 1. Pair of orthogonal ETCs of ST_4

Question 2.1 is answered below in the affirmative for the star transposition graph ST_4 [8]

represented as a straight cutout both on the left and right of Figure 1, with the edges traced thick black and the uniformly colored bright areas (green, blue, yellow) having each a central 6-cycle with darker interior (denoted a3, b3, c3, respectively). The shared straight borders between differently colored bright areas are indicated as dashed-traced segments, with their ends being the members of an efficient dominating set S given by the coinciding open neighborhoods $N(\mathrm{a}3) = N(\mathrm{b}3) = N(\mathrm{c}3) = \{3120, 3102, 3201, 3210, 3012, 3021\}$, with vertex notation as in Remark 1.7. Each 6-cycle xj mentions the value $j \in \{0, 1, 2, 3\}$ assumed by position $x \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ in the six existing vertices of S, where $x \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ indicates respectively the second, third and fourth entries of the 4-permutations $y_0 y_\mathrm{a} y_\mathrm{b} y_\mathrm{c}$ such that $\{y_0, y_\mathrm{a}, y_\mathrm{b}, y_\mathrm{c}\} = \{0, 1, 2, 3\}$.

Theorem 2.3. *The star transposition graph ST_4 is a 3-regular graph of girth 6 admitting EhGTCs via the color set $\{0, 1, 2, 3\}$ in two vertex-fixed edge-orthogonal instances that combine to produce an EGC of the prism graph $ST_4 \square K_2$, whose girth is 4.*

Proof. Throughout the proof, each vertex $y_0 y_\mathrm{a} y_\mathrm{b} y_\mathrm{c}$ of ST_4 is assigned color y_0, which insures that $V(ST_4)$ has a partition into efficient dominating sets [8]. Each edge of ST_4 separates two faces of adjacent 6-cycles. Also, the edges of each 6-cycle form two 2-factors of three edges each. An edge coloring of ST_4 is obtained by selecting in each 6-cycle xj one of its 1-factors, say F, and assigning color j to the three edges of F, with the outcome that all edges of ST_4 get a well-defined color. More specifically, the 6-cycle xj has the edges of its remaining 1-factor F' (so $F \cup F' = E(xj)$), having the colors j_1, j_3, j_5 of the three alternate adjacent 6-cycles xj_1, xj_3, xj_5 of F', where the associated edge color sequence of xj is $(j_0, j_1, j_2, j_3, j_4, j_5)$, with j_0, j_2, j_4 being the colors of the edges of F. This produces an EhGTC C of ST_4 depending on a specific determination of which 2-factors of the 6-cycles xj of ST_4 take the roles of F and of F'. Now, by exchanging the roles of the 1-factors F and F' in the argument above, a second EhGTC of ST_4 is obtained, that behaves as a similar vertex coloring as that of C but that is edge-orthogonal to that of C in the sense that the roles of the 2-factors F and F' are exchanged. The resulting pair of vertex-fixed edge-orthogonal ETCs of ST_4 is shown in Figure 1, where the red edge color numbers behave orthogonally while the red vertex color numbers remain fixed.

The edge orthogonality property here allows to consider the prism $ST_4 \square K_2$, assign the vertex colors of ST_4 to the corresponding edges of the copies $\{v\} \square K_2$ of K_2, $(v \in V(ST_4))$, where the end-vertices of each such edge have a common color, and take the orthogonality property to justify the claimed EGC. $\square$

Figure 2 shows a modification of Figure 1 stressing the difference between the two vertex-fixed edge-orthogonal colorings of Theorem 2.3 by means of colored triangles whose vertices are the midpoints of edges with a common color number, such that on the left of the figure these triangles are pointing down while on the right they are pointing up. The colors of these triangles are; light-gray in a0, b0, c0; light-green in a1, b1, c1; light-blue in a2, b2, c2; yellow in a3, b3, c3.

Corollary 2.4. *The conclusion of Theorem 2.3 can be extended periodically to any larger*

straight cutout in the plane formed by a rectangular continuation of the straight cutout in Theorem 2.3, up to the whole euclidean plane.

Proof. A rectangular continuation of the two orthogonal ETCs of ST_4 to any case as in the statement is sufficient to complete its proof. $\qquad\square$
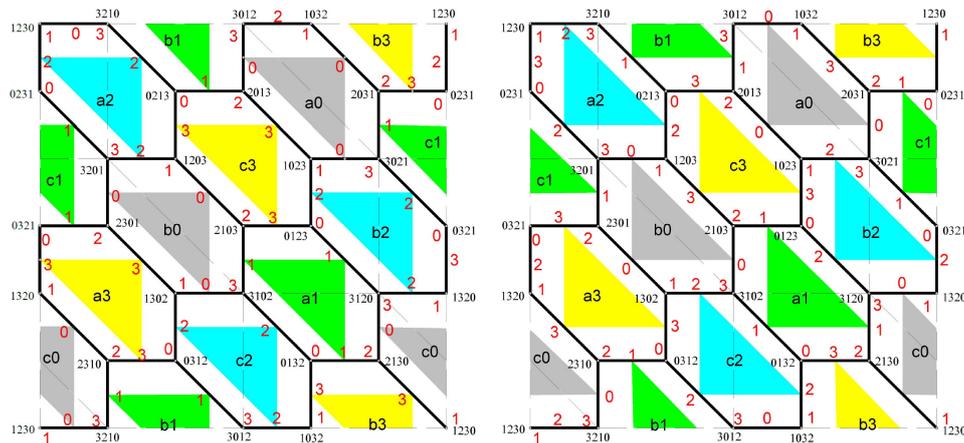

Fig. 2. A view of Figure 1 by means of triangles whose vertices are edge midpoints

2.2.   *Generalized Petersen graphs*

Total coloring of generalized Petersen graphs was addressed in [4].

A pair of orthogonal EpGTC for the generalized Petersen graph $GP(8,3)$ is shown in Figure 3, where in either case there are four of each of the following vertex-color 6-cycles:

$$(012013), (023021), (031032), (120123), (130132), (230231), \tag{1}$$

with for example a case each of color cycles (012013), (023021) and (023021) represented in the figure with "apparent face interior" in area color yellow, blue and gray, respectively.
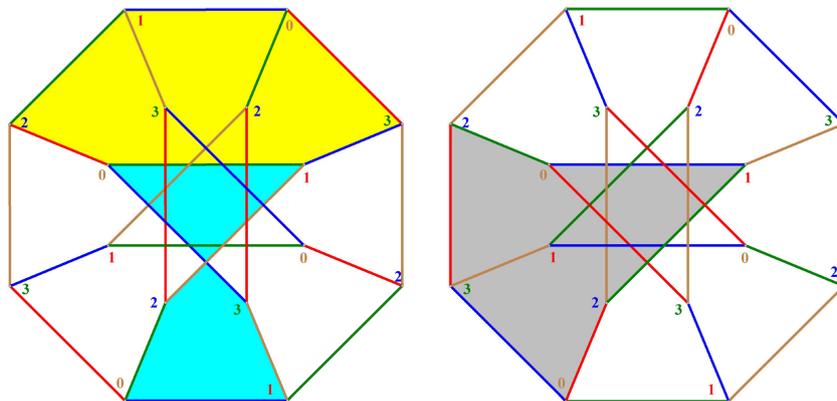

Fig. 3. Orthogonal EpGTCs of generalized Petersen graph $GP(8,3)$

A pair of orthogonal EpGTC for the generalized Petersen graph $GP(16,3)$ is shown in Figure 4, where in either case there are four of each of the following vertex-color 6-cycles:

$$(012312), (123023), (230130), (301201), \tag{2}$$

with for example a case each of these color cycles represented in the figure with "apparent face interior" in area colors yellow, blue and gray.
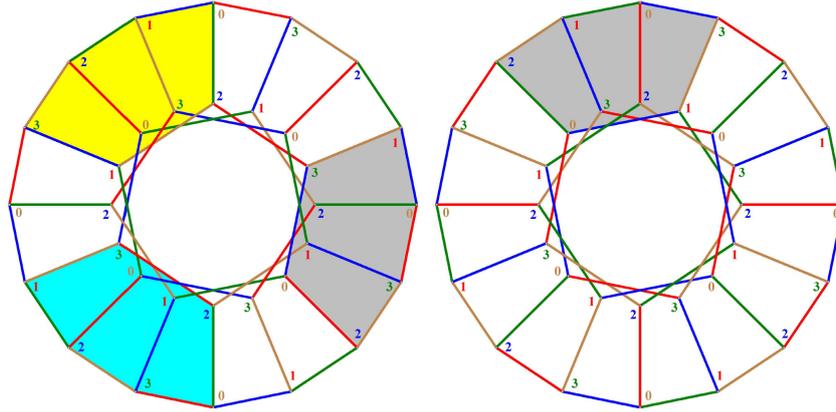

Fig. 4. Orthogonal EpGTCs of generalized Petersen graph $GP(16,3)$

This EpGTC generalizes as follows.

Theorem 2.5. *For $k > 1$, there is a pair of orthogonal EpGTCs in $GP(8k,3)$ with $4k$ concatenated sequences of four vertex-color 6-cycles as in display (2).*

Proof. This is a straight generalization of the argument previous to the statement of the theorem. In fact, the generalized "external" vertex-color cycle" formed by the concatenation of $2k$ copies of the number string $(0,1,2,3)$, of the form $(0,1,2,3)^{2k}$, has its successive member vertices having respective neighbors colored $(2,3,0,1)^{2k}$ but forming a cycle of the form $(2,1,0,3)^{2k}$ by "jumping" two such neighbors in each instance, so that the vertex-color 6-cycles in display 2 are effectively constituted. No more vertex-color 6-cycles exist for $k > 1$. However, as it is shown in display 1 and illustrated in Figure 3, there are 24 vertex-color 6-cycles in $GP(8,3)$, sixteen more than assigned by the statement for $k > 1$. More specifically, by completing the vertex-color cycles above by citing via subindices the intervening edge-colors, we rewrite those "external" and "internal" 6-cycles above as $(0_2 1_3 2_0 3_1)^{2k}$ and $(0_3 1_0 2_1 3_2)^{2k}$ for the first coloring of $GP(8k,3)$ (represented on the left side of the figures) and as $(2_0 1_3 0_2 3_1)^{2k}$ and $(2_3 1_2 0_1 3_0)^{2k}$ for the second coloring of $GP(8k,3)$ (represented on the right side of the figures). The 1-factor1 from the "external" onto the corresponding "internal" coloring in each case is formed by the edge-color period $(0_3 2)$, $(1_0 3)$, $(2_1 0)$, $(3_2 1)$ on the left and by the edge-color period $(0_1 2)$, $1_2 3$, $2_3 0$, $3_0 1$ on the right. Clearly the orthogonality property holds in this case. $\square$

2.3.   Hamming shells of star transposition graphs

By extension from the concept of a Hamming shell in [5], we say that given an efficient dominating set Σ in a graph G, the complement $G \setminus \Sigma$ of Σ in G is a *Hamming shell* of G with respect to Σ.

Theorem 2.6. *The Hamming shell of the star transposition graph ST_n, ($5 \leq n \in \mathbb{Z}$), has an ETC. For $n = 5$, such TC is an EhGTC.*

Proof. The vertices of ST_n are the permutations $y_0 y_a y_b \cdots y_x$ as in Remark 1.7. An edge of ST_n with one endvertex $y_0 y_a y_b \cdots y_x$ has the other endvertex obtained from $y_0 y_a y_b \cdots y_x$ by just transposing y_0 with y_h, for some $h \in \{a, \ldots, x\}$. Thus, each vertex $y_0 y_a y_b \cdots y_x$ has $n - 1$ neighbors via $n - 1$ corresponding edges considered with colors $a, b, \ldots, x$. Now, a partition into efficient dominating sets $\mathcal{S}_0$, $\mathcal{S}_1$, ..., $\mathcal{S}_{n-1}$ is given by defining $\mathcal{S}_j$ as the subset of permutations $y_0 y_a y_b \cdots y_x$ with $y_0 = j$, for $j \in [n]$. Then, the Hamming shell $ST_n \setminus \mathcal{S}_0$ has each vertex v at distance 1 from $\mathcal{S}_0$ via an edge colored with some $h \in \{a, \ldots, x\}$. We color v with color h and this vertex color selection guarantees the existence of the claimed ETC of ST_n that becomes EhGTC for $n = 5$ since $ST_5 \setminus \mathcal{S}_0$ is a cubic graph with girth 6. $\square$

3.   Girth = degree: Hamming shells of Hamming cubes

A modification of Definition 1.4 is given in this section as Definition 3.1 and applied subsequently just to connected simple k-regular graphs of girth k, instead of $2k$, (as in Definition 1.4).

Definition 3.1. A *vertex $\frac{1}{2}$-girth TC*, or $V\frac{1}{2}GTC$, of a connected simple k-regular graph Γ of girth k is a TC of Γ in which the vertices of each girth cycle C are colored with k colors, each such color used twice on $V(C)$. In addition, an ETC of Γ that is also $VV\frac{1}{2}GTC$ will be said to be an *efficient partial-girth total coloring*, or $EV\frac{1}{2}GTC$, of Γ.

Based on [5] we have the following.

Theorem 3.2. *The Hamming shell $Q_n \setminus \mathcal{S}_0$ has an ETC via the coordinate directions associated to the STS in Remark 1.8 as colors for the edges and the coordinate directions of missing edges (due to the removal of S_0 from Q_n) to corresponding Hamming words as colors for the vertices. Moreover, that TC becomes $EV\frac{1}{2}GTC$ for $n = 7$ as $Q_7 \setminus \mathcal{S}_0$ is a 6-regular graph of girth 6.*

Proof. An ETC T of $Q_n \setminus \mathcal{S}_0$ is obtained by coloring the vertices of each $\mathcal{S}_i$ with color i, for $i = 1, 2, 3, , 4, 5, 6, 7$, and by coloring the edges of $Q_n \setminus \mathcal{S}_0$ so that the color of each edge and the colors of its end-vertices form one of the triples of $STS(n)$. $\square$

Example 3.3. We have that $STS(7)$ can be considered to consist cyclically [3] of the following successive triples:

$$f(7) = 124, f(1) = 235, f(2) = 346, f(3) = 457, f(4) = 561, cf(5) = 672, f(6) = 713, \quad (3)$$

that is the cyclic (mod 7) Steiner triple system $STS(7)$ interpreted as triples of coordinate directions (represented by corresponding parallel 1-factors in Q_7), with f being the duality bijection from points to lines of the Fano plane. Then, $\mathcal{S}_0$ is composed by the 16 7-tuples:

$$
\begin{array}{ccccccc}
0000000 & 1101000 & 0110100 & 0011010 & 0001101 & 1000110 & 0100011, 1010001 \\
1111111 & 0010111 & 1001011 & 1100101 & 1110010 & 0111001 & 1011100, 0101110.
\end{array}
\quad (4)
$$

Moreover, the efficient dominating sets $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$, $\mathcal{S}_5$,$\mathcal{S}_6$ and $\mathcal{S}_7$, are obtained from $\mathcal{S}_0$ by traversing the edges along the coordinate directions, 1, 2, 3, 4, 5, 6 and 7, respectively, by departing from the 16 vertices of $\mathcal{S}_0$ in (4). By removing $\mathcal{S}_0$ from Q_7, the Hamming shell $Q_7 \setminus \mathcal{S}_0$ [3] is obtained that is a 6-regular graph whose girth is 6.

Corollary 3.4. *The Hamming shell $Q_7 \setminus \mathcal{S}_0$ is a 6-regular graph of girth 6 admitting $EV\frac{1}{2}GTCs$ via the color set $\{1,2,3,4,5,6,7\}$ in two vertex-fixed edge-orthogonally instances. These instances can be combined to produce an EGC of the prism graph $(Q_7 \setminus \mathcal{S}_0)\square K_2$.*

Proof. Adapting the proof of Theorem 2.3, a second $EV\frac{1}{2}GTC$ T' of $Q_7 \setminus \mathcal{S}_0$ is obtained by replacing the edge colors in T by the points representing the lines of the Fano plane in (3). The final assertion of the statement is established in a similar fashion to that of the last part in Theorem 2.3. $\qquad \square$

Corollary 3.5. [3] *The Hamming shell $Q_7 \setminus \mathcal{S}_0$ is the union of two vertex-spanning self-complementary 112-vertex graphs cubic $G_r \equiv G_b$ of girth 10 that are edge-transitive but not vertex-transitive (known as Ljubljana graph [14]). Both G_r, G_b inherit the property from $Q_7 \setminus \mathcal{S}_0$ of having $EV\frac{1}{2}GTCs$ from the $EV\frac{1}{2}GTC$ of $Q_7 \setminus \mathcal{S}_0$.*

Proof. Clearly, the restriction of a total coloring on a graph to a subgraph is still a total coloring. Each edge of $Q_7 \setminus \mathcal{S}_0$ is of the form (v,w), with the weight of v being odd and that of w even. If the color of v is i and that of w is j, let us make vw be an edge of G_r if $j - i \in \{1,2,4\}$ (mod 7) and an edge of G_b, otherwise. Clearly, Definition 1.1 is satisfied both for G_r and G_b. $\qquad \square$

Declarations

The author declares no conflict of interest.

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