CONGRESSUS NUMERANTIUM 236 (2025) 115–122

Congressus Numerantium
www.combinatorialpress.com/cn

Combinatorial Press

The star-critical Gallai-Ramsey number $gr_*(F_{3,2}, K_3, K_3)$ and related critical colorings

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ABSTRACT

Recently, it was shown that the Gallai-Ramsey number satisfies $gr(F_{3,2}, K_3, K_3) = 31$, where $F_{3,2}$ is the generalized fan $F_{3,2} := K_1 + 2K_3$. In this paper, we show that the star-critical Gallai-Ramsey number satisfies $gr_*(F_{3,2}, K_3, K_3) = 27$. We also prove that the critical colorings for $r_*(K_3, K_3)$, $gr(F_{3,2}, K_3, K_3)$, and $gr_*(F_{3,2}, K_3, K_3)$ are unique.

Keywords: Ramsey number, Gallai coloring, generalized fan

2020 Mathematics Subject Classification: Primary: 05C55, 05C15; Secondary: 05D10.

1. Introduction

A t-coloring of a graph G is a map $f : E(G) \longrightarrow \{1, 2, \ldots, t\}$. When f satisfies

$$|\{f(xy), f(xz), f(yz)\}| \leq 2, \qquad (1)$$

for all distinct $x, y, z \in V(G)$, it is called a *Gallai t-coloring*. For graphs $G_1, G_2, \ldots, G_t$, the *Ramsey number* $r(G_1, G_2, \ldots, G_t)$ is the least $p \in \mathbb{N}$ such that every t-coloring of K_p contains a subgraph that is isomorphic to G_i in color i, for some $i \in \{1, 2, \ldots, t\}$. A *critical coloring* for $r(G_1, G_2, \ldots, G_t)$ is a t-coloring of $K_{r(G_1,G_2,\ldots,G_t)-1}$ that lacks a monochromatic copy of G_i in color i, for all $i \in \{1, 2, \ldots, t\}$. If we replace "$t$-coloring" with "Gallai t-coloring" and "$r(G_1, G_2, \ldots, G_t)$" with "$gr(G_1, G_2, \ldots, G_t)$" in these definitions, we obtain the defintions of the *Gallai-Ramsey number* $gr(G_1, G_2, \ldots, G_t)$ and a *critical coloring* for $gr(G_1, G_2, \ldots, G_t)$.

Star-critical Ramsey numbers were first introduced by Hook and Isaak (see [9] and [10]) in 2010 and measure the number of edges that one must add between a vertex

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Received 09 Jun 2025; Revised 24 Nov 2025; Accepted 10 Dec 2025; Published Online 24 Dec 2025.

DOI: 10.61091/cn236-08

and a critical coloring for a Ramsey number in order for the Ramsey property to be established. To be precise, denote by $K_n \sqcup K_{1,k}$ the graph formed by taking the disjoint union of K_n and K_1 and joining them together using exactly k edges. The *star-critical Ramsey number* $r_*(G_1, G_2, \ldots, G_t)$ is then defined to be the least k such that every t-coloring of $K_{r(G_1,G_2,\ldots,G_t)-1} \sqcup K_{1,k}$ contains a monochromatic copy of G_i in color i, for some $i \in \{1, 2, \ldots, t\}$. Here,

$$0 \leq r_*(G_1, G_2, \ldots, G_t) \leq r(G_1, G_2, \ldots, G_t) - 1,$$

with $r_*(G_1, G_2, \ldots, G_t) \neq 0$ when every G_i lacks an isolated vertex. A *critical coloring* for $r_*(G_1, G_2, \ldots, G_t)$ is a t-coloring of

$$K_{r(G_1,G_2,\ldots,G_t)-1} \sqcup K_{1,r_*(G_1,G_2,\ldots,G_t)-1},$$

that lacks a monochromatic copy of G_i in color i, for all $i \in \{1, 2, \ldots, t\}$.

The concept of a star-critical Ramsey number was first extended to Gallai t-colorings by Su and Liu [12] in 2022. The *star-critical Gallai-Ramsey number* $gr_*(G_1, G_2, \ldots, G_t)$ is the least k such that every Gallai t-coloring of $K_{gr(G_1,G_2,\ldots,G_t)-1} \sqcup K_{1,k}$ contains a monochromatic copy of G_i in color i, for some $i \in \{1, 2, \ldots, t\}$. A *critical coloring* for $gr_*(G_1, G_2, \ldots, G_t)$ is a Gallai t-coloring of

$$K_{gr(G_1,G_2,\ldots,G_t)-1} \sqcup K_{1,gr_*(G_1,G_2,\ldots,G_t)-1},$$

that lacks a monochromatic copy of G_i in color i, for all $i \in \{1, 2, \ldots, t\}$. Note that when $t = 2$,

$$gr(G_1, G_2) = r(G_1, G_2) \quad \text{and} \quad gr_*(G_1, G_2) = r_*(G_1, G_2).$$

For $t \geq 2$ and $n \geq 2$, the *generalized fan* $F_{t,n}$ is defined by $F_{t,n} := K_1 + nK_t$, where $G_1 + G_2$ denotes the join of G_1 and G_2. It has order $tn + 1$ and minimum degree t.

The Gallai-Ramsey number $gr(F_{3,2}, K_3, K_3) = 31$ was recently determined in [2]. In this paper, we build upon this result, first showing that there exist unique critical colorings for both $r_*(K_3, K_3)$ and $gr(F_{3,2}, K_3, K_3)$ in Section 2. In Section 3, we turn our attention to the evaluation of $gr_*(F_{3,2}, K_3, K_3)$ and also prove that it has a unique critical coloring. Additional results involve general lower bounds for other related star-critical Gallai-Ramsey numbers and we offer a conjecture concerning the expected value of $gr_*(F_{t,n}, K_3, K_3, \ldots, K_3)$.

2. Critical colorings

When all of the arguments are complete graphs, it can be shown that

$$gr_*(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = gr(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) - 1, \tag{2}$$

(cf. [5] and Theorem 1.1 of [1]). It is well-known that the only critical coloring for $r(K_3, K_3) = 6$ is the 2-coloring of K_5 whose subgraphs spanned by edges in each color form a C_5 (i.e., a cycle of length 5). In the next theorem, we prove that $r_*(K_3, K_3) = 5$ has a unique critical coloring.

Theorem 2.1. *Up to isomorphism, the coloring shown in Figure 1 is the unique critical coloring for* $r_*(K_3, K_3)$.

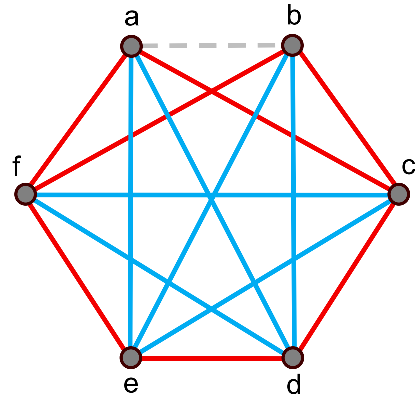

Fig. 1. The unique critical coloring for $r_*(K_3, K_3)$

Proof. Let $V = \{a, b, c, d, e, f\}$ be the vertex set for a 2-colored $K_5 \sqcup K_{1,4}$ in which ab is the "missing" edge and a monochromatic K_3-subgraph is avoided. Consider the subgraph induced by $V \setminus \{b\}$ and note that if any vertex is incident with at least three edges of the same color, then the vertices at the other ends of those edges induce a subgraph containing an edge of the same color, or all three edges are the other color. Either way, a monochromatic K_3-subgraph is formed. Thus, every vertex is incident with exactly two red edges and exactly two blue edges.

Without loss of generality, assume that ac and af are red and ad and ae are blue. Avoiding a monochromatic K_3-subgraph then requires cf to be blue and de to be red. Now c must join to one of d and e with a red edge and the other with a blue edge. Without loss of generality, suppose that cd is red and ce is blue. Then ef must be red and fd must be blue. So, $acdefa$ forms a red C_5 and $adfcea$ forms a blue C_5.

Now consider the possibility that bc is blue. Then bf and be must be red, and $\{b, e, f\}$ induces a red K_3. It follows that bc must be red, bd must be blue, bf must be red, and be must be blue, resulting in the coloring shown in Figure 1. □

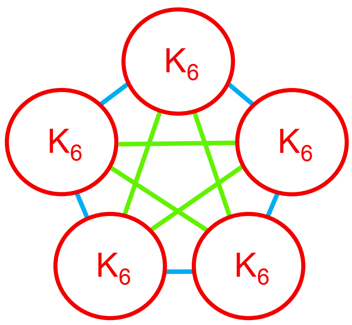

Fig. 2. A Gallai 3-coloring $\mathcal{G}$ of K_{30} that avoids a red $F_{3,2}$, a blue K_3, and a green K_3

When $t = 3$, we will represent colors 1, 2, and 3, by red, blue, and green, respectively.

The critical coloring for $gr(F_{3,2}, K_3, K_3) = 31$ that was given in [2] was formed by taking the unique critical coloring for $r(K_3, K_3)$ (a K_5 containing a blue C_5 and a green C_5) and replacing each of its vertices with a red K_6-subgraph (see Figure 2). We refer to this critical coloring for $gr(F_{3,2}, K_3, K_3)$ as $\mathcal{G}$.

Before we prove that $\mathcal{G}$ is unique, recall the well-known structure theorem for Gallai colorings, which reinterprets a classic result of Gallai [7].

Theorem 2.2 ([8]). *Every Gallai-colored complete graph can be formed by replacing the vertices of a 2-colored complete graph of order at least 2 with Gallai-colored complete graphs.*

The 2-colored complete graph in Theorem 2.2 is called the *base graph* for the Gallai coloring, and the Gallai-colored complete graphs that replace the vertices in the base graph are called the *blocks*.

Proving that $\mathcal{G}$ is unique will require us to consider the concept of a connected Ramsey number. A 2-coloring of a graph G is called a *connected* 2*-coloring* of G if the subgraphs spanned by the edges in each of the colors are connected. The *connected Ramsey number* $r_c(G_1, G_2)$ is the least $p \in \mathbb{N}$ such that every connected 2-coloring of K_p contains a subgraph isomorphic to G_i in color i, for some $i \in \{1, 2\}$ (see [13]). Since every connected 2-coloring of K_p is a 2-coloring of K_p, it follows that

$$r_c(G_1, G_2) \leq r(G_1, G_2),$$

for all graphs G_1 and G_2, with exceptions for the cases where $r(G_1, G_2) \leq 3$ (see [13]).

Theorem 2.3. *Up to isomorphism, $\mathcal{G}$ is the only critical coloring for $gr(F_{3,2}, K_3, K_3)$.*

Proof. It was shown in Theorem 3.6 of [2] that $gr(F_{3,2}, K_3, K_3) = 31$. We must show that, up to isomorphism, $\mathcal{G}$ is the only Gallai 3-coloring of K_{30} that avoids a red $F_{3,2}$, a blue K_3, and a green K_3. The proof of $gr(F_{3,2}, K_3, K_3) \leq 31$ given in [2] considered a Gallai 3-coloring of K_{31}, and then used Theorem 2.2 to break the proof into ten cases, based on the order of the base graph $\mathcal{B}$ for the Gallai coloring, which was chosen to have minimal order. In most of the cases, it was shown that avoiding a red $F_{3,2}$, a blue K_3, and a green K_3 resulted in a Gallai 3-coloring of a complete graph that contained at most 29 vertices. The only cases that allowed for up to 30 vertices were Subcase 3.2 ($|V(\mathcal{B})|= 5$ with the edges in $\mathcal{B}$ being blue and green) and Case 9 ($|V(\mathcal{B})|= 11$), so those are the only cases we must conisder here.

When $|V(\mathcal{B})|= 5$ and the edges in $\mathcal{B}$ are blue and green, then $\mathcal{B}$ must consist of a blue C_5 and a green C_5, corresponding with the unique critical coloring for $r(K_3, K_3)$. If any block contains a blue or green edge, then a blue or green K_3 can be formed. So, each block must be a red complete graph. If a block contains 7 or more vertices, then a red K_7 contains a red $F_{3,2}$ as a subgraph. It follows that each block is a red K_6, resulting in the critical coloring $\mathcal{G}$.

Now suppose that $|V(\mathcal{B})|= 11$. Note that $\mathcal{B}$ cannot be blue and green as $r(K_3, K_3) = 6$ would then force there to be a blue K_3 or a green K_3. So, assume that $\mathcal{B}$ is red and (without loss of generality) blue. Also, the red/blue coloring of $\mathcal{B}$ must be a connected 2-coloring, otherwise the Gallai partition can be reduced to a partition with a smaller order base graph, which has already been handled. Since $r_c(K_{1,6}, K_3) = 11$ (see [6] and [11]), some vertex v in $\mathcal{B}$ has red degree at least 6 and blue degree at most 4. Also note that v must be incident with at least one red and one blue edge since the 2-coloring of $\mathcal{B}$ is connected.

Since a blue K_3 is avoided, the vertices in $\mathcal{B}$ that are blue adjacent to v form a red complete subgraph. By Lemmas 3.2-3.5 of [2], the blocks corresponding to these vertices contain a total of at most 8 vertices. Since $r(2K_3, K_3, K_3) \leq 14$ (see Lemma 1 of [2]), the blocks correponding to the vertices that are red-adjacent to v contain a total of at most 13 vertices. Finally, Lemma 3.2 of [2] implies that the block corresponding to vertex v contains at most 7 vertices. In total, the complete graph has at most 29 vertices, contradicting the assumption that it was a K_{30}. Hence, this case does not occur.

Thus, the only critical coloring for $gr(F_{3,2}, K_3, K_3)$ is the construction that corresponds with $\mathcal{G}$. □

In the next section, we turn our attention to the evaluation of the star-critical Gallai-Ramsey number $gr_*(F_{3,2}, K_3, K_3)$.

3. The star-critical Gallai-Ramsey number $gr_*(F_{3,2}, K_3, K_3)$

In this section, we prove that $gr_*(F_{3,2}, K_3, K_3) = 27$ and that the critical coloring for $gr_*(F_{3,2}, K_3, K_3)$ is unique. First, we consider lower bounds for a more general class of Gallai-Ramsey numbers. In [2], it was noted that the Gallai-Ramsey number

$$gr^s(K_3) := gr(\underbrace{K_3, K_3, \ldots, K_3}_{s \text{ terms}}) = \begin{cases} 5^{s/2} + 1 & \text{if } s \text{ is even,} \\ 2 \cdot 5^{(s-1)/2} + 1 & \text{if } s \text{ is odd,} \end{cases}$$

for all $s \geq 1$ (due to Chung and Graham [4]), along with a general lower bound from [3], implies that

$$gr(F_{t,n}, \underbrace{K_3, K_3, \ldots, K_3}_{s \text{ terms}}) \geq \begin{cases} tn \cdot 5^{s/2} + 1 & \text{if } s \text{ is even,} \\ 2tn \cdot 5^{(s-1)/2} + 1 & \text{if } s \text{ is odd,} \end{cases} \quad (3)$$

for all $t \geq 2$ and $s \geq 1$. When this inequality gives the exact value of the Gallai-Ramsey number, we obtain the following lower bound for the corresponding star-critical Gallai-Ramsey number.

Theorem 3.1. *For any $t \geq 2$, $n \geq 2$, and $s \geq 1$ for which the inequality given in (3) is exact, the corresponding star-critical Gallai-Ramsey number satisfies*

$$gr_*(F_{t,n}, \underbrace{K_3, K_3, \ldots, K_3}_{s \text{ terms}}) \geq \begin{cases} tn(5^{s/2-1} - 1) + t & \text{if } s \text{ is even,} \\ tn(2 \cdot 5^{(t-1)/2} - 1) + t & \text{if } s \text{ is odd.} \end{cases}$$

Proof. By Eq. (2), There exists an s-coloring of $K_{gr^s(K_3)} - e$ that avoids a monochromatic copy of K_3 in colors $2, 3, \ldots, s+1$. Denote its vertex set by $\{x_1, x_2, \ldots, x_{gr^s(K_3)}\}$ and assume that x_1x_2 is the missing edge. For example, the $s = 2$ case is shown in Image (A) of Figure 3.

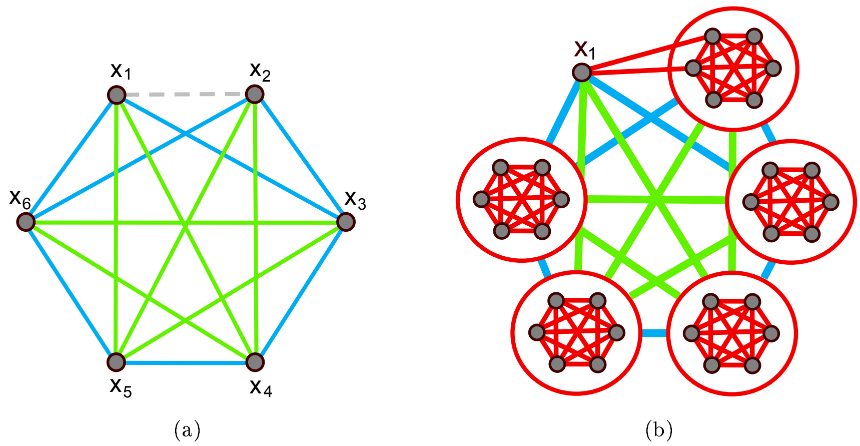

Fig. 3. The construction of a Gallai 3-coloring $\mathcal{G}'$ of $K_{30} \sqcup K_{1,26}$ that avoids a red $F_{3,2}$, a blue K_3, and a green K_3

Replace each of the vertices $x_2, x_3, \ldots, x_{gr^s(K_3)}$ with K_{tn}-subgraphs in color 1. Join exactly $t-1$ edges in color 1 from x_1 to the block that replaced x_2 (see Image (B) of Figure 3). Now, no $F_{t,n}$ exists in color 1 since the color 1 blocks do not contain enough vertices, and the color 1 degree of x_1 is not large enough for it to be included in an $F_{t,n}$. We have also avoided a monochromatic copy of K_3 in colors $2, 3, \ldots, s+1$. If s is even, then

$$gr_*(F_{3,2}, K_3, K_3) > tn5^{s/2} - tn + (t-1).$$

If s is odd, then

$$gr_*(F_{3,2}, K_3, K_3) > 2tn5^{(s-1)/2} - tn + (t-1),$$

resulting in the inequality given in the statement of the theorem. □

Theorem 3.2. *The star-critical Gallai-Ramsey number satisfies*

$$gr_*(F_{3,2}, K_3, K_3) = 27.$$

Proof. Theorem 3.1 (and in particular, Image (B) in Figure 3) implies that $gr_*(F_{3,2}, K_3, K_3) \geq 27$. To prove the reverse inequality, consider a Gallai 3-coloring of $K_{30} \sqcup K_{1,27}$ and let v be the vertex of degree 27. If we remove vertex v, and assume that the resulting K_{30} avoids a red $F_{3,2}$, a blue K_3, and a green K_3, then by Theorem 2.3, it is the unique critical coloring $\mathcal{G}$.

We can add at most two red edges from v to a single block in $\mathcal{G}$, as adding three red edges from v to a single block forces a red $F_{3,2}$. Since v joins to $\mathcal{G}$ with 27 edges, v must join to each block with at least three edges. If a red $F_{3,2}$ is avoided, then v joins to every block in $\mathcal{G}$ with either a blue or green edge. Suppose that vx_1, vx_2, vx_3, vx_4, and vx_5 are blue or green edges, where each x_i comes from a different block in $\mathcal{G}$. Then the subgraph induced by $\{v, x_1, x_2, x_3, x_4, x_5\}$ is a blue/green K_6. Since $r(K_3, K_3) = 6$, there is a blue K_3 or a green K_3. It follows that $r_*(F_{3,2}, K_3, K_3) \le 27$. □

Image (B) of Figure 3 provided the lower bound for $gr_*(F_{3,2}, K_3, K_3)$ in Theorem 3.2. We refer to the graph in this image as $\mathcal{G}'$. In the next theorem, we prove that $\mathcal{G}'$ is the unique critical coloring for the star-critical Gallai-Ramsey number $gr_*(F_{3,2}, K_3, K_3)$.

Theorem 3.3. *Up to isomorphism, $\mathcal{G}'$ is the only critical coloring for $gr_*(F_{3,2}, K_3, K_3)$.*

Proof. Consider a Gallai 3-coloring of $K_{30} \sqcup K_{1,26}$ that lacks a red $F_{3,2}$, a blue K_3, and a green K_3. Let v be the vertex of degree 26. If we remove v, then we have a Gallai 3-coloring of K_{30} that is a critical coloring for $gr(F_{3,2}, K_3, K_3)$. By Theorem 2.3, it is isomorphic to $\mathcal{G}$. If vx_1, vx_2, vx_3, vx_4, and vx_5 are all blue and green edges with each x_i coming from a different block in $\mathcal{G}$, then the subgraph induced by $\{v, x_1, x_2, x_3, x_4, x_5\}$ is a blue/green K_6. The Ramsey number $r(K_3, K_3) = 6$ then implies that there is a blue K_3 or a green K_3. So, some block can only join to v via red edges. Since joining v to a block with three or more red edges leads to a red $F_{3,2}$, it follows that exactly two red edges from v join to a single block (with the other four edges to that block being the missing edges) and v joins to all other blocks with six edges each.

Denote the block that joins to v with two red edges (and four missing edges) by Y and denote the other blocks in $\mathcal{G}$ by X_1, X_2, X_3, X_4. If v joins to any X_i with a red edge, then vx_ix_j is a rainbow triangle when $x_j \in V(X_j)$ and $i \neq j$. Select $y \in V(Y)$ such that vy is a missing edge and select $x_i \in V(X_i)$, for each $i \in \{1, 2, 3, 4\}$ such that vx_i is either blue or green. Then the subgraph induced by $\{v, y, x_1, x_2, x_3, x_4\}$ is a critical coloring for $r_*(K_3, K_3)$. By Theorem 2.1, vx_i receives the same color as edge yx_i. Thus, the critical coloring for $gr_*(F_{3,2}, K_3, K_3)$ must be isomorphic to $\mathcal{G}'$. □

4. Conclusion

In [2], it was conjectured that

$$gr(F_{t,n}, \underbrace{K_3, K_3, \ldots, K_3}_{s \text{ terms}}) = \begin{cases} tn \cdot 5^{s/2} + 1 & \text{if } s \text{ is even,} \\ 2tn \cdot 5^{(s-1)/2} + 1 & \text{if } s \text{ is odd.} \end{cases}$$

If this is true, then Theorems 3.1 and 3.2 support the following conjecture regarding the corresponding star-critical Gallai-Ramsey number.

Conjecture 4.1. *For all $t \ge 2$ and $s \ge 1$,*

$$gr_*(F_{t,n}, \underbrace{K_3, K_3, \ldots, K_3}_{s \text{ terms}}) = \begin{cases} tn(5^{s/2-1} - 1) + t & \text{if } s \text{ is even,} \\ tn(2 \cdot 5^{(t-1)/2} - 1) + t & \text{if } s \text{ is odd.} \end{cases}$$

The methods used in this paper and [2] may assist in proving specific cases of these two conjectures, but they do not appear to be strong enough to prove the conjectures in general.

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