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On local antimagic chromatic numbers of the join of two special families of graphs

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ABSTRACT

It is known that null graphs are the only (regular) graphs with local antimagic chromatic 1 and 1-regular graphs are the only regular graphs without local antimagic chromatic number. In this paper, we first use matrices of size $(2m+1)\times(2k+1)$ to completely determine the local antimagic chromatic number of the join of null graphs O_m and 1-regular graphs $(2k+1)P_2$ for all $k\geq 1, m\geq 2$. We then make use of other matrices of same size to obtain the local antimagic chromatic number of another family of tripartite graphs. Consequently, we obtained infinitely many (possibly disconnected) regular tripartite graphs with local antimagic chromatic number 3.

Keywords: local antimagic chromatic number, null graphs, 1-regular graphs, join product

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1. Introduction

Let $G=(V,E)$ be a connected graph of order p and size q. A bijection $f:E\to\{1,2,\ldots,q\}$ is called a *local antimagic labeling* if $f^+(u)\neq f^+(v)$ whenever $uv\in E$, where $f^+(u)=\sum_{e\in E(u)} f(e)$ and $E(u)$ is the set of edges incident to u. The mapping f^+ which is also denoted by f^+_G is called a *vertex labeling of* G *induced by* f, and the labels assigned to vertices are called *induced colors* under f. The *color number* of a local antimagic labeling f is the number of distinct induced colors under f, denoted by $c(f)$. Moreover, f is called a *local antimagic* $c(f)$*-coloring* and G is *local antimagic* $c(f)$*-colorable*. The *local antimagic chromatic number* $\chi_{la}(G)$ is defined to be the minimum number of colors taken over all colorings of G induced by local antimagic labelings of G [1] (also see [3]

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that introduced the concept of local antimagic labeling independently). Let $G+H$ and mG denote the disjoint union of graphs G and H, and m copies of G, respectively. For integers $c<d$, let $[c,d]=\{n\in\mathbb{Z}\mid c\le n\le d\}$. Very few results on the local antimagic chromatic number of regular graphs are known (see [1, 6]). Throughout this paper, we let O_m and aP_2 be the families of 0-regular (or null) and 1-regular graphs, respectively with $V(aP_2\vee O_m)=\{u_i,v_i,x_j\mid 1\le i\le a, 1\le j\le m\}$ and $E(aP_2\vee O_m)=\{u_ix_j,v_ix_j,u_iv_i\mid 1\le i\le a, 1\le j\le m\}$. We also let $V(a(P_2\vee O_m))=\{u_i,v_i,x_{i,j}\mid 1\le i\le a, 1\le j\le m\}$ and $E(a(P_2\vee O_m))=\{u_ix_{i,j},v_ix_{i,j},u_iv_i\mid 1\le i\le a, 1\le j\le m\}$.

In [5], the author proved that all connected graphs without a P_2 component admit a local antimagic labeling. Thus, local antimagic chromatic number is well defined for all graphs without a P_2 component. Clearly, $O_m, m\ge 1$, is the only family of (regular) graphs with local antimagic chromatic number 1 while $aP_2, a\ge 1$, is the only family of regular graphs without local antimagic chromatic number. In [1], it was shown that $\chi_{la}(aP_2\vee O_1)=1$ for $a\ge 1$. In the following sections, we extend the ideas in [14] to prove that $\chi_{la}((2k+1)P_2\vee O_m)=3$ for all $k\ge 1$, $m\ge 2$. Moreover, we also obtain other families of (possibly regular) tripartite graphs with local antimagic chromatic number 3. Interested readers may refer to [2, 4, 6, 11, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18] for the many results on the join of graphs with a regular graph or some results on trees.

2. Joins with even empty graphs

Consider the graph $(2k+1)(P_2\vee O_{2n})$ of order $(2k+1)(2n+2)$ and size $(2k+1)(4n+1)$.

Table 1. Matrix for $i\in[1,k+1]$

i	1	2	3	⋯	k	k+1	common diff.
$f(u_ix_{i,1})$	k+1+n(8k+4)	k +n(8k+4)	k-1+n(8k+4)	⋯	2 +n(8k+4)	1 +n(8k+4)	-1
$f(u_ix_{i,2})$	-3k-1 +n(8k+4)	-3k +n(8k+4)	-3k+1 +n(8k+4)	⋯	-2k-2 +n(8k+4)	-2k-1 +V	$+1$
⋮	⋮	⋮	⋮	⋯	⋮	⋮	
$f(u_ix_{i,2n-2j+1})$	k+1+j(8k+4)	k +j(8k+4)	k-1+j(8k+4)	⋯	2 +j(8k+4)	1 +j(8k+4)	-1
$f(u_ix_{i,2n-2j+2})$	-3k-1 +j(8k+4)	-3k +j(8k+4)	-3k+1 +j(8k+4)	⋯	-2k-2 +j(8k+4)	-2k-1 +j(8k+4)	$+1$
⋮	⋮	⋮	⋮	⋯	⋮	⋮	
$f(u_ix_{i,2n-1})$	10k+5	10k+3	10k+1	⋯	8k+7	8k+5	-2
$f(u_ix_{i,2n})$	5k+3	5k+4	5k+5	⋯	6k+2	6k+3	+1
$f(u_iv_i)$	1	2	3	⋯	k	k+1	+1
$f(v_ix_{i,1})$	3k+2	3k+3	3k+4	⋯	4k+1	4k+2	+1
$f(v_ix_{i,2})$	8k+4	8k+2	8k	⋯	6k+6	6k+4	-2
⋮	⋮	⋮	⋮	⋯	⋮	⋮	
$f(v_ix_{i,2j-1})$	-5k-2 +j(8k+4)	-5k-1 + j(8k+4)	-5k +j(8k+4)	⋯	-4k-3 +j(8k+4)	-4k-2 +j(8k+4)	$+1$
$f(v_ix_{i,2j})$	-k +j(8k+4)	-k-1 +j(8k+4)	-k-2 +j(8k+4)	⋯	-2k+1 + j(8k+4)	-2k +j(8k+4)	-1
⋮	⋮	⋮	⋮	⋯	⋮	⋮	
$f(v_ix_{i,2n-1})$	-5k-2 +n(8k+4)	-5k-1 +n(8k+4)	-5k +n(8k+4)	⋯	-4k-3 +n(8k+4)	-4k-2 +n(8k+4)	$+1$
$f(v_ix_{i,2n})$	-k +n(8k+4)	-k-1 + n(8k+4)	-k-2 +n(8k+4)	⋯	-2k+1 +n(8k+4)	-2k +n(8k+4)	-1

Thus, for $i \in [1, 2k+1]$ and $j \in [2, n]$, we need the following $(4n+1) \times (2k+1)$ matrix with entries in $[1, (4n+1)(2k+1)]$ that correspond to $f(u_i x_{i,j})$, $f(u_i v_i)$, $f(v_i x_{i,j})$, $1 \le i \le 2k+1$, $1 \le j \le 2n$, bijectively. For $n = 1$, the required $5 \times (2k+1)$ matrix has entries $f(u_i x_{i,2n-1})$, $f(u_i x_{i,2n})$, $f(u_i v_i)$, $f(v_i x_{i,1})$ and $f(v_i x_{i,2})$, $1 \le i \le 2k+1$, given by the middle five rows. In the following, we split the matrix into Table 1 (for $i \in [1, k+1]$) and Table 2 (for $i \in [k+2, 2k+1]$) with $2 \le j \le n$.

Table 2. Matrix for $i \in [k+2, 2k+1]$

i	k+2	k+3	$\cdots$	2k-1	2k	2k+1	common diff.
$f(u_i x_{i,1})$	2k+1+ n(8k+4)	2k+ n(8k+4)	$\cdots$	k+4+n(8k+4)	k+3+n(8k+4)	k+2+n(8k+4)	-1
$f(u_i x_{i,2})$	-4k-1+n(8k+4)	-4k+n(8k+4)	$\cdots$	-3k-4+ n(8k+4)	-3k-3+n(8k+4)	-3k-2+ n(8k+4)	$+1$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	
$f(u_i x_{i,2n-2j+1})$	2k+1+j(8k+4)	2k+j(8k+4)	$\cdots$	k+4+j(8k+4)	k+3+j(8k+4)	k+2+j(8k+4)	-1
$f(u_i x_{i,2n-2j+2})$	-4k-1+j(8k+4)	-4k+ j(8k+4)	$\cdots$	-3k-4+j(8k+4)	-3k-3+j(8k+4)	-3k-2+j(8k+4)	$+1$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	
$f(u_i x_{i,2n-1})$	10k+4	10k+2	$\cdots$	8k+10	8k+8	8k+6	-2
$f(u_i x_{i,2n})$	4k+3	4k+4	$\cdots$	5k	5k+1	5k+2	+1
$f(u_i v_i)$	k+2	k+3	$\cdots$	2k-1	2k	2k+1	+1
$f(v_i x_{i,1})$	2k+2	2k+3	$\cdots$	3k-1	3k	3k+1	+1
$f(v_i x_{i,2})$	8k+3	8k+1	$\cdots$	6k+9	6k+7	6k+5	-2
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	
$f(v_i x_{i,2j-1})$	-6k-2+j(8k+4)	-6k-1+ j(8k+4)	$\cdots$	-5k-5+j(8k+4)	-5k-4+j(8k+4)	-5k-3 +j(8k+4)	+1
$f(v_i x_{i,2j})$	0+ j(8k+4)	-1+j(8k+4)	$\cdots$	-k+3+j(8k+4)	-k+2+j(8k+4)	-k+1+ j(8k+4)	-1
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	
$f(v_i x_{i,2n-1})$	-6k-2+n(8k+4)	-6k-1+n(8k+4)	$\cdots$	-5k-5+n(8k+4)	-5k-4+n(8k+4)	-5k-3 +n(8k+4)	+1
$f(v_i x_{i,2n})$	0+n(8k+4)	-1+n(8k+4)	$\cdots$	-k+3+n(8k+4)	-k+2+n(8k+4)	-k+1+n(8k+4)	-1

We now have the following observations.

(1) For a fixed $j \in [2, n]$, $\{f(u_i, x_{i,2n-2j+1}), f(u_i, x_{i,2n-2j+2}), f(v_i, x_{i,2j-1}), f(v_i, x_{i,2j}) \mid 1 \le i \le 2k+1\} = [-6k-2+j(8k+4),\ 2k+1+j(8k+4)]$. Thus, when j runs through $[2, n]$, the integers from $10k+6$ to $2k+1+n(8k+4)$ are used. From the middle 5 rows, one may see that the integers from 1 to $10k+5$ are used. Thus all integers in $[1, (4n+1)(2k+1)]$ are used once.

(2) For each $i \in [1, 2k+1]$, the sum of the first $2n+1$ entries is $f^+(u_i) = 8n^2k + 6nk + 4n^2 + 4n + k + 1$.

(3) For each $i \in [1, 2k+1]$, the sum of the last $2n+1$ entries is $f^+(v_i) = 8n^2k + 2nk + 4n^2 + 2n + k + 1$.

(4) Suppose $n = 1$. For $j = 1, 2$, we let $S_j = \{f^+(x_{i,j}) = f(u_i x_{i,j}) + f(v_i x_{i,j}) \mid 1 \le i \le 2k+1\}$. Thus, the elements of S_j form an arithmetic sequence with first term $13k+7$ and last term $11k+7$ with common difference -1. The sum of all the elements in S_j is $(2k+1)(12k+7)$.

(5) Suppose $n \ge 2$. For $j \in [1, 2n]$, we also let $S_j = \{f^+(x_{i,j}) = f(u_i x_{i,j}) + f(v_i x_{i,j}) \mid 1 \le i \le 2k+1\}$. If $j = 2, 2n-1$, the elements of S_j form an arithmetic sequence with first term $5k+3+n(8k+4)$ and last term $3k+3+n(8k+4)$ with common difference -1. For each $j \in [1, 2n] \setminus \{2, 2n-1\}$, $S_j = \{(4k+3)+n(8k+4), \ldots, (4k+3)+n(8k+4)\}$ with multiplicity $2k+1$. Thus, the sum of all the elements in each S_j, $1 \le j \le 2n$, is $(2k+1)[(4k+3)+n(8k+4)]$.

(6) Suppose $2k+1=(2r+1)(2s+1)$, $r,s\geq 1$. Note that, each S_j is an arithmetic sequence with common difference either 0 or -1. If S_j is an arithmetic sequence with common difference -1, then S_j can be partitioned into $2r+1$ blocks of size $2s+1$ such that sum of all elements in each block is $(2k+1)[(4k+3)+n(8k+4)]/(2r+1) = (2s+1)[(4k+3)+n(8k+4)]$ by the existence of $(2r+1)\times(2s+1)$ magic rectangle. If S_j is an arithmetic sequence with common difference 0, then the partition is obvious. This corresponds to partition the $2k+1$ columns into $2r+1$ blocks of $2s+1$ columns so that for each $j\in[1,2n]$, $\sum[f(u_ix_{i,j})+f(v_ix_{i,j})]=(2s+1)[(4k+3)+n(8k+4)]$ over all the $2s+1$ columns in the same block. Moreover, the partition of S_1 to S_{2n} may be all distinct.

Theorem 2.1. *For $n,k\geq 1$, $\chi_{la}((2k+1)P_2\vee O_{2n})=3$.*

Proof. Note that $\chi_{la}((2k+1)P_2\vee O_{2n})\geq \chi((2k+1)P_2\vee O_{2n})=3$. Let $G=(2k+1)(P_2\vee O_{2n})$ be defined at the beginning of this section. We now define a bijection $f: E(G)\to[1,(2k+1)(4n+1)]$ according to the table above. Clearly, for $1\leq i\leq 2k+1$, $f^+(u_i)=8n^2k+6nk+4n^2+4n+k+1>f^+(v_i)=8n^2k+2nk+4n^2+2n+k+1$. Now, for each $j\in[1,2n]$, merging the vertices $x_{i,j}, 1\leq i\leq 2k+1$, to form new vertex x_j of degree $4k+2$, we get the graph $(2k+1)P_2\vee O_{2n}$. From Observations (2) to (5) above, we get that $(2k+1)P_2\vee O_{2n}$ that admits a bijective edge labeling f with

(a) $f^+(x_j)=(2k+1)[(4k+3)+n(8k+4)]$,

(b) $f^+(u_i)=8n^2k+6nk+4n^2+4n+k+1$ and

(c) $f^+(v_i)=8n^2k+2nk+4n^2+2n+k+1$.

Now,

$$\begin{aligned}(a)-(b)&=16k^2n-8kn^2+8k^2+10kn-4n^2+9k+2\\&=8kn(2k-n)+8k^2+2kn+4n(2k-n)+9k+2\\&=(8kn+4n)(2k-n)+8k^2+2kn+9k+2\\&>0\quad\text{if } 2k\geq n.\end{aligned}$$

Otherwise, if $n\geq 2k+1$ (equivalently, $-n\leq -2k-1$) we get that $(a)-(b)\leq -8kn-4n+8k^2+2kn+9k+2\leq 6k(-2k-1)+4(-2k-1)+8k^2+9k+2<0$. Thus, $f^+(x_j)\neq f^+(u_i)$. Similarly,

$$\begin{aligned}(a)-(c)&=16k^2n-8kn^2+8k^2+14kn-4n^2+9k+2n+2\\&=8kn(2k-n)+8k^2+6kn+4n(2k-n)+9k+2n+2\\&=(8kn+4n)(2k-n)+8k^2+6kn+9k+2n+2\\&>0\quad\text{if } 2k\geq n.\end{aligned}$$

If $n=2k+1$, we can get $(a)-(c)=4k^2+3k>0$. Otherwise, if $n\geq 2k+2$ (equivalently, $-n\leq -2k-2$) we get that $(a)-(c)\leq -16kn+8k^2+6kn-8n+9k+2n+2\leq 10k(-2k-2)+8k^2+6(-2k-2)+9k+2<0$. Thus, $f^+(x_j)\neq f^+(v_i)$. Consequently, f is a local antimagic 3-coloring and $\chi_{la}((2k+1)P_2\vee O_{2n})\leq 3$. This completes the proof. □

Example 2.2. Consider $n = 2, k = 4$. The 9×9 matrix and the $9(P_2 \vee O_4)$ with the defined edge labeling are given below. For each $j \in [1, 4]$, merge the vertices in $\{x_{i,j} \mid 1 \leq i \leq 9\}$ to get the vertex x_j of degree 18 gives the $9P_2 \vee O_4$ with the defined labeling as required. The induced vertex labels of u_i, v_i, x_j are 205, 169, 819 respectively.

Table 3. Matrix for $n = 2, k = 4$

i	1	2	3	4	5	6	7	8	9
$f(u_ix_{i,1})$	77	76	75	74	73	81	80	79	78
$f(u_ix_{i,2})$	59	60	61	62	63	55	56	57	58
$f(u_ix_{i,3})$	45	43	41	39	37	44	42	40	38
$f(u_ix_{i,4})$	23	24	25	26	27	19	20	21	22
$f(u_iv_i)$	1	2	3	4	5	6	7	8	9
$f(v_ix_{i,1})$	14	15	16	17	18	10	11	12	13
$f(v_ix_{i,2})$	36	34	32	30	28	35	33	31	29
$f(v_ix_{i,3})$	50	51	52	53	54	46	47	48	49
$f(v_ix_{i,4})$	68	67	66	65	64	72	71	70	69

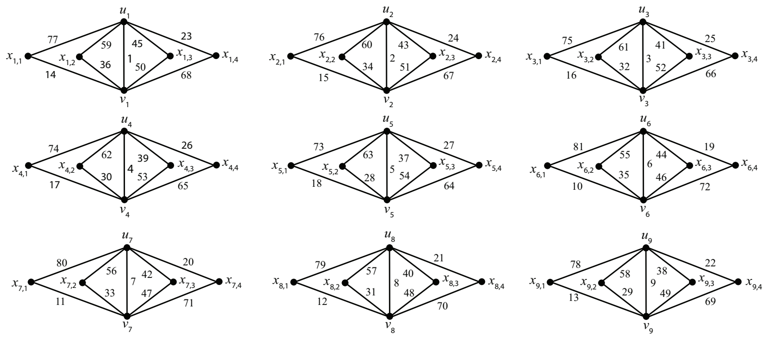

Fig. 1. The graph $9(P_2 \vee O_4)$ for the graph $9P_2 \vee O_4$.

Let $2k+1 = (2r+1)(2s+1)$ for $r, s \geq 1$. Consider $(2k+1)(P_2 \vee O_2)$ with the bijective edge labeling as defined in the proof of Theorem 2.1. Recall that the i-th $P_2 \vee O_2$ has two degree 3 vertices u_i, v_i and two degree 2 vertices $x_{i,j}$, $j = 1, 2$, with induced vertex labels $f^+(u_i) = 15k+9$, $f^+(v_i) = 11k+7$, and $f^+(x_{i,j}) = 13k+8-i$ for $1 \leq i \leq 2k+1$. We now have $S_j = \{f^+(x_{i,j}) \mid 1 \leq i \leq 2k+1\}$. By Observations (4) and (6) above, we can now partition S_j into $2r+1$ blocks of size $2s+1$ each such that in each block, the sum of the induced vertex labels is the constant $(2s+1)(12k+7)$. Now, let $\mathcal{G}_2(2r+1, 2s+1)$ be the set of all the graphs obtained by merging all the $2s+1$ vertices of degree 2 with induced vertex labels in the same block to get a new vertex of degree $4s+2$ with induced vertex label $(2s+1)(12k+7)$. Since the partition of each S_j may not be unique, $\mathcal{G}_2(2r+1, 2s+1)$ may contain more than one graph (see Example 2.5).

Theorem 2.3. *Let $2k+1 = (2r+1)(2s+1)$ for $r, s \geq 1$. If $G \in \mathcal{G}_2(2r+1, 2s+1)$ is defined as above, then $\chi_{la}(G) = 3$.*

Proof. By the discussion above, we know each graph in $\mathcal{G}_2(2r+1, 2s+1)$ admits a local antimagic 3-coloring with distinct induced vertex labels $15k+9$, $11k+7$, $(2s+1)(12k+7)$. Thus, $\chi_{la}(G) \le 3$. Since $\chi_{la}(G) \ge \chi(G) = 3$, the theorem holds. □

Corollary 2.4. *For $r, s \ge 2$, $\chi_{la}((2r+1)[(2s+1)P_2 \vee O_2]) = 3$.*

Proof. Let the partition of S_1 and S_2 be equal, then the resulting graph is $(2r+1)[(2s+1)P_2 \vee O_2]$. □

Example 2.5. Consider $k = 4$ so that $r = s = 1$. We have the following 5×9 matrix and a magic square M of order 3.

Table 4. Matrix for $k = 4, r = s = 1$

i	1	2	3	4	5	6	7	8	9
$f(u_ix_{i,1})$	45	43	41	39	37	44	42	40	38
$f(u_ix_{i,2})$	23	24	25	26	27	19	20	21	22
$f(u_iv_i)$	1	2	3	4	5	6	7	8	9
$f(v_ix_{i,1})$	14	15	16	17	18	10	11	12	13
$f(v_ix_{i,2})$	36	34	32	30	28	35	33	31	29

$$M = \begin{array}{|c|c|c|}\hline 8 & 3 & 4 \\ \hline 1 & 5 & 9 \\ \hline 6 & 7 & 2 \\ \hline \end{array}$$

For $j = 1, 2$, partition S_j into blocks $\{f^+(x_{1,j}), f^+(x_{5,j}), f^+(x_{9,j})\}$, $\{f^+(x_{2,j}), f^+(x_{6,j}), f^+(x_{7,j})\}$, $\{f^+(x_{3,j}), f^+(x_{4,j}), f^+(x_{8,j})\}$ by using the rows of M. We get $G = 3(3P_2 \vee O_2) \in \mathcal{G}_2(3,3)$ as required. The induced vertex labels are $69, 51, 165$ respectively.

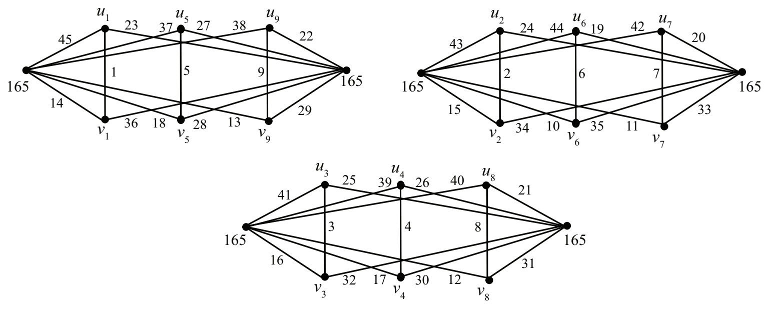

Fig. 2. The graph $3(3P_2 \vee O_2)$

If we keep the partition of S_1 and partition S_2 into blocks $\{f^+(x_{1,2}), f^+(x_{6,2}), f^+(x_{8,2})\}$, $\{f^+(x_{2,2}), f^+(x_{4,2}), f^+(x_{9,2})\}$ and $\{f^+(x_{3,2}), f^+(x_{5,2}), f^+(x_{7,2})\}$ by using the columns of M, then we get a connected graph as follow with the same induced vertex labels.

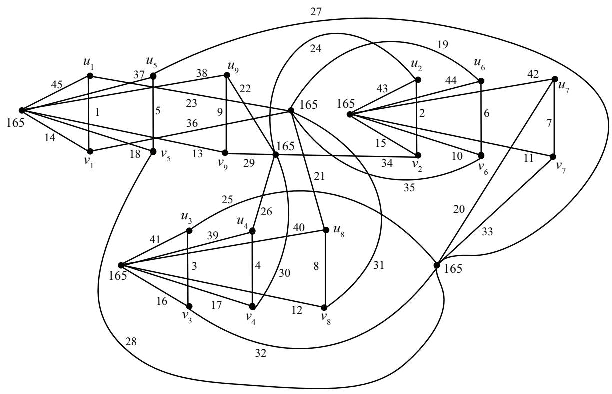

Fig. 3. A connected graph in $\mathcal{G}_2(3,3)$

The degree 6 vertices are indicated with their induced vertex label 165.

Let $2k+1=(2r+1)(2s+1)$ for $r,s\geq 1$. Consider $(2k+1)(P_2\vee O_{2n}), n\geq 2$, with the bijective edge labeling as defined in the proof of Theorem 2.1. Similar to the discussion above for $(2k+1)(P_2\vee O_2)$, and by Observations (5) and (6), we also have the set of graphs $\mathcal{G}_{2n}(2r+1,2s+1)$. Each of the graph has vertices u_i and v_i, $1\leq i\leq 2k+1$, of degree $2n+1$ with induced vertex labels $f^+(u_i)=8n^2k+6nk+4n^2+4n+k+1$, $f^+(v_i)=8n^2k+2nk+4n^2+2n+k+1$, and $2n(2r+1)$ vertices of degree $2(2s+1)$ with induced vertex label $(2s+1)[(4k+3)+n(8k+4)]$. Note that, since S_1 is a constant sequence, there are at least two different partitions of S_1. Thus $\mathcal{G}_{2n}(2r+1,2s+1)$ contains at least two graphs.

Theorem 2.6. *For $n\geq 2$, $r,s\geq 1$ and $G\in\mathcal{G}_{2n}(2r+1,2s+1)$, $\chi_{la}(G)=3$.*

Proof. By the discussion above, we know each $G\in\mathcal{G}_{2n}(2r+1,2s+1)$ admits a local antimagic 3-coloring with each degree $2(2s+1)$ vertex has induced vertex label $(a)=(2s+1)[(4k+3)+n(8k+4)]$ whereas for $1\leq i\leq 2k+1$, the vertices u_i,v_i of degree $2n+1$ have induced vertex labels $(b)=f^+(u_i)=8n^2k+6nk+4n^2+4n+k+1$ and $(c)=f^+(v_i)=8n^2k+2nk+4n^2+2n+k+1$.

Clearly, $(b)>(c)$. Now,

$$\begin{aligned}(a)-(b)&=16kns-8kn^2+2kn+8ks+8ns-4n^2+3k+6s+2\\&=(8kn+4n)(2s-n)+2kn+8ks+3k+6s+2\\&>0\quad\text{if } 2s\geq n.\end{aligned}$$

Otherwise, if $n\geq 2s+1$ (equivalently, $-n\leq -2s-1$), $(a)-(b)\leq -6kn-4n+8ks+3k+6s+2\leq 6k(-2s-1)+4(-2s-1)+8ks+3k+6s+2<0$. Thus, $(a)\neq(b)$. Similarly,

$$\begin{aligned}(a)-(c)&=16kns-8kn^2+6kn+8ks-4n^2+8ns+3k+2n+6s+2\\&=(8kn+4n)(2s-n)+6kn+8ks+3k+2n+6s+2\\&>0\quad\text{if } 2s\geq n.\end{aligned}$$

If $n = 2s + 1$, $(a) - (c) = (-2k - 2)(2s + 1) + 8ks + 3k + 6s + 2 > 0$. Otherwise, if $n \geq 2s + 2$ (equivalently, $2s - n \leq -2$), $(a) - (c) \leq -10kn - 6n + 8ks + 3k + 6s + 2 \leq (10k + 6)(-2s - 2) + 8ks + 3k + 6s + 2 < 0$. Thus, $(a) \neq (c)$.

Consequently, $\chi_{la}(G) \leq 3$. Since $\chi_{la}(G) \geq \chi(G) = 3$, the theorem holds. □

Corollary 2.7. *For* $n \geq 2, r, s \geq 1$, $\chi_{la}((2r + 1)[(2s + 1)P_2 \vee O_{2n}]) = 3$.

Proof. By using the rows of a $(2r + 1) \times (2s + 1)$ magic rectangle to partition S_2. Partition other S_j so that they have the same partition as S_2. The resulting graph is $(2r + 1)[(2s + 1)P_2 \vee O_{2n}]$. □

Example 2.8. Consider $n = 2, k = 4$ as in Example 2.2. We can only have $r = s = 1$. For $1 \leq j \leq 4$, if we partition S_j into blocks as in Example 2.5, we can get $G = 3(3P_2 \vee O_4) \in \mathcal{G}_4(3, 3)$ as follow. The induced vertex labels are 205, 169, 273 respectively.

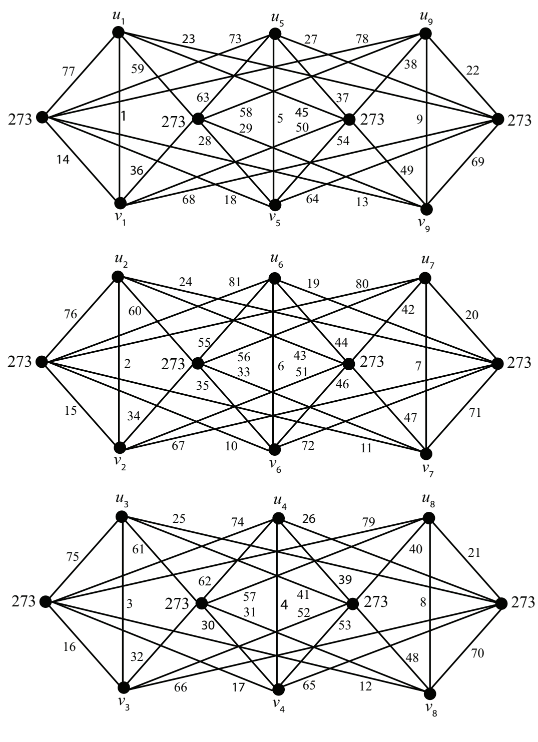

Fig. 4. Graph $3(3P_2 \vee O_4) \in \mathcal{G}_4(3, 3)$.

If we keep the partitions of S_1, S_2, S_4 and partition S_3 into blocks $\{f^+(x_{1,3}), f^+(x_{6,3}), f^+(x_{8,3})\}$, $\{f^+(x_{2,3}), f^+(x_{4,3}), f^+(x_{9,3})\}$ and $\{f^+(x_{3,3}), f^+(x_{5,3}), f^+(x_{7,3})\}$, then we get a connected graph as follow with the same induced vertex labels.

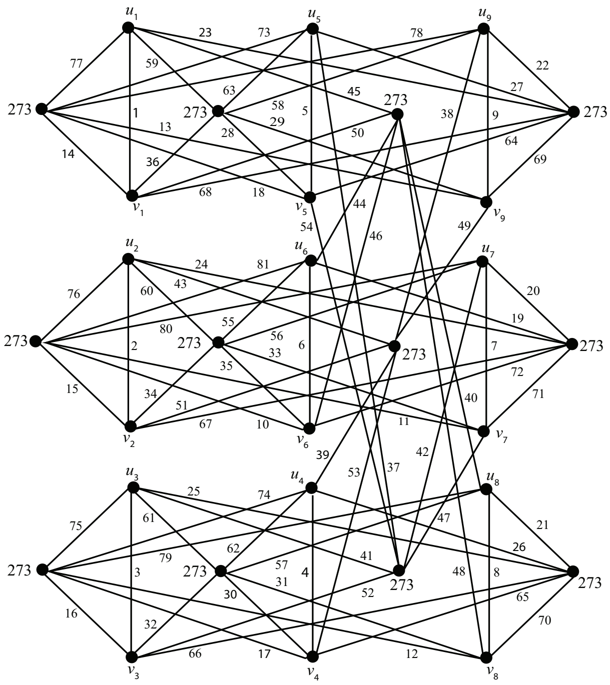

Fig. 5. A connected graph in $\mathcal{G}_4(3,3)$.

The degree 6 vertices are indicated with their induced vertex label 273. Note that there are many more ways to do the partitioning of S_1 to S_4 accordingly to get more non-isomorphic graphs all with local antimagic chromatic number 3.

Recall that for $n \ge 2$, $1 \le i \le 2k+1$ and each $j \in [1,2n] \setminus \{2, 2n-1\}$, the bijective edge labeling of $(2k+1)(P_2 \vee O_{2n})$ defined above has the property that $f^+(x_{i,j}) = (4k+3) + n(8k+4) = f^+(x_{k+1,2}) = f^+(x_{k+1,2n-1})$. Thus, each $G \in \mathcal{G}_{2n}(2r+1, 2s+1)$ has $(2n-2)(2r+1)$ vertices of degree $4s+2$ each of with is incident to $2s+1$ pairs of edges with labels sum $(4k+3)+n(8k+4)$ and also 2 vertices of degree $4s+2$ each of which is incident to a pair of edges with labels sum $(4k+3)+n(8k+4)$.

For $n \ge 2$, $r, s \ge 1$, and a graph $G \in \mathcal{G}_{2n}(2r+1, 2s+1)$, choose two distinct vertices of degree $4s+2$, say x and x', such that

(a) xu_i, xv_i are edges with labels sum $(4k+3)+n(8k+4)$, while $x'u_{i'}, x'v_{i'}$ are also edges with labels sum $(4k+3)+n(8k+4)$, and

(b) x and x' do not have common neighbors.

Delete the edges xu_i, xv_i, $x'u_{i'}$ and $x'v_{i'}$; and add the edges $xu_{i'}$ and $xv_{i'}$ with labels $f(x'u_{i'})$ and $f(x'v_{i'})$ respectively; and add the edges $x'u_i$ and $x'v_i$ with labels $f(xu_i)$ and $f(xv_i)$ respectively. Repeat this *delete-add* process so long as we get another new graph not isomorphic to G. Let $\mathcal{H}_{2n}(2r+1, 2s+1)$ be the set of all the graphs such obtained. Note that the graphs in $\mathcal{H}_{2n}(2r+1, 2s+1)$ may not be connected.

Theorem 2.9. *For $n \ge 2, r, s \ge 1$, if $H \in \mathcal{H}_{2n}(2r+1, 2s+1)$, then $\chi_{la}(H) = 3$.*

Proof. By definition and the discussion above, we immediately have each graph $H \in \mathcal{H}_{2n}(2r+1, 2s+1)$ also admits a a local antimagic 3-coloring with induced vertex labels as for each graphs $G \in \mathcal{G}_{2n}(2r+1, 2s+1)$. Moreover, $\chi(H) = 3$. This completes the proof. □

Example 2.10. Using the $3(3P_2 \vee O_3)$ in Example 2.8, we may let x be the vertex adjacent to u_1, v_1 and x' be the vertex adjacent to u_7, v_7, satisfying $f(xu_1) + f(xv_1) = f(x'u_7) + f(x'v_7) = 91$ and x, x' do not have common neighbors. Note that this graph is not a possible graph of $\mathcal{G}_4(3,3)$ since two of the degree 6 vertices are obtained by merging vertices with induced vertex labels in $\{f^+(x_{7,4}), f^+(x_{5,1}), f^+(x_{9,1})\}$ and $\{f^+(x_{1,1}), f^+(x_{2,4}), f^+(x_{6,4})\}$ which are not subsets of S_1 nor S_4.

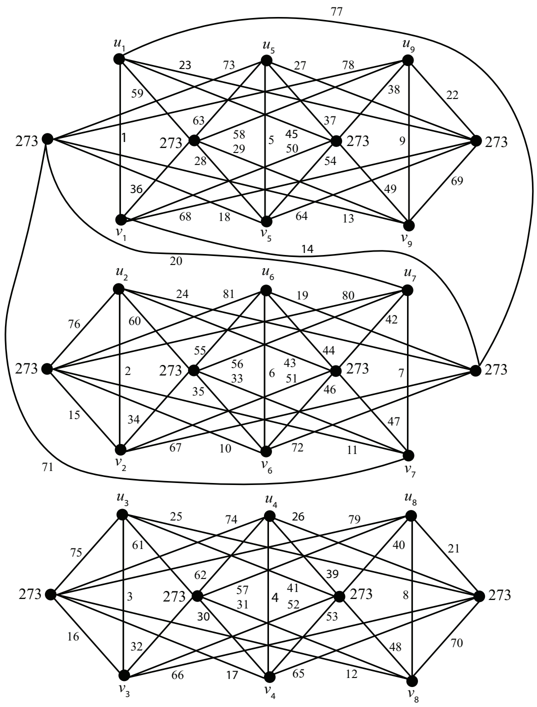

Fig. 6. A disconnected graph in $\mathcal{H}_4(3,3)$.

Note that a new connected graph is obtained if we apply the same process to the graph in Figure 6 by choosing the vertices incident to edges with labels $76, 15$ and $21, 70$.

3. Joins with odd empty graphs

Consider $(2k+1)(P_2 \vee O_{2n+1})$ of order $(2k+1)(2n+3)$ and size $(2k+1)(4n+3)$ for $k, n \geq 1$. Thus, for $1 \in [1, 2k+1]$ and $j \in [1, n]$, we need the following $(4n+3) \times (2k+1)$ matrix with entries in $[1, (4n+3)(2k+1)]$ bijectively. For $n = 1$, the required $7 \times (2k+1)$ matrix has entries $f(u_i x_{i,1})$, $f(u_i x_{i,2})$, $f(u_i x_{i,3})$, $f(u_i v_i)$, $f(v_i x_{i,1})$, $f(v_i x_{i,2})$, $f(v_i x_{i,3})$ given by the first 2, middle 3 and last 2 rows.

Table 5. $(4n+3) \times (2k+1)$ matrix

i	1	2	3	$\cdots$	2k	2k+1	common diff.
$f(u_ix_{i,1})$	6k+3+ 2n(4k+2)	6k+2 + 2n(4k+2)	6k+1 2n(4k+2)	$\cdots$	4k+4 + 2n(4k+2)	4k+3 + 2n(4k+2)	-1
$f(u_ix_{i,2})$	2k+2 + 2n(4k+2)	2k+3 + 2n(4k+2)	2k+4 + 2n(4k+2)	$\cdots$	4k+1 + 2n(4k+2)	4k+2 + 2n(4k+2)	$+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
$f(u_ix_{i,2n-2j+1})$	6k+3 + (n+j)(4k+2)	6k+2 + (n+j)(4k+2)	6k+1 (n+j)(4k+2)	$\cdots$	4k+4 + (n+j)(4k+2)	4k+3 + (n+j)(4k+2)	-1
$f(u_ix_{i,2n-2j+2})$	2k+2 + (n+j)(4k+2)	2k+3 + (n+j)(4k+2)	2k+4 + (n+j)(4k+2)	$\cdots$	4k+1 + (n+j)(4k+2)	4k+2 + (n+j)(4k+2)	$+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
$f(u_ix_{i,2n+1})$	2k+1 + (n+1)(4k+2)	2k+ (n+1)(4k+2)	(2k-1)+ (n+1)(4k+2)	$\cdots$	2 + (n+1)(4k+2)	1 + (n+1)(4k+2)	-1
$f(u_iv_i)$	1	2	3	$\cdots$	2k	2k+1	+1
$f(v_ix_{i,1})$	4k+2	4k+1	4k	$\cdots$	2k+3	2k+2	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
$f(v_ix_{i,2j})$	2k+1 + j(4k+2)	2k + j(4k+2)	2k-1 + j(4k+2)	$\cdots$	2 + j(4k+2)	1 + j(4k+2)	-1
$f(v_ix_{i,2j+1})$	2k+2 + j(4k+2)	2k+3 + j(4k+2)	2k+4 + j(4k+2)	$\cdots$	4k+1 + j(4k+2)	4k +2 + j(4k+2)	$+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
$f(v_ix_{i,2n})$	2k+1 + n(4k+2)	2k + n(4k+2)	2k-1 + n(4k+2)	$\cdots$	2 + n(4k+2)	1 + n(4k+2)	-1
$f(v_ix_{i,2n+1})$	2k+2 + n(4k+2)	2k+3 + n(4k+2)	2k+4 + n(4k+2)	$\cdots$	4k+1 + n(4k+2)	4k +2 + n(4k+2)	$+1$

We now have the following observations.

(1) For a fixed $j \in [1,n]$, $\{f(v_i, x_{i,2j}, f(v_i, x_{i,2j+1}) \mid 1 \le i \le 2k+1\} = [1+j(4k+2),\ 4k+2+j(4k+2)]$. Thus, when j runs through $[1,n]$, the integers from $4k+3$ to $(n+1)(4k+2)$ are used. Similarly, for a fixed $j \in [1,n]$, $\{f(u_i, x_{i,2n-2j+1}), f(u_i, x_{i,2n-2j+2}) \mid 1 \le i \le 2k+1\} = [2k+2+(n+j)(4k+2),\ 6k+3+(n+j)(4k+2)]$. Thus, when j runs through $[1,n]$, the integers from $2k+2+(n+1)(4k+2)$ to $2k+1+(2n+1)(4k+2) = (4n+3)(2k+1)$ are used. From rows $f(u_iv_i)$ and $f(v_ix_{i,1})$ with entries 1 to $4k+2$, and rows $f(u_ix_{i,2n+1})$ with entries $1+(n+1)(4k+2)$ to $2k+1+(n+1)(4k+2)$, we see that all integers in $[1,(4n+3)(2k+1)$ are used once.

(2) For each column, the sum of the first $2n+2$ entries is $f^+(u_i) = 12n^2k + 16nk + 6n^2 + 9n + 6k + 4$.

(3) For each column, the sum of the last $2n+2$ entries is $f^+(v_i) = 4n^2k + 8nk + 2n^2 + 5n + 4k + 3$.

(4) For $S_1 = \{f^+(x_{i,1}) = f(u_ix_{i,1}) + f(v_ix_{i,1}) \mid 1 \le i \le 2k+1\}$, the elements form an arithmetic sequence with first term $10k+5+2n(4k+2)$, last term $6k+5+2n(4k+2)$ and common difference 2. The sum of all these terms is $(2k+1)[(8k+5)+2n(4k+2)]$.

(5) For $j \in [2, 2n+1]$, $S_j = \{f^+(x_{i,j}) = f(u_ix_{i,j}) + f(v_ix_{i,j}) \mid 1 \le i \le 2k+1\} = \{8k+5+2n(4k+2), \ldots, 8k+5+2n(4k+2)\}$ with multiplicity $2k+1$. The sum of all the terms is $(2k+1)[(8k+5)+2n(4k+2)]$.

Theorem 3.1. *For $n, k \geq 1$, $\chi_{la}((2k+1)P_2 \vee O_{2n+1}) = 3$.*

Proof. Note that $\chi_{la}((2k+1)P_2 \vee O_{2n+1}) \geq \chi((2k+1)P_2 \vee O_{2n}) = 3$. Let $G = (2k+1)(P_2 \vee O_{2n+1})$ for $n, k \geq 1$. We now define a bijection $f : E(G) \to [1, (4n+3)(2k+1)]$ according to the table above. Now, for each $j \in [1, 2n+1]$, merging the vertices in $\{x_{i,j} \mid 1 \leq i \leq 2k+1\}$, to form new vertex x_j of degree $4k+2$, we get the graph $(2k+1)P_2 \vee O_{2n+1}$. From Observations (2) to (5) above, we get that $(2k+1)P_2 \vee O_{2n+1}$ admits a bijective edge labeling f with

(a) $f^+(x_j) = (2k+1)[(8k+5) + 2n(4k+2)]$,

(b) $f^+(u_i) = 12n^2k + 16nk + 6n^2 + 9n + 6k + 4$, and

(c) $f^+(v_i) = 4n^2k + 8nk + 2n^2 + 5n + 4k + 3$,

for $1 \leq i \leq 2k+1$ and $1 \leq j \leq 2n+1$.

Clearly, for $1 \leq i \leq 2k+1$, $f^+(u_i) > f^+(v_i)$. Now,

$$\begin{aligned}(a) - (b) &= 16k^2n - 12kn^2 + 16k^2 - 6n^2 + 12k - 5n + 1 \\ &= (4kn + 4k + 2n + 3)(4k - 3n) + 4kn + 4n + 1 \\ &> 0 \text{ if } 4k \geq 3n.\end{aligned}$$

If $3n = 4k+1$, then $(a)-(b) = -(4kn+4k+2n+3)+4kn+4n+1 = -4k+2n-2 = -n-1 < 0$. Otherwise, if $3n \geq 4k+2$, we get that $(a)-(b) \leq -2(4kn+4k+2n+3)+4kn+4n+1 = -4kn - 8k - 5 < 0$. Thus, $f^+(x_j) \neq f^+(u_i)$. Similarly,

$$\begin{aligned}(a) - (c) &= 16k^2n - 4kn^2 + 16k^2 + 8kn - 2n^2 + 14k - n + 2 \\ &= (4kn + 2n + 1)(4k - n) + 16k^2 + 10k + 2 \\ &> 0 \text{ if } n \leq 4k.\end{aligned}$$

Otherwise, if $n \geq 4k+1$ (equivalent, $-n \leq -4k-1$), we get that $(a) - (c) \leq -4kn - 2n + 16k^2 + 10k + 1 \leq (-4k-1)(4k+2) + 16k^2 + 10k + 1 < 0$. Thus, $f^+(x_j) \neq f^+(v_i)$.

Consequently, f is a local antimagic 3-coloring and $\chi_{la}((2k+1)P_2 \vee O_{2n+1}) \leq 3$. This completes the proof. □

Example 3.2. Consider $n = 2, k = 4$. The 11×9 matrix and the $9(P_2 \vee O_5)$ with the defined edge labeling are given below. For each $j \in [1, 5]$, merge the vertices in $\{x_{i,j} \mid 1 \leq i \leq 9\}$ to get the vertex x_j of degree 18 gives the $9P_2 \vee O_5$ as required. The induced vertex labels of u_i, v_i, x_j are $390, 165, 981$ respectively.

Table 6. Matrix for $n = 2, k = 4$

i	1	2	3	4	5	6	7	8	9
$f(u_ix_{i,1})$	99	98	97	96	95	94	93	92	91
$f(u_ix_{i,2})$	82	83	84	85	86	87	88	89	90
$f(u_ix_{i,3})$	81	80	79	78	77	76	75	74	73
$f(u_ix_{i,4})$	64	65	66	67	68	69	70	71	72
$f(u_ix_{i,5})$	63	62	61	60	59	58	57	56	55
$f(u_iv_i)$	1	2	3	4	5	6	7	8	9
$f(v_ix_{i,1})$	18	17	16	15	14	13	12	11	10
$f(v_ix_{i,2})$	27	26	25	24	23	22	21	20	19
$f(v_ix_{i,3})$	28	29	30	31	32	33	34	35	36
$f(v_ix_{i,4})$	45	44	43	42	41	40	38	38	37
$f(v_ix_{i,5})$	46	47	48	49	50	51	52	53	54

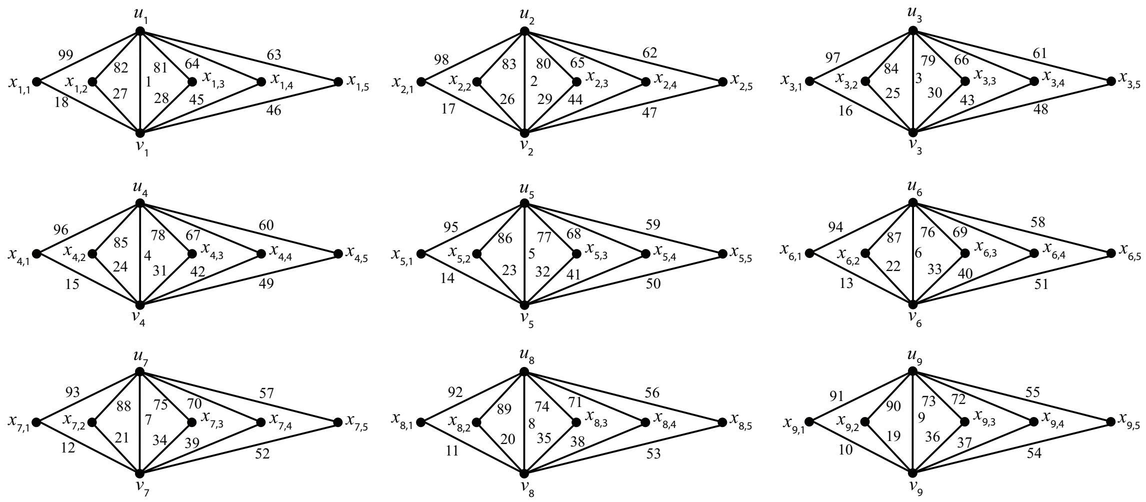

Fig. 7. Graph $9(P_2 \vee O_5)$.

Similar to Theorem 2.6, we also define $\mathcal{G}_{2n+1}(2r+1, 2s+1)$ accordingly for $n, r, s \geq 1$.

Theorem 3.3. *For $n, r, s \geq 1$ and $G \in \mathcal{G}_{2n+1}(2r+1, 2s+1)$, $\chi_{la}(G) = 3$.*

Proof. Using the ideas as in the proof of Theorem 4.4, we can conclude that G admits a bijective edge labeling with each degree $2(2s+1)$ vertex has induced vertex label $(a) = (2s+1)[(8k+5) + 2n(4k+2)]$ whereas for $1 \leq i \leq 2k+1$, the vertices u_i, v_i of degree $2n+2$ have induced vertex labels $(b) = f^+(u_i) = 12n^2k + 16nk + 6n^2 + 9n + 6k + 4$ and $(c) = f^+(v_i) = 4n^2k + 8nk + 2n^2 + 5n + 4k + 3$. Clearly, $(b) > (c)$. Now,

$$\begin{aligned}(a) - (b) &= 16kns - 12kn^2 - 8kn + 16ks - 6n^2 + 8ns + 2k - 5n + 10s + 1 \\ &= (4kn + 4k + 2n + 2)(4s - 3n) + 4kn + 2k + 2s + n + 1 > 0 \text{ if } 4s \geq 3n.\end{aligned}$$

Otherwise, if $4s \le 3n - 1$, $(a) - (b) \le 2s - 2k - n - 1 < 0$. Thus, $(a) \neq (b)$. Similarly,

$$\begin{aligned}(a) - (c) &= 16kns - 4kn^2 + 16ks + 8ns - 2n^2 + 4k - n + 10s + 2 \\ &= (4kn + 2n + 1)(4s - n) + 16ks + 6s + 4k + 2 \\ &> 0 \text{ if } 4s \ge n.\end{aligned}$$

If $4s \le n - 1$ (equivalent, $-n \le -4s - 1$), $(a) - (c) \le -4kn - 2n + 16ks + 6s + 4k + 1 \le (-4s - 1)(4k + 2) + 16ks + 6s + 4k + 1 < 0$. Thus, $(a) \neq (c)$.

Consequently, $\chi_{la}(G) \le 3$. Since $\chi_{la}(G) \ge \chi(G) = 3$, the theorem holds. □

Corollary 3.4. *For $n, r, s \ge 1$, $\chi_{la}((2r+1)[(2s+1)P_2 \vee O_{2n+1}]) = 3$.*

Proof. Partition S_1 so that each of the block corresponsds to a block of the partition of $S_j, 2 \le j \le n$, with elements in the same column of the matrix. □

Example 3.5. Consider $n = 2$, $k = 4$. We can only have $r = s = 1$. For $1 \le j \le 5$, partition S_j into blocks $\{f^+(x_{1,j}), f^+(x_{5,j}), f^+(x_{9j})\}$, $\{f^+(x_{2,j}), f^+(x_{6,j}), f^+(x_{7,j})\}$, $\{f^+(x_{3,j}), f^+(x_{4,j}), f^+(x_{8,j})\}$, we get 6-regular $3(3P_2 \vee O_5) \in \mathcal{G}_5(3,3)$ as required. The induced vertex labels are $390, 165, 327$ respectively as in Figure 8. The unlabeled vertices have induced vertex label 327. Similar to Figure 5 in Example 2.8, we can also keep the partition of $S_j, 1 \le j \le 4$ and partition S_5 into blocks $\{f^+(x_{i,5}) \mid 1 \le i \le 3\}$, $\{f^+(x_{i,5}) \mid 4 \le i \le 6\}$, and $\{f^+(x_{i,5}) \mid 7 \le i \le 9\}$ to get a connected graph in $\mathcal{G}_5(3,3)$.

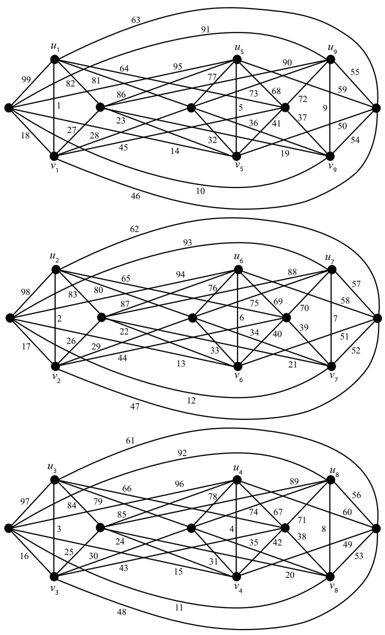

Fig. 8. Graph $3(3P_2 \vee O_5) \in \mathcal{G}_5(3,3)$

Similar to Theorem 2.9, we also have the following theorem without proof.

Theorem 3.6. *For $n, r, s \ge 1$, if $H \in \mathcal{H}_{2n+1}(2r+1, 2s+1)$, then $\chi_{la}(H) = 3$.*

Similar to Example 2.10, a (disconnected) graph in $\mathcal{H}_{2n+1}(2r+1, 2s+1)$ can be obtained by applying the delete-add process to a graph in $\mathcal{G}_{2n+1}(2r+1, 2s+1)$ that has two vertices without common neighbors with a pair of incident edges with equal labels sum. For example, the $3(3P_2 \vee O_5) \in \mathcal{G}_5(3, 3)$ and the two vertices with incident edges labels $54, 55$ and $52, 57$, respectively.

Corollary 3.7. *Each graph (i) $G = (2k+1)P_2 \vee O_{4k+1}$, $k \geq 1$, is a connected $(4k+2)$-regular graph of order $8k+3$ and size $(2k+1)(8k+3)$, $G \in \mathcal{G}_{4s+1}(2r+1, 2s+1)$ or $G \in \mathcal{H}_{4s+1}(2r+1, 2s+1)$, $s \geq 1$, is a (possibly disconnected) $(4s+2)$-regular graph of order $(2r+1)(8s+3)$ and size $(2r+1)(2s+1)(8s+3)$ with $\chi_{la}(G) = 3$.*

Proof. (i) Let $n = 2k$, each $G = (2k+1)P_2 \vee O_{4k+1}$ is a connected $(4k+2)$-graph. (ii) Let $n = 2s$, each graph in $\mathcal{G}_{4s+1}(2r+1, 2s+1)$ or $\mathcal{H}_{4s+1}(2r+1, 2s+1)$ is a $(4s+2)$ regular graph. □

4. New $(4n+1) \times (2k+1)$ matrix

For $k \geq 1$, we now consider the following $(4n+1) \times (2k+1)$ matrix for $2 \leq j \leq n$ that we split into Table 7 and Table 8. Note that when $n = 1$, the required $5 \times (2k+1)$ matrix is given by rows $f(u_i, x_{i,1})$, $f(u_i, x_{i,2})$, $f(u_iv_i)$, $f(v_ix_{i,1})$ and $f(v_1x_{i,2})$ of the matrix below. Moreover, the entries in column $k+1$ appears in both Tables 7 and 8.

Table 7. Matrix for $i \in [1, k+1]$

i	1	2	3	···	k-1	k	k+1
$f(u_ix_{i,1})$	k+2+ n(8k+4)	k+3 + n(8k+4)	k+4+ n(8k+4)	···	2k+ n(8k+4)	2k+1+ n(8k+4)	1+ n(8k+4)
$f(u_ix_{i,2})$	-2k-2+ n(8k+4)	-2k-4 + n(8k+4)	-2k-6+ n(8k+4)	···	-4k+2+ n(8k+4)	-4k n(8k+4)	- 2k-1+ n(8k+4)
⋮	⋮	⋮	⋮	···	⋮	⋮	⋮
$f(u_ix_{i,2j-1})$	9k+6 (n-j)(8k+4)	9k+7 (n-j)(8k+4)	9k+8 (n-j)(8k+4)	···	10k+4 (n-j)(8k+4)	10k+5 (n-j)(8k+4)	8k+5 (n-j)(8k+4)
$f(u_ix_{i,2j})$	5k+2 (n-j)(8k+4)	5k+1 (n-j)(8k+4)	5k (n-j)(8k+4)	···	4k+4 (n-j)(8k+4)	4k+3 (n-j)(8k+4)	6k+3 (n-j)(8k+4)
⋮	⋮	⋮	⋮	···	⋮	⋮	⋮
(u_iv_i)	1	2	3	···	k-1	k	k+1
$f(v_ix_{i,1})$	3k+2	3k+3	3k+4	···	4k	4k+1	4k+2
$f(v_ix_{i,2})$	8k+4	8k+2	8k	···	6k+8	6k+6	6k+4
⋮	⋮	⋮	⋮	···	⋮	⋮	⋮
$f(v_ix_{i,2j-1})$	-5k-2+ j(8k+4)	-5k-1+ j(8k+4)	-5k+ j(8k+4)	···	-4k-4+ j(8k+4)	-4k-3+ j(8k+4)	-4k-2 + j(8k+4)
$f(v_ix_{i,2j})$	-k+ j(8k+4)	-k-1+ j(8k+4)	-k-2+ j(8k+4)	···	-2k+3+ j(8k+4)	-2k+1+ j(8k+4)	-2k+ j(8k+4)
⋮	⋮	⋮	⋮	···	⋮	⋮	⋮

Table 8. Matrix for $i \in [k+1, 2k+1]$

i	k+1	k+2	k+3	⋯	2k-1	2k	2k+1
$f(u_i x_{i,1})$	1+ n(8k+4)	2+ n(8k+4)	3+ n(8k+4)	⋯	k-1+ n(8k+4)	k+ n(8k+4)	k+1+ n(8k+4)
$f(u_i x_{i,2})$	-2k-1+ n(8k+4)	-2k-3+ n(8k+4)	-2k-5+ n(8k+4)	⋯	-4k+3+ n(8k+4)	-4k+1+ n(8k+4)	-4k-1+ n(8k+4)
⋮	⋮	⋮	⋮	⋯	⋮	⋮	⋮
$f(u_i x_{i,2j-1})$	8k+5 (n-j)(8k+4)	8k+6 (n-j)(8k+4)	8k+7 (n-j)(8k+4)	⋯	9k+3 (n-j)(8k+4)	9k+4 (n-j)(8k+4)	9k+5 (n-j)(8k+4)
$f(u_i x_{i,2j})$	6k+3 (n-j)(8k+4)	6k+2 (n-j)(8k+4)	6k+1 (n-j)(8k+4)	⋯	5k+5 (n-j)(8k+4)	5k+4 (n-j)(8k+4)	5k+3 (n-j)(8k+4)
⋮	⋮	⋮	⋮	⋯	⋮	⋮	⋮
$f(u_i v_i)$	k+1	k+2	k+3	⋯	2k-1	2k	2k+1
$f(v_i x_{i,1})$	4k+2	2k+2	2k+3	⋯	3k-1	3k	3k+1
$f(v_i x_{i,2})$	6k+4	8k+3	8k+1	⋯	6k+9	6k+7	6k+5
⋮	⋮	⋮	⋮	⋯	⋮	⋮	⋮
$f(v_i x_{i,2j-1})$	-4k-2+ j(8k+4)	-6k-2+ j(8k+4)	-6k-1+ j(8k+4)	⋯	-5k-5+ j(8k+4)	-5k-4+ j(8k+4)	-5k-3+ j(8k+4)
$f(v_i x_{i,2j})$	-2k+ j(8k+4)	0+ j(8k+4)	-1+ j(8k+4)	⋯	-k+3+ j(8k+4)	-k+2+ j(8k+4)	-k+1+ j(8k+4)
⋮	⋮	⋮	⋮	⋯	⋮	⋮	⋮

We now have the following observations.

(a) For $n \geq 2$ and each $i \in [1, 2k+1]$, the sum of the first $2n+1$ row entries is $f^+(u_i) = 2n(8k+4) - k + 1 + \sum_{j=2}^{n}[2(n-j)(8k+4) + 14k + 8] = 8kn^2 + 6kn + 4n^2 + k + 4n + 1$. Note that, this formula also holds when $n = 1$.

(b) For $n \geq 2$ and each $i \in [1, 2k+1]$, the sum of the last $2n+1$ row entries is $f^+(v_i) = 11k + 7 + \sum_{j=2}^{n}[2j(8k+4) - 6k - 2] = 8kn^2 + 2kn + 4n^2 + k + 2n + 1$. Note that, this formula also holds when $n = 1$.

(c) For each $i \in [1, k]$ and $j \in [1, 2n]$, each of $f(u_i x_{i,j}) + f(v_{2k+2-i} x_{2k+2-i,j})$, $f(v_i x_{i,j}) + f(u_{2k+2-i} x_{2k+2-i,j})$ and $f(u_{k+1} x_{k+1,j}) + f(v_{k+1} x_{k+1,j})$ is a constant $n(8k+4) + 4k + 3$.

(d) Suppose $2k+1 = (2r+1)(2s+1)$, $r, s \geq 1$. For each $a \in [1, r]$ and j $\in [1, 2n]$, each of

$$\sum_{b=1}^{2s+1}[f(u_{(a-1)(2s+1)+b} x_{(a-1)(2s+1)+b,j}) + f(v_{2k+2-(a-1)(2s+1)-b} x_{2k+2-(a-1)(2s+1)-b,j})], \quad (1)$$

$$\sum_{b=1}^{2s+1}[f(v_{(a-1)(2s+1)+b} x_{(a-1)(2s+1)+b,j}) + f(u_{2k+2-(a-1)(2s+1)-b} x_{2k+2-(a-1)(2s+1)-b,j})], \quad (2)$$

$$\sum_{b=1}^{2s+1}[f(u_{r(2s+1)+b} x_{r(2s+1)+b,j}) + f(v_{2k+2-r(2s+1)-b} x_{2k+2-r(2s+1)-b,j})], \quad (3)$$

is a constant $(2s+1)[n(8k+4) + 4k + 3]$.

Consider $G = (2k+1)P_2 \vee O_{2n}$. By Observations (a) and (b) above, we can now define a bijection $f : E(G) \to [1, (4n+1)(2k+1)]$ according to the table above. Clearly, for $1 \le i \le 2k+1$, $f^+(u_i) > f^+(v_i)$.

Now, for each $i \in [1,k]$ and $j \in [1,2n]$, first delete the edges $v_i x_{i,j}$ and $v_{2k+2-i}x_{2k+2-i,j}$, and then add the edges $v_{2k+2-i}x_{i,j}$ and $v_i x_{2k+2-i,j}$ with labels $f(v_{2k+2-i}x_{2k+2-i,j})$ and $f(v_i x_{i,j})$, respectively. Finally, we rename $x_{i,j}$ by $y_{i,j}$ and $x_{2k+2-i,j}$ by $z_{i,j}$. We still denote this new labeling by f. By Observation (c), $f^+(y_{i,j}) = f^+(z_{i,j}) = n(8k+4)+4k+3$. It is easy to verify that $f^+(u_i) \ne f^+(v_i) \ne f^+(y_{i,j})$ for all possible n, k. We denote the resulting graph by $G_{2n}(k+1)$. Note that $G_{2n}(k+1)$ has $k+1$ components.

Theorem 4.1. *For $n, k \ge 1$, we have $\chi_{la}(G_{2n}(k+1)) = 3$.*

Proof. From the above discussion, we know that $G_{2n}(k+1)$ is a tripartite graph with $k+1$ components that admits a local antimagic 3-coloring. The theorem holds. □

Example 4.2. Consider $n = 2$ and $k = 4$. We have the following table.

Table 9. Matrix for $n = 2, k = 4$

i	1	2	3	4	5	6	7	8	9
$f(u_i x_{i,1})$	78	79	80	81	73	74	75	76	77
$f(u_i x_{i,2})$	62	60	58	56	63	61	59	57	55
$f(u_i x_{i,3})$	42	43	44	45	37	38	39	40	41
$f(u_i x_{i,4})$	22	21	20	19	27	26	25	24	23
$f(u_i v_i)$	1	2	3	4	5	6	7	8	9
$f(v_i x_{i,1})$	14	15	16	17	18	10	11	12	13
$f(v_i x_{i,2})$	36	34	32	30	28	35	33	31	29
$f(v_i x_{i,3})$	50	51	52	53	54	46	47	48	49
$f(v_i x_{i,4})$	68	67	66	65	64	72	71	70	69

By the construction above Theorem 4.1, we have the graph $G_4(5)$ as shown below.

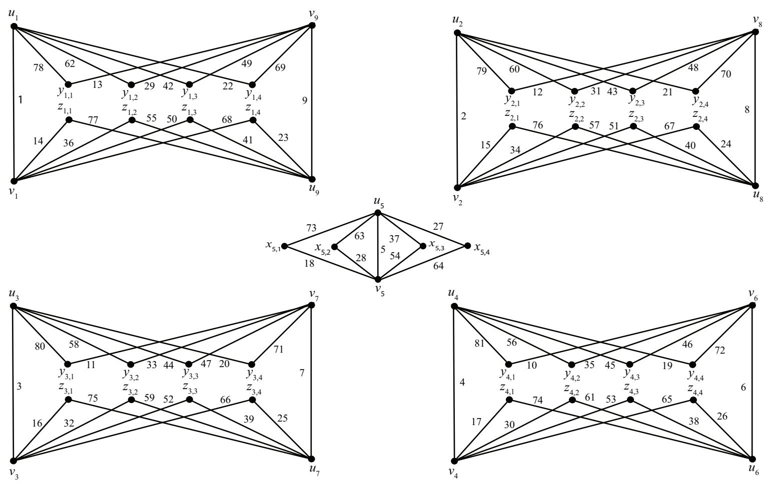

Fig. 9. Graph $G_4(5)$.

We may make use of Observation (d) to construct a new graph with local antimagic chromatic number 3 from $G_{2n}(k+1)$. Let us show an example first. Suppose $2k+1 = (2r+1)(2s+1)$, $r, s \geq 1$.

Example 4.3. Consider $n = 2$, $k = 4$ again. Now we have $r = s = 1$. Consider the graph $G = G_{2n}(k+1)$. Now $V(G) = \{u_i, v_i \mid 1 \leq i \leq 9\} \cup \{y_{i,j}, z_{i,j} \mid 1 \leq i \leq 4, 1 \leq j \leq 4\}$. From Observation (d) we have

$$\begin{aligned}
f^+(y_{1,j}) + f^+(y_{2,j}) + f^+(y_{3,j}) &= [f(u_1x_{1,j}) + f(v_9x_{9,j})] + [f(u_2x_{2,j}) + f(v_8x_{8,j})] \\
&\quad + [f(u_3x_{3,j}) + f(v_7x_{7,j})] = 273, \\
f^+(z_{1,j}) + f^+(z_{2,j}) + f^+(z_{3,j}) &= [f(v_1x_{1,j}) + f(u_9x_{9,j})] + [f(v_2x_{2,j}) + f(u_8x_{8,j})] \\
&\quad + [f(v_3x_{3,j}) + f(u_7x_{7,j})] = 273, \\
f^+(y_{4,j}) + f^+(x_{5,j}) + f^+(z_{4,j}) &= [f(u_4x_{1,j}) + f(v_6x_{2,j})] + [f(u_5x_{5,j}) + f(v_5x_{5,j})] \\
&\quad + [f(u_6x_{6,j}) + f(v_4x_{4,j})] = 273.
\end{aligned}$$

For each $j \in [1, 4]$, we (i) merge the vertices $y_{1,j}, y_{2,j}, y_{3,j}$ as a new vertex (still denote by $y_{1,j}$) of degree 6; (ii) merge the vertices $z_{1,j}, z_{2,j}, z_{3,j}$ as a new vertex (still denote by $z_{1,j}$) of degree 6; and (iii) merge $y_{4,j}, x_{5,j}, z_{4,j}$ (denote by $x_{5,j}$) of degree 6.

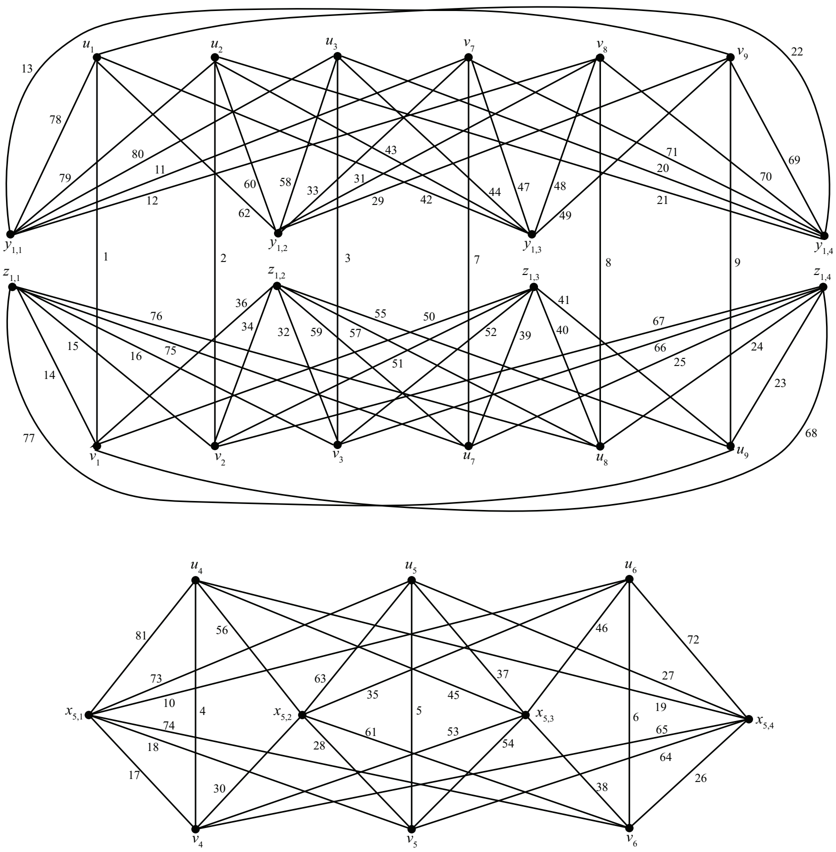

Fig. 10. Graph $G_4(3,3)$

Suppose $2k+1 = (2r+1)(2s+1)$, $r, s \geq 1$. Consider the graph $G_{2n}(k+1)$. For each $a \in [1, r]$ and $j \in [1, 2n]$, we can merge all $2s+1$ vertices in $\{y_{(a-1)(2s+1)+b,j} \mid b \in [1, 2s+1]\}$,

$\{z_{(a-1)(2s+1)+b,j} \mid b \in [1, 2s+1]\}$, and $\{x_{r(2s+1)+b,j} \mid b \in [1, 2s+1]\}$. The new vertices are denoted by $y_{(a-1)(2s+1)+1,j}$, $z_{(a-1)(2s+1)+1,j}$ and $x_{k+1,j}$, respectively. By Eqs. (1), (2) and (3), we have $f^+(y_{(a-1)(2s+1)+1,j}) = f^+(z_{(a-1)(2s+1)+1,j}) = f^+(x_{k+1,j}) = (2s+1)[n(8k+4)+4k+3]$. Let the graph just obtained be $G_{2n}(2r+1, 2s+1)$. Note that $G_{2n}(2r+1, 2s+1)$ has $r+1$ components.

Theorem 4.4. *For $n, r, s \geq 1$, we have $\chi_{la}(G_{2n}(2r+1, 2s+1)) = 3$.*

Proof. From the above discussion, we know that $2k+1 = (2r+1)(2s+1)$, $r, s \geq 1$ and $G_{2n}(2r+1, 2s+1)$ is a tripartite graph with $r+1$ components that admits a bijective edge labeling f with induced vertex labels $(1) = (2s+1)[n(8k+4)+4k+3]$, $(2) = 8kn^2+6kn+4n^2+k+4n+1$, and $(3) = 8kn^2+2kn+4n^2+k+2n+1$. Clearly, $(2) > (3)$. We now show that $(1) \neq (2), (3)$. Now,

$$\begin{aligned}(1)-(2) &= 16kns - 8kn^2 + 2kn + 8ks - 4n^2 + 8ns + 3k + 6s + 2\\ &= (8kn+4n+3)(2s-n) + 2kn + 8ks + 3k + 3n + 2\\ &> 0 \quad \text{if } 2s \geq n.\end{aligned}$$

Otherwise, $2s-n \leq -1$ (equivalently, $-n \leq -2s-1$), $(1)-(2) \leq -6kn-n-1+8ks+3k = -n(6k+1)-1+8ks+3k \leq (-2s-1)(6k+1)-1+8ks+3k = -4ks-3k-2s-2 < 0$. Thus, $(1) \neq (2)$. Similarly,

$$\begin{aligned}(1)-(3) &= 16kns - 8kn^2 + 6kn + 8ks - 4n^2 + 8ns + 3k + 2n + 6s + 2\\ &= (8kn+4n+3)(2s-n) + 6kn + 8ks + 3k + 5n + 2\\ &> 0 \quad \text{if } 2s \geq n.\end{aligned}$$

If $2s-n = -1$, $(1)-(3) = -2kn-n-1+8ks+3k = -n(2k+1)-1+8ks+3k = (-2s-1)(2k+1)-1+8ks+3k = 4ks+k-2s-2 > 0$ since $k \geq 4$. Otherwise, $2s-n \leq -2$ (equivalently, $-n \leq -2s-2$), $(1)-(3) \leq -10kn-3n-4+8ks+3k \leq (-2s-2)(10k+3)-4+8ks+3k < 0$. Thus, $(1) \neq (3)$. Therefore, f is a local antimagic 3-coloring. The theorem holds. □

5. New $(4n+3) \times (2k+1)$ matrix

In what follows, we refer to the following $(4n+3) \times (2k+1)$ matrix to obtain results similar to Theorems 4.1 and 4.4. For $1 \leq j \leq n$, we have

Table 10. Another $(4n+3)\times(2k+1)$ matrix

i	1	2	3	⋯	2k	2k+1
⋮	⋮	⋮	⋮	⋯	⋮	⋮
$f(u_ix_{i,2j-1})$	10k+5 + (2n-j)(4k+2)	10k+4 + (2n-j)(4k+2)	10k+3 (2n-j)(4k+2)	⋯	8k+6 + (2n-j)(4k+2)	8k+5 + (2n-j)(4k+2)
$f(u_ix_{i,2j})$	6k+4 + (2n-j)(4k+2)	6k+5 + (2n-j)(4k+2)	6k+6 + (2n-j)(4k+2)	⋯	8k+3 + (2n-j)(4k+2)	8k+4 + (2n-j)(4k+2)
⋮	⋮	⋮	⋮	⋯	⋮	⋮
$f(u_ix_{i,2n+1})$	2k+1 + (n+1)(4k+2)	2k+ (n+1)(4k+2)	(2k-1)+ (n+1)(4k+2)	⋯	2 + (n+1)(4k+2)	1 + (n+1)(4k+2)
$f(u_iv_i)$	1	2	3	⋯	2k	2k+1
$f(v_ix_{i,1})$	4k+2	4k+1	4k	⋯	2k+3	2k+2
⋮	⋮	⋮	⋮	⋯	⋮	⋮
$f(v_ix_{i,2j})$	4k+3 + (j-1)(4k+2)	4k+4 + (j-1)(4k+2)	4k+5 + (j-1)(4k+2)	⋯	6k+2 + (j-1)(4k+2)	6k+3 + (j-1)(4k+2)
$f(v_ix_{i,2j+1})$	8k + 4 + (j-1)(4k+2)	8k+3 + (j-1)(4k+2)	8k+2 + (j-1)(4k+2)	⋯	6k+5 + (j-1)(4k+2)	6k+4 + (j-1)(4k+2)
⋮	⋮	⋮	⋮	⋯	⋮	⋮

We now have the following observations.

(1) For each column, the sum of the first $2n+2$ entries is $f^+(u_i) = (n+1)(3n+1)(4k+2)+n+2k+2$.

(2) For each column, the sum of the last $2n+2$ entries is $f^+(v_i) = (n+1)^2(4k+2)+n+1$.

(3) For each $i\in[1,k]$ and $j\in[1,2n+1]$, each of $f(u_ix_{i,j})+f(v_{2k+2-i}x_{2k+2-i,j})$, $f(v_ix_{i,j})+f(u_{2k+2-i}x_{2k+2-i,j})$, and, $f(u_{k+1}x_{k+1,j})+f(v_{k+1}x_{k+1,j})$ is a constant $(2n+2)(4k+2)+1$.

(4) Suppose $2k+1=(2r+1)(2s+1)$, $r,s\ge 1$. For each $a\in[1,r]$ and $j\in[1,2n+1]$, each of

$$\sum_{b=1}^{2s+1}[f(u_{(a-1)(2s+1)+b}x_{(a-1)(2s+1)+b,j})+f(v_{2k+2-(a-1)(2s+1)-b}x_{2k+2-(a-1)(2s+1)-b,j})], \quad (4)$$

$$\sum_{b=1}^{2s+1}[f(v_{(a-1)(2s+1)+b}x_{(a-1)(2s+1)+b,j})+f(u_{2k+2-(a-1)(2s+1)-b}x_{2k+2-(a-1)(2s+1)-b,j})], \quad (5)$$

$$\sum_{b=1}^{2s+1}[f(u_{r(2s+1)+b}x_{r(2s+1)+b,j})+f(v_{2k+2-r(2s+1)-b}x_{2k+2-r(2s+1)-b,j})], \quad (6)$$

is a constant $(2s+1)[(2n+2)(4k+2)+1]$.

Similar to graph $G_{2n}(k+1)$ in Theorem 4.1, we also define $G_{2n+1}(k+1)$ of $k+1$ components similarly such that the i-th component has vertex set $\{u_i, v_i, u_{2k+2-i}, v_{2k+2-i}, y_{i,j}, z_{i,j} \mid 1\le j\le 2n+1\}$ and edge set $\{u_iv_i, u_{2k+2-i}v_{2k+2-i}, u_iy_{i,j}, v_{2k+2-i}y_{i,j}, v_iz_{i,j}, u_{2k+2-i}z_{i,j} \mid 1\le j\le 2n+1\}$ for $1\le i\le k$, and the $(k+1)$-st component is the $P_2\vee O_{2n+1}$ with vertex set $\{u_{k+1}, v_{k+1}, x_{k+1,j} \mid 1\le j\le 2n+1\}$ and edge set $\{u_{k+1}v_{k+1}, u_{k+1}x_{k+1,j}, v_{k+1}x_{k+1,j} \mid 1\le$

$j \le 2n+1\}$. Moreover, by Observation (3), $f^+(y_{i,j}) = f^+(z_{i,j}) = (2n+2)(4k+2)+1$. It is easy to verify that $f^+(u_i) \ne f^+(v_i) \ne f^+(y_{i,j})$ for all possible n, k.

Theorem 5.1. *For $n, k \ge 1$, $\chi_{la}(G_{2n+1}(k+1)) = 3$.*

Proof. From the discussion above, we know $G_{2n+1}(k+1)$ is a tripartite graph with $k+1$ components that admits a local antimagic 3-coloring. The theorem holds. $\square$

For $2k+1 = (2r+1)(2s+1), r, s \ge 1$, by Observation (4) above, we also define $G_{2n+1}(2r+1, 2s+1)$ as in Theorem 4.4 with $r+1$ components and similar vertex set with vertices $y_{(a-1)(2s+1)+1,j}$, $z_{(a-1)(2s+1)+1,j}$ and $x_{k+1,j}$ for $1 \le a \le 2r+1$, $1 \le j \le 2n+1$. By equations (4), (5) and (6), we have $f^+(y_{(a-1)(2s+1)+1,j}) = f^+(z_{(a-1)(2s+1)+1,j}) = f^+(x_{k+1,j}) = (2s+1)[(2n+2)(4k+2)+1]$.

Theorem 5.2. *For $n, r, s \ge 1$, we have $\chi_{la}(G_{2n+1}(2r+1, 2s+1)) = 3$.*

Proof. Similar to the proof of Theorem 4.4, we know $2k+1 = (2r+1)(2s+1)$, $r, s \ge 1$ and $G_{2n+1}(2r+1, 2s+1)$ is a tripartite graph with $r+1$ components that admits a bijective edge labeling f with induced vertex labels $(1) = (2s+1)[(2n+2)(4k+2)+1]$, $(2) = (n+1)(2n+1)(4k+2)+n+2k+2$ and $(3) = (n+1)^2(4k+2)+n+1$. Clearly, $(2) > (3)$. We now show that $(1) \ne (2), (3)$.

Now,

$$\begin{aligned}(1)-(2) &= -8kn^2 + 16kns - 4kn + 16ks - 4n^2 + 8ns + 2k - 3n + 10s + 1\\ &= (8kn + 4n + 4k + 5)(2s - n) + 2n + 8ks + 2k + 1\\ &> 0 \quad \text{if } 2s \ge n.\end{aligned}$$

If $2s-n \le -1$, $(1)-(2) \le -8kn-2n-2k-4+8ks \le (-2s-1)(8k+2)-2k-4+8ks < 0$. Thus, $(1) \ne (2)$. Similarly,

$$\begin{aligned}(1)-(3) &= -4kn^2 + 16kns + 16ks - 2n^2 + 8ns + 4k - n + 10s + 2\\ &= (4kn + 2n + 2)(4s - n) + n + 16ks + 2s + 4k + 2\\ &> 0 \quad \text{if } 4s \ge n.\end{aligned}$$

If $4s-n \le -1$, $(1)-(3) \le -4kn-n+16ks+2s+4k \le (-4s-1)(4k+1)+16ks+2s+4k = -2s-1 < 0$. Thus, $(1) \ne (3)$. Therefore, f is a local antimagic 3-coloring. The theorem holds. $\square$

Example 5.3. Take $n = 2$, $k = 4$, we have the following table and graph $G_5(5)$ with the defined labeling.

Table 11. Matrix for $n = 2, k = 4$

i	1	2	3	4	5	6	7	8	9
$f(u_i x_{i,1})$	99	98	97	96	95	94	93	92	91
$f(u_i x_{i,2})$	82	83	84	85	86	87	88	89	90
$f(u_i x_{i,3})$	81	80	79	78	77	76	75	74	73
$f(u_i x_{i,4})$	64	65	66	67	68	69	70	71	72
$f(u_i x_{i,5})$	63	62	61	60	59	58	57	56	55
$f(u_i v_i)$	1	2	3	4	5	6	7	8	9
$f(v_i x_{i,1})$	18	17	16	15	14	13	12	11	10
$f(v_i x_{i,2})$	19	20	21	22	23	24	25	26	27
$f(v_i x_{i,3})$	36	35	34	33	32	31	30	29	28
$f(v_i x_{i,4})$	37	38	39	40	41	42	43	44	45
$f(v_i x_{i,5})$	54	53	52	51	50	49	48	47	46

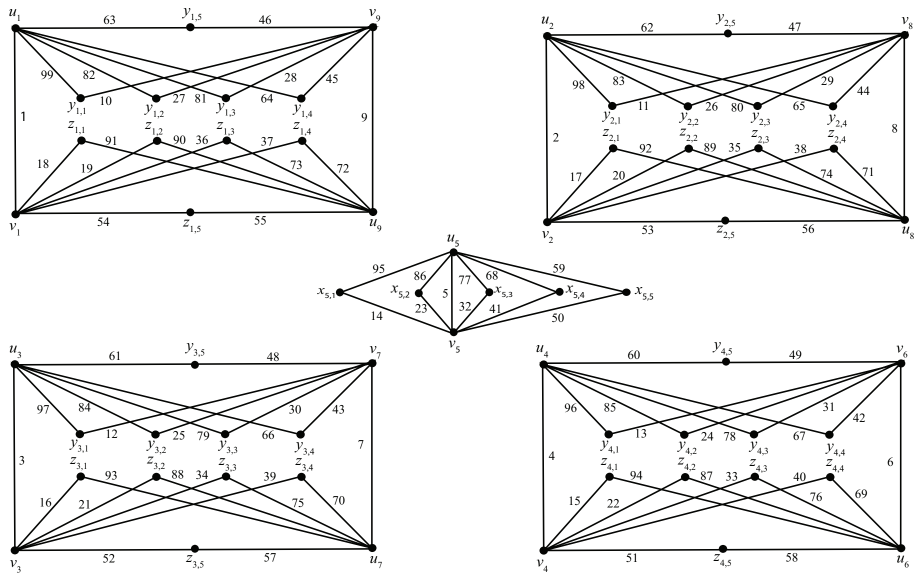

Fig. 11. Graph $G_5(5)$

If we take $r = s = 1$, we can get $G_5(3,3)$ which is a 6-regular graph.

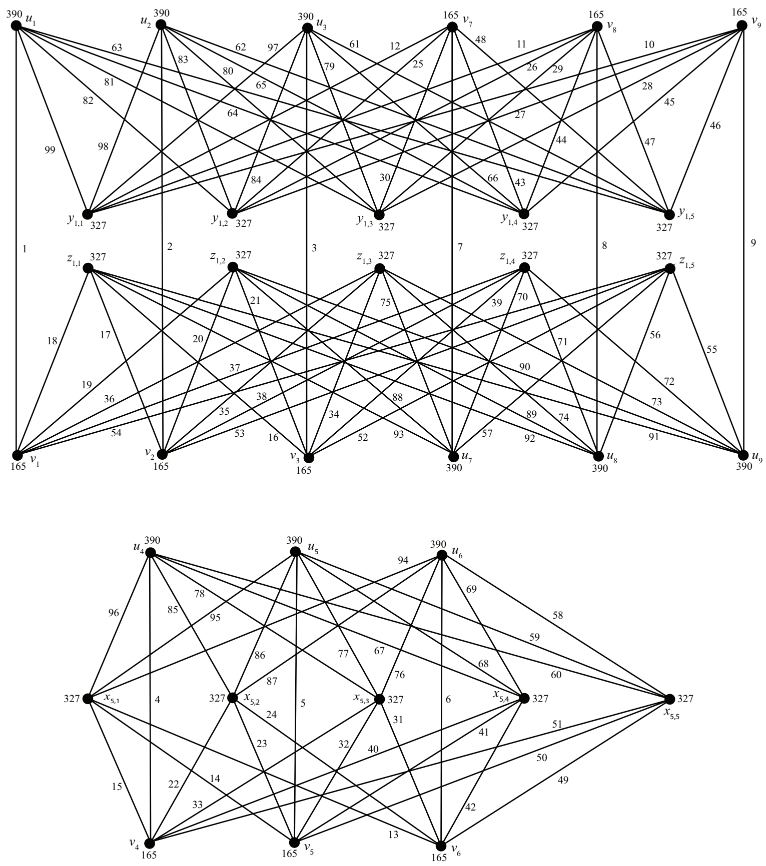

Fig. 12. $G_5(3,3)$ is a 6-regular tripartite graph

Note that we may also apply the delete-add process that gives us Theorem 2.9 to the graphs $G_{2n}(2r+1,2s+1)$ and $G_{2n+1}(2r+1,2s+1)$ to obtain two new families of (possibly connected or regular) tripartite graphs with local antimagic chromatic number 3. Denote the respective families of graph as $\mathcal{R}_{2n}(2r+1,2s+1)$ and $\mathcal{R}_{2n+1}(2r+1,2s+1)$. For example, from graph $G_4(3,3)$, we may remove the edges $v_9y_{1,1}$, $u_1y_{1,1}$ with labels $13, 78$ and $u_4x_{5,1}, u_6x_{5,1}$ with labels $81, 10$ respectively; and add the edges $v_9x_{5,1}$ with label 13, $u_1x_{5,1}$ with label 78, $u_4y_{1,1}$ with label 81, and $u_6y_{1,1}$ with label 10. The new graph is in $\mathcal{R}_4(3,3)$ and is connected. If we apply this process to $G_5(3,3)$ involving the edges with labels $99, 10$ and $96, 13$ respectively, we get a connected 6-regular graph is in $\mathcal{R}_5(3,3)$. Thus, we have the following corollary with the proof omitted.

Corollary 5.4. *For $n, r, s \geq 1$, if $n = 2s$, $\mathcal{R}_{2n+1}(2r+1,2s+1)$ is a family of (possibly connected) $(2n+2)$-regular tripartite graphs with local antimagic chromatic number* 3.

6. Conclusions and Discussion

In this paper, we obtained $\chi_{la}((2k+1)P_2 \vee O_m) = 3$ for all $k \geq 1, m \geq 2$. The local antimagic chromatic number of many other families of tripartite graphs are also obtained. As a natural extension, in [11], we make use of matrices of size $(2m+1) \times 2k$ to show that $\chi_{la}((2k)P_2 \vee O_m) = 3$ for all $k \geq 1, m \geq 2$. Similarly, we also obtain the local antimagic chromatic number of many families of bipartite and tripartite graphs. Interested readers may refer to [10] for new families of (regular) tripartite graphs having same size as $(2k+1)P_2 \vee O_{2n+1}$ $(k, n \geq 1)$, with local antimagic chromatic number 3.

Declarations

The authors declare no conflict of interest.

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