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Some series of tactical decomposable regular group divisible designs

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ABSTRACT

Some methods of constructions of square tactical decomposable regular group divisible designs are described. These designs are useful in threshold schemes. An L_2 design is also identified as square tactical decomposable. This completes spectrum of the solutions of entire L_2 designs listed in Clatworthy [2] using matrix approaches.

Keywords: tactical decomposable design, regular group divisible design, L_2 design, resolvability, threshold scheme

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1. Introduction

1.1. *Tactical decomposable design*

A block design $D(v, b, r, k)$ is an arrangement of a set X of v elements into b subsets (blocks) each of size k such that each element of X appears r times. The parameters satisfy the relation $bk = vr$.

Let a (0, 1) – matrix N have a decomposition

$$N = [N_{ij}] \begin{matrix} i = 1, 2, \ldots, s \\ j = 1, 2, \ldots, t \end{matrix}$$

where N_{ij} are submatrices of N of suitable sizes. The decomposition is called row tactical if row sum of N_{ij} is r_{ij} and column tactical if the column sum of N_{ij} is k_{ij} and tactical if it is row as well as column tactical. If N is the incidence matrix of a block design D, then D is called row (column) tactical decomposable design. D is called uniform row (column)

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tactical decomposable if $r_{ij} = \alpha\,(k_{ij} = \beta)$, for all i, j. Further if each N_{ij} is an $m \times m$ matrix, D is called square tactical decomposable design, STD (m) [1, 10].

Interested readers can find several recent constructions of tactical decomposable designs in [8, 9, 7, 10, 5].

We now introduce the notation used in this paper. I_n denotes the identity matrix of order n, and $0_{m\times n}$ the null matrix of order $m \times n$. The juxtaposition $[A \mid B]$ represents the matrix obtained by placing matrices A and B side by side. The vector e_n is the $n \times 1$ column vector with all entries equal to 1, and $J_{v\times b}$ is the $v \times b$ matrix with all entries equal to 1 (in particular, $J_{v\times v} = J_v$). For a matrix N, we write N^T for its transpose.

The matrix $\alpha = \mathrm{circ}(0\;1\;0\;0\ldots0)$ is a permutation circulant matrix of order n such that $\alpha^n = I_n$. A square block matrix is called **block–circulant** if each block is a circulant matrix of smaller order. The RX and LSX numbers are taken from Clatworthy [2].

Further if $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is a $p \times q$ matrix, then their Kronecker product $A \otimes B$ is the $mp \times nq$ block matrix:

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

1.2. Regular group divisible design

Let $v = mn$ elements be arranged in an $m \times n$ array. A *group divisible* (GD) *design* is an arrangement of the $v = mn$ elements in b blocks each of size k such that:

1. Every element occurs at most once in a block;
2. Every element occurs in r blocks;
3. Every pair of elements, which are in the same row of the $m \times n$ array, occur together in λ_1 blocks whereas remaining pair of elements occur together in λ_2 blocks.

If $r - \lambda_1 > 0$, $rk - v\lambda_2 > 0$, the GD design is regular (R). Further let N be $v \times b$ incidence matrix of a block design such that $J_v N = kJ_{v\times b}$ and satisfies the following conditions (i) or (ii):

(i)
$$NN^T = (r - \lambda_1)\,(I_m \otimes I_n) + (\lambda_1 - \lambda_2)\,(I_m \otimes J_n) + \lambda_2\,(J_m \otimes J_n)\,. \tag{1}$$

Let R_i and R_j be any two rows of blocks of N. Then from (1), their inner product is

$$R_i \bullet R_j = \begin{cases} rI_n + \lambda_1(J-I)_n; i = j, \\ \lambda_2 J_n; i \neq j, \end{cases} = \begin{cases} (r-\lambda_1)I_n + \lambda_1 J_n; i = j, \\ \lambda_2 J_n; i \neq j. \end{cases}$$

(ii)
$$NN^T = (r - \lambda_2)\,(I_n \otimes I_m) + \lambda_2\,(J_n \otimes J_m) + (\lambda_1 - \lambda_2)\,\{(J_n - I_n) \otimes I_m\}\,. \tag{2}$$

Then (2) implies,

$$R_i \bullet R_j = \begin{cases} rI_m + \lambda_2(J-I)_m; i = j, \\ \lambda_1 I_m + \lambda_2(J-I)_m; i \neq j, \end{cases} = \begin{cases} (r-\lambda_2)I_m + \lambda_2 J_m; i = j, \\ (\lambda_1 - \lambda_2)I_m + \lambda_2 J_m; i \neq j. \end{cases}$$

Then N represents a GD design with parameters: $v = mn, r, k, b, \lambda_1, \lambda_2, m, n$. For GD association schemes, we refer to Clatworthy [5]. A GD design is called STD(n) or STD(m)

with orthogonal rows if its incidence matrix satisfies the conditions (i) or (ii) respectively. In some cases it may happen that $m = n$.

Example 1.1. Consider a RGD design with parameters: $v = 8, r = 10, k = 2, b = 40, \lambda_1 = 2, \lambda_2 = 1, m = 2, n = 4$, which is a STD($n = 4$), with a pair of orthogonal rows. This design is listed as R32 in Clatworthy [2] whose incidence matrix N is given below.

$$N = \left(\begin{array}{cc|cc|c|ccc} M & 0_{4\times6} & M & 0_{4\times6} & I_4 & I_4 & I_4 & I_4 \\ 0_{4\times6} & M & 0_{4\times6} & M & I_4 & \alpha & \alpha^2 & \alpha^3 \end{array} \right),$$

where M represents a balanced incomplete block design with parameters: $v' = 4, b' = 6, r' = 3, k' = 2, \lambda = 1$ and $\alpha = circ\ (0\ 1\ 0\ 0)$ is a permutation circulant matrix of order four [see Clatworthy [5]].

Example 1.2. Consider a RGD design with parameters: $v = 22, r = 10, k = 5, b = 44, \lambda_1 = 0, \lambda_2 = 2, m = 11, n = 2$, which is a STD($m = 11$), with a pair of orthogonal rows. This design is listed in Freeman [3] whose incidence matrix N is given below.

$$N = \begin{pmatrix} \alpha + \alpha^2 + \alpha^3 + \alpha^6 & \alpha^9 & \alpha + \alpha^3 + \alpha^8 & \alpha^2 + \alpha^{10} \\ \alpha^9 & \alpha + \alpha^2 + \alpha^3 + \alpha^6 & \alpha^2 + \alpha^{10} & \alpha + \alpha^3 + \alpha^8 \end{pmatrix},$$

where $\alpha = circ\,(0\ 1\ 0\ 0 \ldots 0)$ is permutation circulant matrix of order 11.

1.3. L_2 design

Let $v = n^2$elements be arranged in an $n \times n$ array. An L_2 design is an arrangement of the $v = n^2$ elements in b blocks each of size k such that:

1. Every element occurs at most once in a block;
2. Every element occurs in r blocks;
3. Every pair of elements, which are in the same row or in the same column of the $n \times n$ array, occur together in λ_1 blocks whereas every other pair of elements occur together in λ_2 blocks.

The non–negative integers $v = n^2, b, r, k, \lambda_1$and λ_2 are known as parameters of the L_2 design and they satisfy the relations: $bk = vr; 2\,(n-1)\,\lambda_1 + (n-1)^2\lambda_2 = r\,(k-1)\,.$

1.4. $\mu-$resolvable design

A block design $D(v, b, r, k)$ whose b blocks can be divided into $t = r/\mu$ classes, each of size $\beta = v\mu/k$ and such that in each class of β blocks every element of D is replicated μ times, is called an $\mu-$ resolvable design. If $\mu = 1$ then the design is said to be resolvable.

Alternatively, if the incidence matrix N of a block design $D(v, b, r, k)$ may be partitioned in to submatrices as: $N = (N_1 \mid N_2 \mid \cdots \mid N_t)$ where each $N_i(1 \le i \le t)$ is a $v \times (v\mu/k)$ matrix such that each row sum of N_i is μ then the design is $\mu-$resolvable.

1.5. $P-$matrix

A $P-$matrix $P_{i_1 i_2 \ldots . i_n}$ introduced by Saurabh and Prasad [7], is an $n \times n$ matrix whose first row is the $i_1 -$th row of I_n, second row is the $i_2 -$th row of I_n and so on up to $n-$th row where $i_1, i_2, \ldots, i_n \in A = \{1, 2, \ldots, n\}$, where the symbols $i_1, i_2, \ldots, i_n$ may or may not be distinct.

Further if all $i'_k s (1 \le k \le n)$ are distinct then $P_{i_1 i_2 \ldots i_n}$ is a permutation as well as an orthogonal matrix and hence $P_{i_1 i_2 \ldots i_n} {P_{i_1 i_2 \ldots i_n}}^T = I_n$.

Example 1.3. $P_{3134} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Clearly first row of P_{3134} is third row of I_4, second row is first row of I_4, third and fourth rows are same as of I_4.

Example 1.4. $P_{1423} {P_{1423}}^T = I_4$.

$$P_{1214} {P_{1314}}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here, some series of STD regular group divisible designs are obtained. An L_2 design is also identified as tactical decomposable. This completes spectrum of the solutions of entire L_2 designs listed in Clatworthy [2] using matrix approaches. Such designs are useful as $(2,\ n)-$threshold schemes [see Saurabh [6]].

2. The constructions

Let $N = (N_{ij})_{\substack{1 \le i \le s \\ 1 \le j \le t}}$ be a block matrix where each N_{ij} is a $P-$matrix. Then N is the incidence matrix of a GD design if its rows of blocks satisfy either condition (i) or (ii) of Subsection 1.2.

The construction methods of RGD designs using $P-$matrices and permutation circulant matrices are described below:

Theorem 2.1. *There exist resolvable STD(5) RGD designs with parameters:*

$$(a) \qquad v = 10, r = 8s + t, k = 2, b = 5(8s + t), \lambda_1 = t, \lambda_2 = s, m = 5, n = 2, \qquad (3)$$

$$(b) \qquad v = 15, r = 6s + t, k = 3, b = 5(4s + t), \lambda_1 = t, \lambda_2 = s, m = 5, n = 3, \qquad (4)$$

where $s \ge 1, t \ge 0$.

Proof. (a) Consider the following block matrix:

$$M_1 = \begin{pmatrix} P_{11234} & P_{12133} & P_{12314} & P_{12341} & P_{12334} & P_{12234} & P_{12323} & P_{12342} \\ P_{53425} & P_{24545} & P_{34255} & P_{45523} & P_{41552} & P_{55143} & P_{45415} & P_{53451} \end{pmatrix}.$$

Then we have $M_1 M_1^T = 7\,(I_2 \otimes I_5) + (J_2 \otimes J_5) - \{(J_2 - I_2) \otimes I_5\}$ and each column sum of M_1 is 2. Hence M_1 represents a RGD design R36 with parameters: $v = 10, b = 40, r = 8, k = 2, \lambda_1 = 0, \lambda_2 = 1, m = 5, n = 2$.

Let N_1 be a block matrix obtained by taking s copies of M_1 i. e. $(\underbrace{M_1\ M_1 \ldots M_1}_{s\ times})$ and N_2 be the block matrix obtained by taking t copies of the block matrix: $e_2 \otimes I_5 = \begin{pmatrix} I_5 \\ I_5 \end{pmatrix}$, arranged columnwise. Then $N = [N_1 \mid N_2]$ represents a RGD design with generalized parameters (3) which may be verified. Further since each row sum of the columns of blocks of N is one, the design is resolvable.

(b) Consider the following block matrix:

$$M_1 = \begin{pmatrix} P_{11133} & P_{13525} & P_{12451} & P_{14314} & P_{14245} & P_{14552} \\ P_{22244} & P_{21453} & P_{23535} & P_{25425} & P_{25112} & P_{25341} \\ P_{34555} & P_{34241} & P_{34124} & P_{31532} & P_{33354} & P_{32413} \end{pmatrix}.$$

Then we have $M_1 M_1^T = 5\,(I_3 \otimes I_5) + (J_3 \otimes J_5) - \{(J_3 - I_3) \otimes I_5\}$ and each column sum of M_1 is 3. Hence M_1 represents RGD design R81 with parameters: $v = 15, b = 30, r = 6, k = 3, \lambda_1 = 0, \lambda_2 = 1, m = 5, n = 3$.

Let N_1 be a block matrix obtained by taking s copies of M_1 and N_2 be the block matrix obtained by taking t copies of the block matrix: $e_3 \otimes I_5 = \begin{pmatrix} I_5 \\ I_5 \\ I_5 \end{pmatrix}$, arranged columnwise. Then $N = [N_1 \mid N_2]$ represents a RGD design with generalized parameters (4) which may be verified. Further since each row sum of the columns of blocks of N is one, the design is resolvable.

A Table 1 of tactical decomposable RGD designs obtained using above results is given below:

Table 1. Resolvable RGD designs under $r, k \leq 10$

No.	$RX : (v, r, k, b, \lambda_1, \lambda_2, m, n)$	(s, t)
1	$R37 : (10,\ 10,\ 2,\ 50,\ 2,\ 1,\ 5,\ 2)$	$(1,\ 2)$
2	$R82 : (15,\ 8,\ 3,\ 40,\ 2,\ 1,\ 5,\ 3)$	$(1,\ 2)$
3	$R84 : (15,\ 9,\ 3,\ 45,\ 3,\ 1,\ 5,\ 3)$	$(1,\ 3)$
4	$R85 : (15,\ 10,\ 3,\ 50,\ 4,\ 1,\ 5,\ 3)$	$(1,\ 4)$

□

Theorem 2.2. *There exists a $(n+1)$ −resolvable STD(5) RGD design with parameters:*

$$v = 5n,\ \ r = 2\,(n+1)\,, k = n+1, b = 10n, \lambda_1 = 2n, \lambda_2 = 1, m = 5, n\ \ (>1)\,. \quad (5)$$

Proof. Let $\alpha = circ\ (0\ 1\ 0\ 0\ 0)$ be a permutation circulant matrix of order 5. Consider the following block matrix:

$$N = (N_1 \mid N_2) = \left(I_n \otimes \left(\alpha + \alpha^2\right) + (J-I)_n \otimes \alpha \;\middle|\; I_n \otimes \left(\alpha + \alpha^3\right) + (J-I)_n \otimes \alpha\right).$$

Then we have $NN^T = (2n+1)(I_n \otimes I_5) + (J_n \otimes J_5) + (2n-1)\{(J_n - I_n) \otimes I_5\}$ and each column sum of N is $n+1$. Hence N represents a RGD design with parameters (5). Further since each row sum of N_1 and N_2 is $(n+1)$, the design is $(n+1)$ −resolvable. □

Remark 2.3. The series (3) may also be found in Raghavarao and Padgett [4]. Here the plug–in matrices are $\alpha, \alpha+\alpha^2$ and $\alpha+\alpha^3$ whereas the plug–in matrices used in Raghavarao and Padgett (2005) are $0_n, I_n$ and J_n. For $t = 2,\ 3$ and 4 in Theorem 2.2, we obtain RGD designs R69 (3–resolvable), R115 (4–resolvable) and R152 (5–resolvable) respectively.

Some STD (m) RGD designs are displayed below where $\alpha = circ\ (0\ 1\ 0 \ldots 0)$ is a permutation circulant matrix of order m and N represents the incidence matrix.

1. $R50 : v = 6, r = 9, k = 3, b = 18, \lambda_1 = 6, \lambda_2 = 3, m = 3, n = 2$ (3−resolvable).

$$N = \left(\begin{array}{cc|cc|cc} I_3 & I_3+\alpha & I_3 & I_3+\alpha^2 & I_3 & I_3+\alpha \\ I_3+\alpha & I_3 & I_3+\alpha^2 & I_3 & I_3+\alpha & I_3 \end{array}\right).$$

2. $R56 : v = 8, r = 9, k = 3, b = 24, \lambda_1 = 6, \lambda_2 = 2, m = 4, n = 2$ (3−resolvable).

$$N = \left(\begin{array}{cc|cc|cc} I_4 & I_4+\alpha & I_4 & I_4+\alpha^2 & \alpha & \alpha+\alpha^2 \\ I_4+\alpha & I_4 & I_4+\alpha^2 & I_4 & \alpha+\alpha^2 & \alpha \end{array}\right).$$

3. $R80 : v = 14, r = 9, k = 3, b = 42, \lambda_1 = 6, \lambda_2 = 1, m = 7, n = 2$ (3−resolvable).

$$N = \left(\begin{array}{cc|cc|cc} \alpha+\alpha^2 & \alpha & \alpha+\alpha^3 & \alpha & \alpha+\alpha^4 & \alpha \\ \alpha & \alpha+\alpha^2 & \alpha & \alpha+\alpha^3 & \alpha & \alpha+\alpha^4 \end{array}\right).$$

4. $R89 : v = 18, r = 9, k = 3, b = 54, \lambda_1 = 2, \lambda_2 = 1, m = 9, n = 2.$

$$N = \left(\begin{array}{cc|cccc} \alpha & \alpha+\alpha^3 & \alpha & \alpha & \alpha & \alpha+\alpha^2+\alpha^5 \\ \alpha^7+\alpha^8 & \alpha^3 & \alpha^2+\alpha^4 & \alpha+\alpha^5 & \alpha^6+I_9 & 0_9 \end{array}\right).$$

5. $R96 : v = 6, r = 8, k = 4, b = 12, \lambda_1 = 4, \lambda_2 = 5, m = 3, n = 2$ (4−resolvable).

$$N = \left(\begin{array}{cc|cc} I_3 & J_3 & I_3+\alpha & I_3+\alpha^2 \\ J_3 & I_3 & I_3+\alpha^2 & I_3+\alpha \end{array}\right).$$

6. $R117 : v = 15, r = 8, k = 4, b = 30, \lambda_1 = 1, \lambda_2 = 2, m = 3, n = 5$ (4−resolvable).

$$N = \left[circ\ (\alpha+\alpha^4,\ I_5, I_5)\ \middle|\ circ\ (\alpha^2+\alpha^3,\ I_5, I_5) \right],$$

where $\alpha = circ\ (0\ 1\ 0\ 0\ 0)$.

Clearly $circ\ (\alpha+\alpha^4,\ I_5, I_5)$ and $circ\ (\alpha^2+\alpha^3,\ I_5, I_5)$ are block–circulant matrices of order 15.

3. A tactical decomposable L_2 design

Saurabh and Sinha [8] obtained some series of tactical decomposable L_2 designs which yielded solutions of all designs listed in Clatworthy [2] except LS4: $v = 16,\ r = 9,\ k = 2,\ b = 72,\ \ n_1 = 6, n_2 = 9, \lambda_1 = 0, \lambda_2 = 1$. The aim was to plug−in suitable $n \times t$ matrices in certain combinatorial structures so that the resultant matrix becomes incidence matrix of the desired L_2 design. Here LS4 is identified as a tactical decomposable design whose solution is obtained using a permutation circulant matrix of order four.

Let $\alpha = circ\ (0\ 1\ 0\ 0)$ and 0 be a null matrix of order four. Consider the following $(0,\ 1)$ −block matrix $N = (N_1 \mid N_2 \mid N_3 \mid N_4 \mid N_5 \mid N_6 \mid N_7 \mid N_8 \mid N_9)$, where:

$$N_1 = \begin{pmatrix} \alpha & 0 \\ \alpha^2 & 0 \\ 0 & \alpha^3 \\ 0 & I_4 \end{pmatrix}, N_2 = \begin{pmatrix} \alpha^2 & 0 \\ \alpha & 0 \\ 0 & I_4 \\ 0 & \alpha^3 \end{pmatrix}, N_3 = \begin{pmatrix} \alpha & 0 \\ 0 & I_4 \\ \alpha^2 & 0 \\ 0 & \alpha^3 \end{pmatrix},$$
$$N_4 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^3 \\ 0 & \alpha \\ \alpha^2 & 0 \end{pmatrix}, N_5 = \begin{pmatrix} 0 & \alpha \\ \alpha^3 & 0 \\ 0 & I_4 \\ I_4 & 0 \end{pmatrix}, N_6 = \begin{pmatrix} \alpha & 0 \\ \alpha^3 & 0 \\ 0 & I_4 \\ 0 & \alpha^2 \end{pmatrix},$$
$$N_7 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^2 \\ \alpha^3 & 0 \\ 0 & I_4 \end{pmatrix}, N_8 = \begin{pmatrix} \alpha & 0 \\ 0 & I_4 \\ 0 & \alpha \\ \alpha^3 & 0 \end{pmatrix}, N_9 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^3 \\ 0 & \alpha^2 \\ I_4 & 0 \end{pmatrix}.$$

Then clearly each column sum of N is two and $NN^T = 9\,(I_4 \otimes I_4) + (J_4 - I_4) \otimes (J_4 - I_4)$. Hence N represents the incidence matrix of LS4. Further since each row sum of block matrices $N_i (1 \le i \le 9)$ is one, the design is resolvable.

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