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Equivalence of labeled graphs and lattices

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ABSTRACT

In 1973, Harary and Palmer posed the problem of enumeration of labeled graphs on $n \geq 1$ unisolated vertices and $l \geq 0$ edges. In 1997, Bender et al. obtained a recurrence relation representing the sequence $A054548$(OEIS) of labeled graphs on $n \geq 0$ unisolated vertices containing $q \geq \frac{n}{2}$ edges. In 2020, Bhavale and Waphare obtained a recurrence relation representing the sequence of fundamental basic blocks on $n \geq 0$ comparable reducible elements, having nullity $l \geq \lfloor \frac{n+1}{2} \rfloor$. In this paper, we prove the equivalence of these two sequences. We also provide an edge labeling for a given vertex labeled finite simple graph.

Keywords: chain, lattice, digraph, labeled graph

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1. Introduction

In 1955, Harary [6] obtained the number of non-isomorphic linear and connected graphs with p points and k lines. In 1956, Gilbert [4] enumerated connected labeled graphs. In 1973, Harary and Palmer [7] obtained the number of ways to label a graph, and also posed the problem of enumeration of labeled graphs on $n \geq 1$ unisolated vertices and $l \geq 0$ edges. In 1997, Bender et al. [1] obtained asymptotic number of labeled graphs on n unisolated vertices with q edges. Bender et al. [1] also obtained the recurrence relation representing the number $d(n,q)$ of labeled graphs on n unisolated vertices, containing q edges as $qd(n,q) = (N-q+1)d(n,q-1) + n(n-1)d(n-1,q-1) + Nd(n-2,q-1)$ with boundary conditions $d(0,0) = 1$, and for non-zero values of n and q, $d(0,q) = d(n,0) = 0$, where $\frac{n}{2} \leq q \leq N = \binom{n}{2}$. In 2008, Tauraso [11] obtained the (triangular) sequence of edge coverings of the complete graph K_n for $n \geq 1$, which is interestingly the sequence $A054548$ (see Sloane [10]). In 2020, Bhavale and Waphare [2] introduced

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the concept of a fundamental basic block. Let $\mathscr{F}_n(l)$ be the set of all non-isomorphic fundamental basic blocks of nullity l, containing n comparable reducible elements. Bhavale and Waphare [2] obtained the recurrence relation representing the number of all non-isomorphic fundamental basic blocks on $n \geq 1$ comparable reducible elements, having nullity l, as $|\mathscr{F}_{n+1}(l)| = \sum_{k=1}^{n}\sum_{j=0}^{k}\binom{n}{j}\binom{n-j}{k-j}|\mathscr{F}_{n-j}(l-k)|$ with $|\mathscr{F}_0(0)| = 1, |\mathscr{F}_1(l)| = 0$, where $\lfloor\frac{n+2}{2}\rfloor \leq l \leq \binom{n+1}{2}$. Bhavale and Waphare [2] also observed that for $n \geq 2$, the sequence $(f(n,l))$ where $f(n,l) = |\mathscr{F}_n(l)|$, is equivalent to the sequence $(d(n,l))$. But this equivalence is yet to establish. Moreover, both of these sequences are equivalent to $A054548$. In this paper, we prove the equivalence of the sequence $(f(n,l))$ with the sequence $(d(n,l))$. We also provide an edge labeling which is uniquely determined for a given vertex labeling of a finite simple graph/directed graph.

A *labeled graph* $G = (V(G), E(G))$ is a finite set of graph vertices $V(G)$ with a set of graph edges $E(G)$ of 2-subsets of $V(G)$. Two labeled graphs G and H are said to be *isomorphic* if and only if there is a one-one (or injective) map from $V(G)$ onto (or surjective) $V(H)$ which preserves not only adjacency but also the labeling. The *nullity of a graph* G is given by $|E(G)| - |V(G)| + c$, where c is the number of connected components of G.

Let $\leq$ be a partial order relation on a non-empty set P, and let $(P, \leq)$ be a poset. A subset S of a poset $(P, \leq)$ is subposet of P if S itself is a poset under the same relation $\leq$ of P. Elements $x, y \in P$ are said to be *comparable*, if either $x \leq y$ or $y \leq x$. Elements $x, y \in P$ are said to be *incomparable*, denoted by $x||y$, if x, y are not comparable. A *chain* (or a *linear order*) is a poset in which any two elements are comparable. An element $c \in P$ is a *lower bound* (*upper bound*) of $a, b \in P$ if $c \leq a, c \leq b$ ($a \leq c, b \leq c$). A *meet* of $a, b \in P$, denoted by $a \wedge b$, is defined as the greatest lower bound of a and b. A *join* of $a, b \in P$, denoted by $a \vee b$, is defined as the least upper bound of a and b. An element b in P *covers* an element a in P if $a < b$, and there is no element c in P such that $a < c < b$. This fact is denoted by $a \prec b$ and called as a *covering* or an *edge*. If $a \prec b$ then a is called a *lower cover* of b, and b is called an *upper cover* of a. The graph on a poset P with edges as coverings is called *cover graph* and is denoted by $C(P)$. The number of coverings in a chain is called *length* of the chain. If C is a chain on n elements, that is, if $|C| = n$ then its length is $n - 1$. Bhavale and Waphare [2] defined *nullity of a poset* P, denoted by $\eta(P)$, to be the nullity of its cover graph $C(P)$.

A poset L is a *lattice* if both $a \wedge b$ and $a \vee b$, exist in L, $\forall\, a, b \in L$. A sublattice of a lattice L is a subset of L which is a lattice under the same operations $\wedge$ and $\vee$ of L. For $a \leq b$, the interval $[a, b] = \{x \in L | a \leq x \leq b\}$ is a sublattice of L. An element x in a lattice L is *join-reducible*(*meet-reducible*) in L, if there exist $y, z \in L$ both distinct from x, such that $y \vee z = x (y \wedge z = x)$. An element x in a lattice L is *reducible*, if it is either join-reducible or meet-reducible. The set of all reducible elements of L is denoted by $\mathrm{Red}(L)$. An element x is *join-irreducible*(*meet-irreducible*), if it is not join-reducible(meet-reducible); An element x is *doubly irreducible*, if it is both join-irreducible and meet-irreducible. Lattices L_1 and L_2 are *isomorphic* (in symbol, $L_1 \cong L_2$), and the map $\phi : L_1 \to L_2$ is an *isomorphism* if and only if ϕ is one-one and onto (or bijective), and $a \leq b$ in L_1 if and only if $\phi(a) \leq \phi(b)$

in L_2.

For the other definitions, notation, and terminology see [5, 7, 13]. The following result is due to Rival [8].

Lemma 1.1. [8] *If L is a lattice then $L \setminus A$ is a sublattice of L for every subset A of all doubly irreducible elements of L.*

In 1974, Rival [8] introduced and studied the class of dismantlable lattices.

Definition 1.2. [8] A finite lattice of order n is called *dismantlable* if there exists a chain $L_1 \subset L_2 \subset \cdots \subset L_n (= L)$ of sublattices of L such that $|L_i| = i$, for all i.

Brucker and Gély [3] characterized dismantlable lattices as follows.

Theorem 1.3. [3] *A lattice L is dismantlable lattice if and only if there exists a chain of lattices $L_1 \subset L_2 \subset \cdots \subset L_n = L$ such that L_1 is a singleton and $L_{i-1} = L_i \setminus \{x\}$, where x is a doubly irreducible element of L_i.*

Thakare et al. [12] introduced the concept of an *adjunct operation of lattices*. Suppose L_1 and L_2 are two disjoint lattices and (a, b) is a pair of elements in L_1 such that $a < b$ and $a \not\prec b$. Define the partial order $\leq$ on $L = L_1 \cup L_2$ with respect to the pair (a, b) as follows: $x \leq y$ in L if $x, y \in L_1$ and $x \leq y$ in L_1, or $x, y \in L_2$ and $x \leq y$ in L_2, or $x \in L_1$, $y \in L_2$ and $x \leq a$ in L_1, or $x \in L_2$, $y \in L_1$ and $b \leq y$ in L_1. The pair (a, b) is called an *adjunct pair* of L. If L is an adjunct of lattices L_1 and L_2 with an adjunct pair (a, b) then it is denoted by $L = L_1]_a^b L_2$.

The following structure theorem is due to Thakare et al. [12].

Theorem 1.4. [12] *A finite lattice is dismantlable if and only if it is an adjunct of chains.*

Suppose a dismantlable lattice L is an adjunct of the chains $C_0, C_1, C_2, \ldots, C_r$, where C_0 is a maximal chain of L. Then an adjunct representation of L is given by $L = C_0]_{a_1}^{b_1} C_1]_{a_2}^{b_2} C_2 \cdots$
$]_{a_r}^{b_r} C_r$, where $(a_i, b_i); 1 \leq i \leq r$ are adjunct pairs. Moreover, an adjunct representation of lattice L is unique up to the order in which adjunct pairs occur.

Proposition 1.5. *If $L = L_1]_a^b L_2$ then $\eta(L) = \eta(L_1) + \eta(L_2) + 1$.*

Proof. Note that $|E(L)| = |E(L_1)| + |E(L_2)| + 2$ and $|V(L)| = |V(L_1)| + |V(L_2)|$. Further, $\eta(L_1) + \eta(L_2) = (|E(L_1)| - |V(L_1)| + 1) + (|E(L_2)| - |V(L_2)| + 1) = (|E(L_1)| + |E(L_2)| + 2) - (|V(L_1)| + |V(L_2)|) = |E(L)| - |V(L)| = \eta(L) - 1$. Thus $\eta(L) = \eta(L_1) + \eta(L_2) + 1$. □

Using Theorem 1.4 and Proposition 1.5, we get the following result.

Corollary 1.6. *A dismantlable lattice with n elements has nullity r if and only if it is an adjunct of $r + 1$ chains.*

Now given a vertex labeling of a simple graph on n unisolated vertices, we obtain an edge labeling of that graph in the following section. Using this edge labeling, in Section 4, we obtain an equivalence between the sequences $(f(n,l))$ and $(d(n,l))$.

2. Edge labeling of graphs

A labeling of a graph G is an assignment of labels to either the vertices or the edges or both subject to certain conditions. The origin of labelings can be attributed to Rosa [9]. In this section, given a vertex labeling of the complete graph K_n, we obtain an edge labeling of K_n which is determined uniquely. In this regard, we begin with the fact that dictionary order is a linear order. That is, if $S = \{(i,j) | 1 \leq i < j \leq n\}$, with a dictionary order as follows for $(i,j),(i',j') \in S, (i,j) \leq (i',j')$, if $i \leq i'$, or if $i = i'$ and $j \leq j'$, then $(S, \leq)$ is a chain. For $1 \leq r < n$, define $S_r = \{(r,j) | r < j \leq n\}$.

Proposition 2.1. *For $1 \leq r < n$, S_r is a subchain of S. Further, $|S_r| = n - r$ and $S_i \cap S_j = \phi, \forall i \neq j$.*

Proof. As $S_r \subseteq S$, S_r is a chain. Now, it follows that $|S_r| = n-r$, since $r < j \leq n$ implies that $r+1 \leq j \leq r+(n-r)$. Also, by the definition of S_r, $S_i \cap S_j = \phi, \forall i \neq j$. □

Proposition 2.2. *For $n \geq 2$, $S = S_1 \oplus S_2 \oplus \cdots \oplus S_{n-1}$.*

Proof. By Proposition 2.1, suppose S_r is the chain $(r,r+1) \prec (r,r+2) \prec \cdots \prec (r,n)$, where $1 \leq r \leq n-1$. It is sufficient to prove that $(r,n) \prec (r+1,r+2), \forall r$, where $1 \leq r \leq n-2$. Suppose $(r,n) \leq (i,j) \leq (r+1,r+2)$. If $(i,j) = (r,n)$ then we are done. Otherwise, $i \neq r$ or $j \neq n$ which implies that $r < i$, since $(r,n) \leq (i,j)$. Also, $(i,j) \leq (r+1,r+2)$ implies that $i \leq r+1$, or $i = r+1$ and $j \leq r+2$. If $i \leq r+1$ then $i = r+1$ as $r < i$. But then we must have $j \leq r+2$. Now $i = r+1 < j \leq r+2$. Therefore $j = r+2$. Hence $(i,j) = (r+1,r+2)$. Thus, $S = S_1 \oplus S_2 \oplus \cdots \oplus S_{n-1}$. □

Proposition 2.3. *For fixed r, where $1 \leq r < n$, if the pair (r,n) is l^{th} element of S then $l = rn - \binom{r+1}{2}$.*

Proof. Observe that for $1 \leq r < n$, (r,n) is the largest element of the subchain S_r of S. So consider a subchain $C^{(r)} = S_1 \oplus S_2 \oplus \cdots \oplus S_r$ of S. Note that, the pair (r,n) is also the largest element of $C^{(r)}$. Therefore (r,n) is the l^{th} element of S, and l is one more than the length of the chain $C^{(r)}$. Therefore $l = (|C^{(r)}|-1)+1 = |C^{(r)}| = |S_1 \oplus S_2 \oplus \cdots \oplus S_r| = \sum_{i=1}^{r} |S_i| = rn - \binom{r+1}{2}$, since by Proposition 2.1, $|S_i| = n - i$ and $S_i \cap S_j = \phi$, for all $i \neq j$. □

Proposition 2.4. *If $(i,j) \in S$ is k^{th} element of S then $1 \leq k = (i-1)n - \binom{i}{2} + j - i \leq N = \binom{n}{2}$.*

Proof. Consider the chain $S_i : (i,i+1) \prec (i,i+2) \prec \cdots \prec (i,n)$. Now j^{th} pair in S_i is $(i,i+j)$. This implies that the pair (i,j) in S_i is $(j-i)^{\text{th}}$ pair in S_i. Now by Proposition 2.3, if a pair $(i-1,n)$ is the l^{th} element of S, then $l = (i-1)n - \binom{i}{2}$. Note that $(i-1,n)$ is the largest element of S_{i-1}. Therefore if (i,j) is k^{th} element of S then $(i,j) \in S_i$ and $k = l + (j-i) = (i-1)n - \binom{i}{2} + j - i$. □

Proposition 2.5. *For $1 \leq k \leq N = \binom{n}{2}$, there exists unique $(i,j) \in S$ such that $k = (i-1)n - \binom{i}{2} + j - i$.*

Proof. For if, suppose there exist $(i,j),(i',j') \in S$ such that $(i,j) \neq (i',j')$ but

$$k = (i-1)n - \binom{i}{2} + j - i = (i'-1)n - \binom{i'}{2} + j' - i'. \tag{1}$$

If $i = i'$ then by Eq. (1), we have $j = j'$, and hence $(i,j) = (i',j')$, which is a contradiction. Therefore $i \neq i'$. Without loss of generality, suppose $i < i'$. Now from Eq. (1), $j - j' = (i'-i)(n-1) + (i'-i)(\frac{1-(i'+i)}{2}) = (i'-i)(n - \frac{1}{2} - \frac{i'}{2} - \frac{i}{2}) = (i'-i)(\frac{n-i'}{2} + \frac{n-i-1}{2}) \geq 0$, as $1 \leq i < i' \leq n$. This implies that $j \geq j'$, a contradiction, since there exists $j \geq i+1$ such that $j < j'$, as $i < j$ and $i < i+1 \leq i' < j'$. Thus $(i,j) \neq (i',j')$ implies that $(i-1)n - \binom{i}{2} + j - i \neq (i'-1)n - \binom{i'}{2} + j' - i'$, which is not possible by Eq. (1). Hence $(i,j) = (i',j')$. □

Let $J_N = \{i | 1 \leq i \leq N = \binom{n}{2}\}$, where $n \geq 2$. Then the following result follows immediately from Proposition 2.5.

Theorem 2.6. *Let $n \geq 2$. Define $f : S \to J_N$ by $f(i,j) = (i-1)n - \binom{i}{2} + j - i, \forall (i,j) \in S$. Then f is bijective, and hence $|S| = N = \binom{n}{2}$.*

Proof. Clearly f is a well defined function, since if $(i,j) = (i',j') \in S$ then $i = i'$ and $j = j'$. Therefore $f(i,j) = f(i',j')$. Further by Proposition 2.5, f is clearly surjective. Note that, if $(i,j) \neq (i',j')$ in S then we get $(i-1)n - \binom{i}{2} + j - i \neq (i'-1)n - \binom{i'}{2} + j' - i'$. Hence f is injective. □

Let K_n be a complete graph on $n \geq 2$ vertices, say $\{1,2,\ldots,n\}$. Let $\overrightarrow{K_n}$ be the digraph associated with K_n such that $1 \leq i < j \leq n$ whenever edge $(i,j) \in E(\overrightarrow{K_n})$. Then using Theorem 2.6, we get the following result.

Corollary 2.7. *Consider a directed graph $\overrightarrow{K_n}$ with vertex labeling $1,2,\ldots,n$. Then for $i < j$, a directed edge $\overrightarrow{(i,j)} \in E(\overrightarrow{K_n})$ can be labeled as $k = (i-1)n - \binom{i}{2} + j - i \in J_N$. Moreover, for any $k \in J_N$, there exists unique directed edge $\overrightarrow{(i,j)} \in E(\overrightarrow{K_n})$.*

In Corollary 2.7, we obtained unique edge labeling of $\overrightarrow{K}_n$. Using this kind of edge labeling, we obtain in the following an edge labeling for a directed subgraph (on n unisolated vertices) of $\overrightarrow{K}_n$.

Theorem 2.8. *Suppose $\overrightarrow{G}$ is a directed subgraph (on n unisolated vertices) of $\overrightarrow{K}_n$ with vertex labeling $\{v_1, v_2, \ldots, v_n\}$. Then for $i < j$, if $\overrightarrow{(v_i, v_j)} \in E(\overrightarrow{G})$ then $\overrightarrow{(v_i, v_j)}$ can be labeled as $k = (i-1)n - \binom{i}{2} + j - i \in J_N$. Moreover, if $k \in J_N$ is a label of an edge $\overrightarrow{e} \in E(\overrightarrow{G})$ then $\overrightarrow{e} = \overrightarrow{(v_i, v_j)}$.*

If we remove direction of each edge of a subgraph $\overrightarrow{G}$ of $\overrightarrow{K}_n$, then we get a simple subgraph G (on n unisolated vertices) of K_n. Thus using Theorem 2.8, for a given finite simple graph G on n unisolated vertices, we get an edge labeling for the graph G, which is uniquely determined.

In order to prove equivalence of the sequences $(f(n,l))$ and $(d(n,l))$, we need to study the class $\mathscr{F}_n(l)$ firstly. For this sake in Section 3, we introduce the concept of a complete fundamental basic block, and obtain in general an adjunct representation of a fundamental basic block.

3. Fundamental basic blocks

Bhavale and Waphare [2] defined the class of “RC-lattices”as class of all lattices such that each member of the class has all the reducible elements comparable.

Theorem 3.1. [2] *A lattice in which all the reducible elements are comparable is a dismantlable lattice.*

Thus an RC-lattice is dismantlable. Bhavale and Waphare [2] also introduced the concepts of a doubly irreducible element, a basic block, and a fundamental basic block. An element of a poset P is *doubly irreducible* in P, if it has at most one upper cover and at most one lower cover in P. Let $\mathrm{Irr}(P)$ denote the set of all doubly irreducible elements in the poset P.

Definition 3.2. [2] A poset B is a *basic block*, if it is one element, or $\mathrm{Irr}(B) = \phi$, or removal of a doubly irreducible element from B reduces nullity of B by one.

Definition 3.3. [2] An RC-lattice F is said to be a *fundamental basic block*, if it is a basic block, and all the adjunct pairs in an adjunct representation of F are distinct.

Definition 3.4. For $n \geq 2$, *complete fundamental basic block*, denoted by $\mathrm{CF}(n)$, is the fundamental basic block on n reducible elements, having nullity $N = \binom{n}{2}$.

Using Definition 3.4, Theorem 1.4 and Proposition 2.5, we get an adjunct representation of a complete fundamental basic block in the following result.

Theorem 3.5. *Let $C : u_1 \leq u_2 \leq \cdots \leq u_n$ be the chain of reducible elements of* $\mathrm{CF}(n)$*. Then an adjunct representation of* $\mathrm{CF}(n)$ *is given by* $\mathrm{CF}(n) = C_0]_{u_1}^{u_2}\{c_1\}]_{u_1}^{u_3}\{c_2\}\cdots]_{u_1}^{u_n}\{c_{n-1}\}$ $]_{u_2}^{u_3}\{c_n\}]_{u_2}^{u_4}\{c_{n+1}\}\cdots]_{u_2}^{u_n}\{c_{2n-3}\}]_{u_3}^{u_4}\{c_{2n-2}\}\cdots]_{u_i}^{u_j}\{c_k\}\cdots]_{u_{n-1}}^{u_n}\{c_N\}$*, where $N = \binom{n}{2}$, $k = (i-1)n - \binom{i}{2} + j - i$, $1 \leq i < j \leq n$, and C_0 is the chain $u_1 \prec x_1 \prec u_2 \prec x_2 \prec \cdots \prec u_{n-1} \prec x_{n-1} \prec u_n$ with $x_i || c_k$ for $k = (i-1)n - \binom{i}{2} + 1$.*

Proof. By Theorem 3.1, $\mathrm{CF}(n)$ is a dismantlable lattice. Therefore by Theorem 1.4 and Corollary 1.6, $\mathrm{CF}(n) = C_0]_{a_1}^{b_1}C_1]_{a_2}^{b_2}C_2\cdots]_{a_N}^{b_N}C_N$, where C_0 is a maximal chain containing the chain $C : u_1 \leq u_2 \leq \cdots \leq u_n$, C_i is a chain and (a_i, b_i) is an adjunct pair for each $i, 1 \leq i \leq N = \binom{n}{2}$. Clearly $|C_k| = 1, \forall k, 1 \leq k \leq N$. For if, suppose $|C_k| \geq 2$ for some k, where $1 \leq k \leq N$. Now removal of a doubly irreducible element from C_k results in removal of an (incident) edge from C_k, as $|C_k| \geq 2$. Therefore there exists $y \in \mathrm{Irr}(C_k)$ such that $\eta(\mathrm{CF}(n) \setminus \{y\}) = \eta(\mathrm{CF}(n))$. This is a contradiction to the fact that $\mathrm{CF}(n)$ is a basic block. Therefore suppose $C_k = \{c_k\}$, where $1 \leq k \leq N$. Now $a_k, b_k \in U = \{u_1, u_2, \ldots, u_n\}, \forall k, 1 \leq k \leq N$. Also by Proposition 2.5, for each $1 \leq k \leq N$, there exists unique $(i, j) \in S$ such that $k = (i-1)n - \binom{i}{2} + j - i$. Hence for each c_k or an adjunct pair (a_k, b_k), there corresponds a unique pair (u_i, u_j) such that $k = (i-1)n - \binom{i}{2} + j - i$, where $1 \leq i < j \leq n$. Moreover, $a_k = u_i$ and $b_k = u_j$. Now if $[u_i, u_{i+1}] \cap C_0$ consists of more than one doubly irreducible elements, then again we get a contradiction to the fact that $\mathrm{CF}(n)$ is a basic block. Therefore for each i, $1 \leq i \leq n-1$, $[u_i, u_{i+1}] \cap C_0$ consists of exactly one doubly irreducible element, say x_i. Thus C_0 is the chain $u_1 \prec x_1 \prec u_2 \prec x_2 \prec \cdots \prec u_{n-1} \prec x_{n-1} \prec u_n$. Moreover $x_i || c_k$ for $k = (i-1)n - \binom{i}{2} + 1$, as $u_i \prec c_k \prec u_{i+1}$. □

Let $X = C_0 \setminus C$ and $Y = \{c_k | k \in J_N\}$. Then from Theorem 3.5, $X = \{x_1, x_2, \ldots, x_{n-1}\}$ and as a set $\mathrm{CF}(n) = C_0 \cup Y$. Moreover, $|\mathrm{CF}(n)| = |C_0| + |Y| = |C| + |X| + |Y| = n + (n-1) + N = 2n - 1 + \binom{n}{2}$. Also using Theorem 3.5, the total number of coverings or edges in $\mathrm{CF}(n)$ is $(2n-2) + 2\binom{n}{2}$. Thus it can be verified that $\eta(\mathrm{CF}(n)) = \binom{n}{2}$.

The following result immediately follows from Theorem 3.5.

Proposition 3.6. *If F is a fundamental basic block obtained from* $\mathrm{CF}(n)$ *by removal of a doubly irreducible element, then $F \in \mathscr{F}_n(N-1)$, where $n > 2$ and $N = \eta(\mathrm{CF}(n)) = \binom{n}{2}$.*

Proof. Let $z \in \mathrm{Irr}(\mathrm{CF}(n))$, where $n > 2$. Let $L = \mathrm{CF}(n) \setminus \{z\}$. Then it is clear that $\mathrm{Red}(L) = \mathrm{Red}(\mathrm{CF}(n))$. By Lemma 1.1, L is a sublattice of $\mathrm{CF}(n)$. By Theorem 3.5, either $z = c_k$ where $1 \leq k \leq N$ or $z = x_i$ where $1 \leq i \leq n-1$. If $z = x_i$ then $L \cong \mathrm{CF}(n) \setminus \{c_k\}$ for $k = (i-1)n - \binom{i}{2} + 1$. Therefore without loss of generality, we assume that $z = c_k$, where $1 \leq k \leq N$. As $k \in J_N$, by Proposition 2.5, there exists unique $(i, j) \in S$ such that $k = (i-1)n - \binom{i}{2} + j - i$, where $1 \leq i < j \leq n$. As $c_k \notin L$, (u_i, u_j) does not remain an adjunct pair in an adjunct representation of L. Therefore by Theorem 3.5, L is an adjunct of N chains, and hence by Corollary 1.6, $\eta(L) = N - 1$. Note that the elements u_i and u_j remains reducible in L, since $\mathrm{Red}(L) = \mathrm{Red}(\mathrm{CF}(n))$. Now if $j > i+1$ then by Definition 3.3 and using Theorem 3.5, L itself is the fundamental basic block on n reducible elements with $\eta(L) = N - 1$. Also, if $j = i + 1$ then by Definition 3.2, L

does not remain a basic block, since $\eta(L \setminus \{x_i\}) = \eta(L)$. But in this case, by Definition 3.3 and using Theorem 3.5, $L \setminus \{x_i\}$ becomes the fundamental basic block on n reducible elements, having nullity same as that of L. Note that by Lemma 1.1, $L \setminus \{x_i\}$ is also a sublattice of L, and hence of $\mathrm{CF}(n)$. Thus removal of a doubly irreducible element from $\mathrm{CF}(n)$ gives rise to the fundamental basic block on n reducible elements, having nullity exactly one less than that of $\mathrm{CF}(n)$. □

Note that, removal of an arbitrary number of doubly irreducible elements from $\mathrm{CF}(n)$ does not guarantee the preservation of all the n reducible elements. However, removal of at most $n-2$ doubly reducible elements from $\mathrm{CF}(n)$ preserves all the n reducible elements, since each reducible element is connected to the remaining all $n-1$ reducible elements via $n-1$ adjunct pairs.

Now by Lemma 1.1 and using the repeated application of the treatment which is used in the proof of Proposition 3.6, we get the following result.

Corollary 3.7. *If F is a fundamental basic block obtained from $\mathrm{CF}(n)$ by removal of $N-l$ doubly irreducible elements such that $\mathrm{Red}(F) = \mathrm{Red}(\mathrm{CF}(n))$ where $n > 2$, then $F \in \mathscr{F}_n(l)$, where $N = \eta(\mathrm{CF}(n)) = \binom{n}{2}$ and $\lfloor \frac{n+1}{2} \rfloor \leq l \leq N$. Further, if $l \geq N-n+2$ then $f(n,l) = |\mathscr{F}_n(l)| = \binom{N}{l}$.*

Proof. In Proposition 3.6, we have seen that removal of a doubly irreducible element from $\mathrm{CF}(n)$ gives rise to the fundamental basic block on n reducible elements, having nullity exactly one less than that of $\mathrm{CF}(n)$. Now by Theorem 1.3, removal of more than $n-2$ doubly irreducible elements from $\mathrm{CF}(n)$ gives rise to the fundamental basic block, but may contain $\leq n$ reducible elements. But $\mathrm{Red}(F) = \mathrm{Red}(\mathrm{CF}(n))$. Therefore removal of $N-l$ doubly irreducible elements from $\mathrm{CF}(n)$ (which are precisely the elements of the set Y) gives rise to the fundamental basic block, say F (on n reducible elements) having nullity $N-(N-l) = l$. Thus $F \in \mathscr{F}_n(l)$.

Now by Lemma 1.1, F is a sublattice of $\mathrm{CF}(n)$. Hence $\eta(F) = l \leq \eta(\mathrm{CF}(n)) = N = \binom{n}{2}$. Also, as far as the lower bound on l is concerned, there are the following two cases.

Case I. Suppose n is even, say $n = 2k, k \in \mathbb{N}$. By Definition 3.3, the multiplicity of each adjunct pair in an adjunct representation of F is one. Moreover, an adjunct pair corresponds to exactly two reducible elements of F. Therefore, if we want to cover all the n reducible elements of F, then at least k adjunct pairs are required. Therefore $l \geq k = \frac{n}{2} = \lfloor \frac{n+1}{2} \rfloor$. Note that, if we consider l adjunct pairs, where $l < k$ then care of all the $2k$ reducible elements can not be taken.

Case II. Suppose n is odd, say $n = 2k+1, k \in \mathbb{N}$. Again in this case, at least k adjunct pairs are required to cover $n-1$ reducible elements. Therefore one more adjunct pair is required to cover all the n reducible elements of F, so that none of the reducible elements is left unassigned. Hence in this case, at least $k+1 = \frac{n-1}{2} + 1 = \frac{n+1}{2} = \lfloor \frac{n+1}{2} \rfloor$ reducible elements are required. Thus $l \geq \lfloor \frac{n+1}{2} \rfloor$.

Further, if $l \geq N-n+2$ then $N-l \leq n-2$. But then $\mathrm{Red}(F) = \mathrm{Red}(\mathrm{CF}(n))$, and hence $F \in \mathscr{F}_n(l)$. Therefore in this situation, $f(n,l) = \binom{N}{N-l}$, which is the number of ways to choose $N-l$ doubly irreducible elements for the removal, out of N doubly irreducible elements of $\mathrm{CF}(n)$ (or the set Y). Thus $f(n,l) = \binom{N}{l}$. □

In Theorem 3.5, we have obtained an adjunct representation of $\mathrm{CF}(n)$ which is the fundamental basic block on n reducible elements, having maximum nullity $N = \binom{n}{2}$. By Lemma 1.1 and by Corollary 3.7, each $F \in \mathscr{F}_n(l)$ where $\lfloor \frac{n+1}{2} \rfloor \leq l \leq N$, is a sublattice of $\mathrm{CF}(n)$, since $\mathrm{Red}(F) = \mathrm{Red}(\mathrm{CF}(n))$. Therefore using Theorem 3.5 and Corollary 3.7, we get an adjunct representation of F in general, by restricting an adjunct representation of $\mathrm{CF}(n)$ to that of F. In fact, we have the following result.

Corollary 3.8. *An adjunct representation of $F \in \mathscr{F}_n(l)$ is given by $F = C_0']_{a_1}^{b_1}\{c_{q_1}\}]_{a_2}^{b_2}\{c_{q_2}\} \cdots]_{a_l}^{b_l}\{c_{q_l}\}$, where for each $s, 1 \leq s \leq l \leq N = \binom{n}{2}$, $a_s = u_i, b_s = u_j$ with $q_s = (i-1)n - \binom{i}{2} + j - i, 1 \leq i < j \leq n$, $(q_1 < q_2 < \cdots < q_l)$, and C_0' is the chain containing chain $C : u_1 \leq u_2 \leq \cdots \leq u_n$ of all the reducible elements of $\mathrm{CF}(n)$ and all those $x_i \in X = C_0 \setminus C$ which satisfy $x_i || c_{q_s}$ in F for $q_s = (i-1)n - \binom{i}{2} + 1$.*

Remark 3.9. Let $Y_F = Y \cap F$, where $Y = \{c_k | k \in J_N\}$. Then by Corollary 3.8, $Y_F = \{c_{q_s} | 1 \leq s \leq l\}$. Also by Corollary 3.7, if $|Y \setminus Y_{F_1}| = |Y \setminus Y_{F_2}| = N - l$, then $F_1, F_2 \in \mathscr{F}_n(l)$. Moreover, $F_1 \cong F_2$ if and only if $Y \setminus Y_{F_1} = Y \setminus Y_{F_2}$. That is, $F_1 \cong F_2$ if and only if $Y_{F_1} = Y_{F_2}$.

4. Equivalence of labeled graphs and lattices

Let $\mathscr{D}(n,q)$ be the set of all non-isomorphic labeled graphs on n unisolated vertices, containing q edges. Then $\mathscr{D}(n,q)$ is precisely the set of all non-isomorphic labeled subgraphs (on n unisolated vertices, containing q edges) of K_n. Clearly $d(n,q) = |\mathscr{D}(n,q)|$. Let $\overrightarrow{\mathscr{D}}(n,q)$ be the set of all non-isomorphic labeled digraphs on n unisolated vertices $v_1, v_2, \ldots, v_n$ such that for $\overrightarrow{G} \in \overrightarrow{\mathscr{D}}(n,q)$, $\overrightarrow{(v_i, v_j)}$ is a directed edge of $\overrightarrow{G}$ whenever $i < j$. Note that, $\overrightarrow{\mathscr{D}}(n,q)$ is precisely the set of all non-isomorphic labeled directed subgraphs (on n unisolated vertices, containing q directed edges) of $\overrightarrow{K}_n$.

Lemma 4.1. *For $n \geq 2$ and for $G \in \mathscr{D}(n,q)$, $\lfloor \frac{n+1}{2} \rfloor \leq q \leq \binom{n}{2}$.*

Proof. Let $G \in \mathscr{D}(n,q)$ and $n \geq 2$. Note that the complete graph K_n is the largest (simple) graph on n unisolated vertices which has $N = \binom{n}{2}$ edges. As G is a subgraph of K_n, $q \leq N$. Now as far as lower bound on q is concerned, there are the following two cases.

Case I. Suppose n is even, say $n = 2k, k \in \mathbb{N}$. Since an edge consists of two vertices, at least $k = \frac{n}{2} = \lfloor \frac{n+1}{2} \rfloor$ edges are required to cover all the vertices of G, so that none of the vertex remains isolated.

Case II. Suppose n is odd, say $n = 2k+1, k \in \mathbb{N}$. Again, in this case at least k edges of G are required to cover $n-1$ vertices out of n. Therefore we require one more edge to cover all the n vertices of G, so that none of the vertex remains isolated. Thus, in this case at least $k + 1 = \frac{n-1}{2} + 1 = \lfloor \frac{n+1}{2} \rfloor$ edges are required. □

The following result follows immediately from Lemma 4.1.

Corollary 4.2. *For* $n \geq 2$ *and for* $\overrightarrow{G} \in \overrightarrow{\mathscr{D}}(n,q)$, $\lfloor \frac{n+1}{2} \rfloor \leq q \leq \binom{n}{2}$.

Consider the chain $C : u_1 \leq u_2 \leq \cdots \leq u_n$ of all the reducible elements of CF(n), where $n \geq 2$. Then an interval $[u_i, u_j]$ is a sublattice of CF(n) for $1 \leq i < j \leq n$. Further, $[u_i, u_j] \cong$ CF(m), where $m = j - i + 1 \leq n$. Let $I = \{I_{ij} = [u_i, u_j] | 1 \leq i < j \leq n\}$. By Theorem 2.6, it clearly follows that $|I| = |S| = \binom{n}{2}$.

By Proposition 2.4, each $I_{ij} \in I$ can be associated with a unique directed edge $\overrightarrow{e}_k \in E(\overrightarrow{K_n})$. Conversely, by Proposition 2.5, each directed edge $\overrightarrow{e}_k \in E(\overrightarrow{K_n})$ can be associated with a unique interval $I_{ij} \in I$.

In fact, using Proposition 2.5 and Theorem 3.5, we get the following result which provides unique association of CF(n) with $\overrightarrow{K_n}$.

Theorem 4.3. *Let* $u_1 \leq u_2 \leq \cdots \leq u_n$ *be the chain of all reducible elements of* CF(n), *where* $n \geq 2$. *Consider the class* $\mathscr{A} = \{u_i, I_{ij} = [u_i, u_j] | 1 \leq i \leq n, I_{ij} \in I\}$ *associated with* CF(n). *Let* $V(\overrightarrow{K_n}) = \{v_1, v_2, \ldots, v_n\}$ *and* $E(\overrightarrow{K_n}) = \{\overrightarrow{e_k} | k \in J_N\}$. *Define* $\psi : \mathscr{A} \to \overrightarrow{K_n}$ *by* $\psi(u_i) = v_i$ *where* $1 \leq i \leq n$, *and* $\psi(I_{ij}) = \psi([u_i, u_j]) = \overrightarrow{(v_i, v_j)} = \overrightarrow{e_k}$ *where* $k = (i-1)n - \binom{i}{2} + j - i$, $1 \leq i < j \leq n$. *Then* ψ *is a well-defined bijective map.*

Proof. To show that ψ is well-defined, it is sufficient to prove $\psi(I_{ij}) = \psi(I_{pq})$ in $E(\overrightarrow{K_n})$ whenever $I_{ij} = I_{pq}$ in I. So suppose $I_{ij} = I_{pq}$ in I. Therefore $u_i = u_p$ and $u_j = u_q$, since $[u_i, u_j] = [u_p, u_q]$. Also $1 \leq i < j \leq n$, $1 \leq p < q \leq n$. This implies that $i = p$ and $j = q$. Therefore $v_i = v_p, v_j = v_q$, and $k = (i-1)n - \binom{i}{2} + j - i = (p-1)n - \binom{p}{2} + q - p$. Therefore $\overrightarrow{(v_i, v_j)} = \overrightarrow{(v_p, v_q)} = \overrightarrow{e_k}$. Thus $\psi(I_{ij}) = \psi(I_{pq})$ in $E(\overrightarrow{K_n})$.

To show that ψ is injective, it is sufficient to prove that $I_{ij} = I_{pq}$ in I whenever $\psi(I_{ij}) = \psi(I_{pq})$ in $E(\overrightarrow{K_n})$. So suppose $\psi(I_{ij}) = \psi(I_{pq})$ in $E(\overrightarrow{K_n})$. But then $\overrightarrow{e_k} = \overrightarrow{(v_i, v_j)} = \overrightarrow{(v_p, v_q)}$, with $k = (i-1)n - \binom{i}{2} + j - i = (p-1)n - \binom{p}{2} + q - p$. Therefore $v_i = v_p$ and $v_j = v_q$. Now by Proposition 2.5, $(i,j) = (p,q)$. This implies that $i = p$ and $j = q$. Hence we have $u_i = u_p$ and $u_j = u_q$. Therefore $I_{ij} = [u_i, u_j] = [u_p, u_q] = I_{pq}$ in I.

To show that ψ is surjective, it is sufficient to prove that for each $\overrightarrow{e_k} \in E(\overrightarrow{K_n})$, there exists $I_{ij} \in I$ such that $\psi(I_{ij}) = \overrightarrow{e_k}$. So suppose $\overrightarrow{e_k} \in E(\overrightarrow{K_n})$, where $1 \leq k \leq \binom{n}{2}$. By Proposition 2.5, there exists unique (i,j) such that $k = (i-1)n - \binom{i}{2} + j - i$, $1 \leq i < j \leq n$. Thus there exists a unique pair (u_i, u_j) of reducible elements of F, and hence a unique interval $[u_i, u_j] = I_{ij} \in I$ such that $\psi(I_{ij}) = \overrightarrow{e_k}$. □

Consider the chain $u_1 \leq u_2 \leq \cdots \leq u_n$ of all reducible elements of $F \in \mathscr{F}_n(l)$, where $n \geq 2$. Let $U = \{u_i | 1 \leq i \leq n\}$. Now if $l < N$ then there exists $(i,j) \in S$ such that (u_i, u_j) need not be an adjunct pair in an adjunct representation of F. So let $I_F = \{I_{ij} = [u_i, u_j] | 1 \leq i < j \leq n$ and (u_i, u_j) is an adjunct pair in $F\}$, and let $\mathscr{A}_F = U \cup I_F$. Then $\mathscr{A}_F = \{u_i, I_{ij} | 1 \leq i \leq n, I_{ij} \in I_F\}$. By Corollary 1.6 and by Corollary 3.8, $|I_F| = l$. Note that for every $F \in \mathscr{F}_n(l)$, there is a unique class $\mathscr{A}_F$ associated with F. Clearly $\mathscr{A}_F \subseteq \mathscr{A}$. Therefore, if we restrict the domain of the map ψ (see Theorem 4.3) to $\mathscr{A}_F$, then we get the following result.

Corollary 4.4. *For $n \geq 2$ and for $F \in \mathscr{F}_n(l)$, the (restricted) map $\psi_F : \mathscr{A}_F \to \overrightarrow{K_n}$ given by $\psi_F(u_i) = v_i$, $1 \leq i \leq n$, and for $I_{ij} \in I_F$, $\psi_F(I_{ij}) = \psi_F([u_i, u_j]) = \overrightarrow{(v_i, v_j)} = \overrightarrow{e_k}$, where $k = (i-1)n - \binom{i}{2} + j - i$, $1 \leq i < j \leq n$. Then ψ_F is a well-defined injective map.*

Using Corollary 3.8 and Corollary 4.4, we get the following result.

Proposition 4.5. *For each $F \in \mathscr{F}_n(l)$ there exists unique $\overrightarrow{G_F} \in \overrightarrow{\mathscr{D}}(n,l)$ such that $\psi_F(\mathscr{A}_F) = \overrightarrow{G_F}$.*

Proof. By Corollary 4.4, for each $F \in \mathscr{F}_n(l)$, $\psi_F : \mathscr{A}_F \to \overrightarrow{K_n}$ is a well defined injective map, where $\mathscr{A}_F = U \cup I_F$. Suppose $\psi_F(\mathscr{A}_F) = \overrightarrow{G_F}$. Then $\overrightarrow{G_F}$ is a (directed) subgraph of $\overrightarrow{K_n}$ with $V(\overrightarrow{G_F}) = \psi_F(U)$, $E(\overrightarrow{G_F}) = \psi_F(I_F)$, and $\psi_F : \mathscr{A}_F \to \overrightarrow{G_F}$ is a bijection. Clearly $|V(\overrightarrow{G_F})| = n$. Also $|E(\overrightarrow{G_F})| = l$, since by Corollary 3.8, $|I_F| = l$. Thus $\overrightarrow{G_F} \in \overrightarrow{\mathscr{D}}(n,l)$, since none of the vertex of $\overrightarrow{G_F}$ is isolated. For if, suppose for some i, there is an isolated vertex v_i in $V(\overrightarrow{G_F})$. Therefore $\overrightarrow{(v_i, v_j)} \notin E(\overrightarrow{G_F})$ for any vertex $v_j \in V(\overrightarrow{G_F})$, where $j > i$. This implies that $\psi_F^{-1}(\overrightarrow{(v_i, v_j)}) = I_{ij} = [u_i, u_j] \notin I_F$ for any reducible element u_j of F, where $j > i$. In other words, (u_i, u_j) is not an adjunct pair in an adjunct representation of F for any u_j, where $j > i$. Similarly, we can prove that (u_j, u_i) is not an adjunct pair in an adjunct representation of F for any reducible element u_j of F, where $j < i$. Thus u_i does not remain as a reducible element of F. This is a contradiction to the fact that F is a fundamental basic block on n reducible elements. Thus for any $F \in \mathscr{F}_n(l)$, there is a unique class $\mathscr{A}_F$, and hence there exists a unique directed graph $\overrightarrow{G_F} \in \overrightarrow{\mathscr{D}}(n,l)$. □

In particular, if $F \in \mathscr{F}_4(4)$ then $\mathscr{A}_F = U \cup I_F$ with $U = \{u_1, u_2, u_3, u_4\}$ and $I_F = \{I_{12}, I_{14}, I_{23}, I_{24}\}$. Also $\overrightarrow{G_F} \in \overrightarrow{\mathscr{D}}(4,4)$ with $V(\overrightarrow{G_F}) = \{v_1, v_2, v_3, v_4\}$ and $E(\overrightarrow{G_F}) = \{\overrightarrow{e_1}, \overrightarrow{e_3}, \overrightarrow{e_4}, \overrightarrow{e_5}\}$ (see Figure 1).

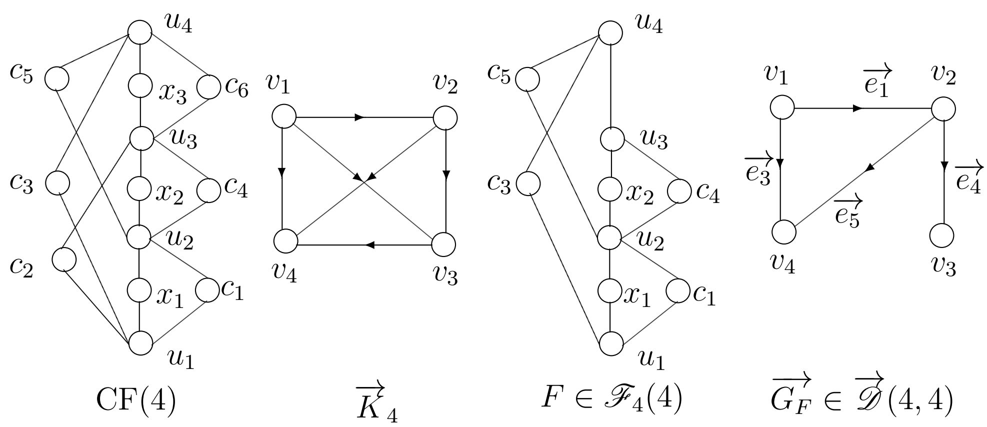

Fig. 1.

Now using Corollary 3.8 and Proposition 4.5, we get the following result.

Theorem 4.6. *For $n \geq 2$ and for fixed l, $\lfloor \frac{n+1}{2} \rfloor \leq l \leq \binom{n}{2}$, there is a one-to-one correspondence between $\mathscr{F}_n(l)$ and $\overrightarrow{\mathscr{D}}(n,l)$.*

Proof. Define $\phi : \mathscr{F}_n(l) \to \overrightarrow{\mathscr{D}}(n,l)$ by $\phi(F) = \overrightarrow{G}$ $(= \overrightarrow{G_F})$, for all $F \in \mathscr{F}_n(l)$. By Proposition 4.5, ϕ is a well defined map. Now using Theorem 4.3, Corollary 4.4 and Proposition 4.5, for each $F \in \mathscr{F}_n(l)$, $\psi_F : \mathscr{A}_F \to \overrightarrow{G}$ is a bijection, where ψ_F is a restriction of the bijective map $\psi : \mathscr{A} \to \overrightarrow{K_n}$.

Let $\overrightarrow{G} \in \overrightarrow{\mathscr{D}}(n,l)$. Then $\overrightarrow{G}$ is a (directed) subgraph of $\overrightarrow{K_n}$. Suppose $V(\overrightarrow{G}) = V(\overrightarrow{K_n}) = \{v_1, v_2, \ldots, v_n\}$ and $E(\overrightarrow{G}) = \{\overrightarrow{e_{q_1}}, \overrightarrow{e_{q_2}}, \ldots, \overrightarrow{e_{q_l}}\} \subseteq E(\overrightarrow{K_n}) = \{\overrightarrow{e_k} | k \in J_N\}$. By Theorem 2.8, $\overrightarrow{e_k} = \overrightarrow{(v_i, v_j)}$ where $k = (i-1)n - \binom{i}{2} + j - i$. Now suppose $U = \psi^{-1}(V(\overrightarrow{G}))$ and $J = \psi^{-1}(E(\overrightarrow{G}))$. Then there exists $F \in \mathscr{F}_n(l)$ with $\mathscr{A}_F = U \cup J$ such that $\psi(\mathscr{A}_F) = \overrightarrow{G}$. Thus there exists $F \in \mathscr{F}_n(l)$ with $\mathscr{A}_F = U \cup J$ such that $\phi(F) = \overrightarrow{G}$. Therefore ϕ is a surjective map.

Let $F_1, F_2 \in \mathscr{F}_n(l)$. Suppose $\phi(F_1) = \overrightarrow{G_1}$, $\phi(F_2) = \overrightarrow{G_2}$, and $\overrightarrow{G_1} \cong \overrightarrow{G_2}$. By Corollary 3.8, suppose $F_1 = C_0']_{a_1}^{b_1}\{c_{q_1}\}]_{a_2}^{b_2}\{c_{q_2}\}\cdots]_{a_l}^{b_l}\{c_{q_l}\}$, where for each $s, 1 \leq s \leq l \leq N = \binom{n}{2}$, $a_s = u_i, b_s = u_j$ with $q_s = (i-1)n - \binom{i}{2} + j - i$, $1 \leq i < j \leq n$, $(q_1 < q_2 < \cdots < q_l)$, and C_0' is the chain containing chain $C : u_1 \leq u_2 \leq \cdots \leq u_n$ of all reducible elements of $\mathrm{CF}(n)$, and all those $x_i \in X = C_0 \setminus C$ which satisfy $x_i || c_{q_s}$ in F_1 for $q_s = (i-1)n - \binom{i}{2} + 1$. Note that C_0 is a maximal chain $u_1 \prec x_1 \prec u_2 \prec x_2 \prec \cdots \prec u_{n-1} \prec x_{n-1} \prec u_n$ of $\mathrm{CF}(n)$. Similarly by Corollary 3.8, suppose $F_2 = C_0'']_{a_1'}^{b_1'}\{c_{r_1}\}]_{a_2'}^{b_2'}\{c_{r_2}\}\cdots]_{a_l'}^{b_l'}\{c_{r_l}\}$, where for each s, $1 \leq s \leq l \leq N = \binom{n}{2}$, $a_s' = u_{i'}, b_s' = u_{j'}$ with $r_s = (i'-1)n - \binom{i'}{2} + j' - i'$, $1 \leq i' < j' \leq n$, $(r_1 < r_2 < \cdots < r_l)$, and C_0'' is the chain containing chain $C : u_1 \leq u_2 \leq \cdots \leq u_n$ of all reducible elements of $\mathrm{CF}(n)$, and all those $x_i \in X = C_0 \setminus C$ which satisfy $x_i || c_{r_s}$ in F_2 for $r_s = (i-1)n - \binom{i}{2} + 1$.

As $\overrightarrow{G_1} \cong \overrightarrow{G_2}$, $\psi_{F_1}(\mathscr{A}_{F_1}) \cong \psi_{F_2}(\mathscr{A}_{F_2})$. As ψ is bijective, $\psi^{-1}(\psi_{F_1}(\mathscr{A}_{F_1})) = \psi^{-1}(\psi_{F_2}(\mathscr{A}_{F_2}))$. That is, $(\psi^{-1} \circ \psi_{F_1})(\mathscr{A}_{F_1}) = (\psi^{-1} \circ \psi_{F_2})(\mathscr{A}_{F_2})$. Hence $\mathscr{A}_{F_1} = \mathscr{A}_{F_2}$, since $\psi^{-1} \circ \psi_{F_1}$ and $\psi^{-1} \circ \psi_{F_2}$ are identity maps on $\mathscr{A}_{F_1}$ and $\mathscr{A}_{F_2}$ respectively. As $\mathscr{A}_{F_1} = \mathscr{A}_{F_2}$, we have $U \cup I_{F_1} = U \cup I_{F_2}$. This implies that $I_{F_1} = I_{F_2}$, *that is*, $\{[a_s, b_s] | 1 \leq s \leq l\} = \{[a_s', b_s'] | 1 \leq s \leq l\}$.

Now suppose for fixed s where $1 \leq s \leq l$, $[a_s, b_s] \cong [a_t', b_t']$ for some $t, 1 \leq t \leq l$. Then $a_s = a_t'$ and $b_s = b_t'$. But $a_s \prec c_{q_s} \prec b_s$ and $a_t' \prec c_{r_t} \prec b_t'$. Therefore $c_{q_s} = c_{r_t}$, and hence $q_s = r_t$. Thus for all $s, 1 \leq s \leq l$, there exists t, where $1 \leq t \leq l$ such that $c_{q_s} = c_{r_t}$. Therefore $Y_{F_1} = Y \cap F_1 \subseteq Y_{F_2} = Y \cap F_2$, where $Y = \{c_k | k \in J_N\}$. Similarly, we can prove that $Y_{F_2} \subseteq Y_{F_1}$. Therefore $Y_{F_1} = Y_{F_2}$. Hence by Remark 3.9, $F_1 \cong F_2$. Thus ϕ is an injective map. □

Note that, for every $\overrightarrow{G} \in \overrightarrow{\mathscr{D}}(n,l)$, there is a unique $G \in \mathscr{D}(n,l)$ which is obtained from $\overrightarrow{G}$ by removing direction of each edge $\overrightarrow{e} \in \overrightarrow{G}$. Also for every $G \in \mathscr{D}(n,l)$, there is a unique $\overrightarrow{G} \in \overrightarrow{\mathscr{D}}(n,l)$ with $V(\overrightarrow{G}) = V(G)$, and $(v_i, v_j) = \overrightarrow{e_k}$ is a directed edge in $\overrightarrow{G}$ whenever $k = (i-1)n - \binom{i}{2} + j - i$, $1 \leq i < j \leq n$. Thus there is a one-to-one correspondence between $\mathscr{D}(n,l)$ and $\overrightarrow{\mathscr{D}}(n,l)$. Thus by Theorem 4.6, we get the following main result.

Theorem 4.7. *For $n \geq 0$ and for fixed l, $\lfloor \frac{n+1}{2} \rfloor \leq l \leq \binom{n}{2}$, there is a one-to-one* *correspondence between* $\mathscr{F}_n(l)$ *and* $\mathscr{D}(n, l)$*, that is,* $f(n, l) = d(n, l)$*.*

Thus from Theorem 4.7, it clearly follows that the sequences $(f(n, l))$ and $(d(n, l))$ are equivalent.

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