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Forbidding monochromatic odd cycles and rainbow cycles in complete graphs

Derrick DeMars* and Peter Johnson

ABSTRACT

A k-edge coloring c of the edge set $E(G)$ of a graph G is a surjective mapping $c : E(G) \to [k] = \{1, 2, \ldots, k\}$. If $\mathcal{F}$ and $\mathcal{H}$ are families of graphs, $\mathrm{MRS}(K_n; \mathcal{F}, \mathcal{H})$ is the set of numbers k such that there is a k-edge coloring of K_n with respect to which there is neither a monochromatic copy of any $F \in \mathcal{F}$ nor a rainbow copy of any $H \in \mathcal{H}$ in K_n. Our main result is that for all $n \geq 2$, $\mathrm{MRS}(K_n; \{\text{odd cycles}\}, \{\text{cycles}\}) = \{\lceil \log_2 n \rceil, \ldots, n-1\}$. The proof will exploit an idea for edge-coloring connected graphs so as to forbid rainbow cycles to be found in [4].

Keywords: Gallai-colorings, jl-colorings, monochromatic odd cycles, rainbow cycles, binary trees, mixed ramsey spectrum

2020 Mathematics Subject Classification: 05C55, 05D10.

1. Introduction

All graphs here are finite and simple. See Abstract for the definition of a k-edge coloring of a graph G.

A subgraph of an edge colored graph is called *monochromatic* if all its edges have the same color. A subgraph of an edge colored graph is called *rainbow* (also referred to as *polychromatic* or *totally multicolored*) if no color appears more than once on any of its edges.

In [4], Hoffman et al. define an edge coloring of a graph G as *rainbow-cycle-forbidding* if no cycle in G is rainbow with respect to that coloring. They also define a *JL-coloring* as a rainbow-cycle-forbidding edge coloring for a given graph G on n vertices with p components in which the maximum possible number of colors, $n - p$, appear. By the

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* Corresponding author.

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main result in [4], JL-colorings forbid monochromatic odd cycles.

Edge colorings of complete graphs which forbid rainbow K_3's are known as *Gallai colorings*. All Gallai colorings are rainbow-cycle-forbidding. We will use the construction of a JL-coloring in [4] to produce non-JL-colorings (i.e., fewer than $n-1$ colors may be used) of $E(K_n)$ which forbid rainbow cycles; it will be clear from the construction that the subgraph of K_n induced by each color class of edges is bipartite, and that, therefore, no color class can contain an odd cycle. Hence, no odd cycle in K_n will be monochromatic.

Suppose F is a graph with no isolated vertices. Suppose that $k \geq 1$ is an integer. The *Ramsey number* $R_k(F)$ is the minimum positive integer n such that for every k-edge coloring of K_n, there is a monochromatic copy of F in K_n.

In [5], Magnant and Nowbandegani define the *Gallai-Ramsey number* $gr_k(G : H)$ to be the minimum integer N such that for all $n \geq N$, for every k-edge coloring of K_n, K_n contains either a rainbow copy of G or a monochromatic copy of H. If $\mathcal{G}, \mathcal{H}$ are non-empty families of graphs, none with isolated vertices, and k is a positive integer, $gr_k(\mathcal{G} : \mathcal{H})$ is the minimum integer N such that for all $n \geq N$, for every k-edge coloring of K_n, K_n contains either a rainbow $G \in \mathcal{G}$ or a monochromatic $H \in \mathcal{H}$. Clearly, for all $G \in \mathcal{G}$ and $H \in \mathcal{H}$,

$$gr_k(\mathcal{G} : \mathcal{H}) \leq \min gr_k(G : H) \leq \min R_k(H).$$

As a corollary of our main result, Corollary 3.10, we shall show that $gr_k(\{\text{odd cycles}\} : \{\text{cycles}\}) \leq 2^k + 1$ for all $k \geq 3$. Defined by Axenovich and Iverson [1], the *mixed Ramsey spectrum* $\mathrm{MRS}(K_n; F, H)$ is the set of numbers k such that for some k-edge coloring of K_n, there is neither a monochromatic copy of $F \subseteq K_n$ nor a rainbow copy of $H \subseteq K_n$. More generally, if $\mathcal{F}, \mathcal{H}$ are families of graphs then $\mathrm{MRS}(K_n; \mathcal{F}, \mathcal{H})$ is the set of numbers k such that for some k-edge coloring of K_n, there is no monochromatic copy of any $F \subseteq \mathcal{F}$ in K_n nor any rainbow copy of any $H \subseteq \mathcal{H}$ in K_n. Our main result, mentioned in the abstract, determines $\mathrm{MRS}(K_n; \{\text{odd cycles}\}, \{\text{cycles}\})$ for all $n \geq 2$.

2. Balanced binary trees

A binary tree is a rooted tree in which every vertex has at most two children. A vertex of a binary tree with no children is called a leaf, and any other vertex is called an internal vertex. In what follows, we focus on a particular class of binary trees defined below.

A *balanced binary tree*, T, is an acyclic connected graph with a root vertex, v, and descendants presenting exclusively in pairs known as *siblings* or *children*. The vertex v is the only vertex to be located on level zero. The children of v, v_0 and v_1, are on level one, v_0's and v_1's children are on level two, and so forth. The vertex v_w (where w is a binary word) does not have children unless all the vertices on the previous level have children. The number of levels is known as the *height* of T. T's final children are known as leaves and they are all located on the last two levels; see Figure 1.

Given two vertices X and Y of a binary tree T, we say X is an ancestor of Y if X appears on a level of T preceding that of Y and there is a path in T from X to Y. (In particular, the root is an ancestor of every other vertex of T.)

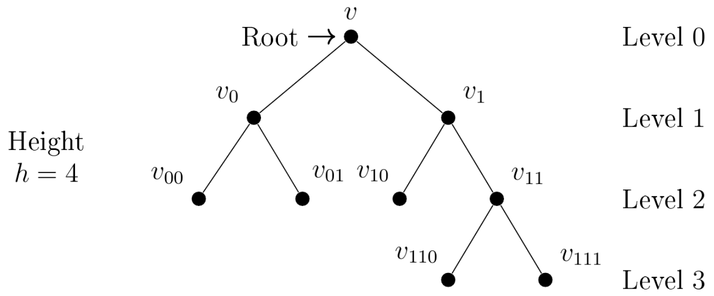

Fig. 1. A balanced binary tree on nine vertices with five leaves, four levels, and a height of four.

Lemma 2.1. *The height of a balanced binary tree with* $n \geq 2$ *leaves is* $\lceil \log_2 n \rceil + 1$.

Proof. Let T be a balanced binary tree with $n \geq 2$ leaves, and let $\ell \geq 1$ denote the deepest level of T. By the definition of a balanced binary tree, every level from 0 to $\ell - 1$ is full (i.e., every vertex on these levels has two children), while level ℓ contains $2t$ vertices, arranged in sibling pairs, for some $t \in \{1, 2, \ldots, 2^{\ell-1}\}$. Among the $2^{\ell-1}$ vertices on level $\ell - 1$, exactly t have children, and the remaining $2^{\ell-1} - t$ are leaves. Hence the total number of leaves is

$$n = \left(2^{\ell-1} - t\right) + 2t = 2^{\ell-1} + t.$$

Since $1 \leq t \leq 2^{\ell-1}$, this gives $2^{\ell-1} + 1 \leq n \leq 2^{\ell}$, and therefore $\lceil \log_2 n \rceil = \ell$. Because we count levels starting at 0, the height of T (the total number of levels) is $\ell + 1 = \lceil \log_2 n \rceil + 1$. (For example, in Figure 1 we have $n = 5$, so $\ell = \lceil \log_2 5 \rceil = 3$ and the height is 4, in agreement with the figure.) □

In a balanced binary tree on more than 3 vertices, the root is the unique vertex of degree 2; every other internal vertex has degree 3 (one parent and two children); and every leaf has degree 1.

Lemma 2.2. *A balanced binary tree with* n *leaves has* $2n - 1$ *vertices.*

Proof. Let q be the number of vertices of the tree. Then summing the degrees of those vertices, will give us $2(q - 1)$, where $q - 1$ is the number of edges. We have $2(q - 1) = n + 2 + 3(q - n - 1)$; solving we get $q = 2n - 1$. □

3. Proof of main result

Our main result is Corollary 3.9 and follows from Theorems 3.5 and 3.8.

It is well known (see [4]) that every edge coloring of K_n in which at least n colors appear contains a rainbow cycle. On the other hand, rainbow-cycle-forbidding colorings of K_n using $n - 1$ colors do exist, and every such coloring forbids monochromatic odd cycles. Hence $n - 1$ is the largest integer in $\mathrm{MRS}(K_n; \{\text{odd cycles}\}, \{\text{cycles}\})$. We show that every integer between $\lceil \log_2 n \rceil$ and $n - 1$ is in $\mathrm{MRS}(K_n; \{\text{odd cycles}\}, \{\text{cycles}\})$. If

$X, Y \subseteq V(K_n)$ are disjoint, then $[X, Y]$ denotes the set of edges in K_n with one end in X and one end in Y.

Theorem 3.1 (Gallai's theorem [3]). *Suppose $n \geq 3$. Let k be a positive integer. In any k-edge coloring of K_n where there is no rainbow $K_3 \subseteq K_n$, there exists a partition of $V(K_n)$ into non empty subsets $V_1, V_2, \ldots, V_t$ ($t \geq 2$) such that*

(1) *for each pair i, j of integers with $1 \leq i < j \leq t$, all edges in $[V_i, V_j]$ are colored the same color;*

(2) *the number of colors on the edges in the set $\bigcup_{1 \leq i < j \leq t} [V_i, V_j]$ is at most 2; and*

(3) *for each $l \in \{1, \ldots, t\}$, no edge within the complete graph induced by V_l is colored with any of the $[V_i, V_j]$ colors.*

Lemma 3.2. *Suppose that G is a connected graph on $n > 1$ vertices. Then $V(G)$ can be partitioned into sets, A and B, satisfying the following*

(1) $||A| - |B|| \leq 1$, *and*

(2) *$G[A]$ and $G[B]$ are connected,*

if and only if G has a spanning tree T such that for some edge $e \in E(T)$ such that, if T_1 and T_2 denote the two components of $T - e$, then $||V(T_1)| - |V(T_2)|| \leq 1$.

Proof. Suppose A and B partition $V(G)$, with $||A| - |B|| \leq 1$, and $G[A]$ and $G[B]$ are connected. Note that $n > 1$ implies that $A \neq \emptyset \neq B$. Let T_1 and T_2 be spanning trees in $G[A]$ and $G[B]$ respectively, so $A = V(T_1)$ and $B = V(T_2)$. Because G is connected, there must exist an edge e with one end in A and the other in B. Then $T = T_1 \cup T_2 \cup e$ is a tree satisfying the requirements given in the Lemma.

Conversely, if T and e satisfy those requirements, let $A = V(T_1)$ and $B = V(T_2)$. Then $||A| - |B|| \leq 1$, $G[A]$ and $G[B]$ have spanning trees T_1, T_2, respectively, and are therefore connected. □

Theorem 3.3 ([4]). *If G is a connected graph on n vertices, then there is a rainbow cycle forbidding edge coloring of G with $n - 1$ colors appearing.*

Lemma 3.4. *Suppose $n > 1$. An edge coloring of K_n is rainbow-cycle-forbidding if and only if it is a Gallai coloring.*

Proof. The forward implication is clear, since K_3 is a cycle. Now suppose that we have a coloring of the edges of K_n which is not rainbow-cycle-forbidding. We aim to show that there is a rainbow $K_3 = C_3$ in K_n, with respect to this coloring.

Let m be the smallest integer such that there is a rainbow C_m in K_n. If $m = 3$, we are done. Otherwise, consider any chord uv of this C_m. The two edge-disjoint paths along C_m with ends u and v, each combined with the chord uv, form two cycles, each of order less than m. The color of uv can appear on at most one of those two uv paths, because the C_m is rainbow; but then there exists a smaller rainbow cycle in K_n, contradicting the choice of m. $\square$

Theorem 3.5. *For positive integers $n > 1$ and k, if $2^{k-1} < n \leq 2^k$ then $[k, \dots, n-1] \subseteq$ $\mathrm{MRS}(K_n; \{\textit{odd cycles}\}, \{\textit{cycles}\})$.*

Proof. We will construct a balanced binary tree T representing a Gallai coloring c. The vertices of T will be subsets of $V(K_n) = V$. The root will be the entire vertex set V. For each vertex $X \subseteq V$ of T, if $|X|= 1$ then X is a leaf of T. Otherwise, if $|X|> 1$, the two "children" of X at the next "level" of T will be sets Y, Z partitioning X, such that $||Y|-|Z||\leq 1$. We will refer to Y and Z as "siblings."

The edges of K_n will be colored as follows: for every pair of sibling vertices Y and Z of T, the edges $[Y, Z]$ will be colored with a single color that does not appear on any previously colored edge incident to a vertex in an ancestor of Y and Z. Such a color always exists: at each step of the construction only finitely many colors have been used so far, so we may always introduce a fresh color (one not previously used) for the pair (Y, Z).

We will enforce this restriction by the requirement that the sets of colors appearing on edges between siblings at different levels be disjoint. Thus, a color may appear on edges between different pairs of siblings, but it may not appear on edges between different pairs of siblings on different levels.

This requirement is stronger than necessary, but it is sufficient for our purposes. We shall see that every such coloring forbids rainbow cycles and monochromatic odd cycles, and the total number of colors appearing can range from $\lceil \log_2 n \rceil$ to $n-1$.

To see this last claim, first note that the binary tree constructed will have n leaves, one for each vertex of K_n. Therefore, by Lemma 2.2, it will have $n-1$ non-leaves, and each of these will have two sibling children, the edges between the sets of vertices corresponding to which will bear one of our colors. Thus we can arrange to have $n-1$ colors appear in the coloring by making the colors assigned to the $n-1$ sibling pairs distinct.

Now we can reduce the number of colors, one at a time, while honoring the requirement that the sets of colors assigned to the sets of sibling pairs at different levels be disjoint, by merging pairs of colors on the same level. For instance, if, at some stage, blue and burgundy both appear on (edges between sibling pairs on) the same level (and therefore on no other level), we can recolor all burgundy edges blue, thus reducing the total number of colors appearing by one while preserving the disjointness of color sets on different levels.

We can continue counting down in this way until on each level after Level 0 only one color is assigned to sibling pairs on that level. At that point the number of different colors deployed is one less than the number of levels. By Lemma 2.1 the number of levels is $\lceil \log_2 n \rceil + 1$, so the number of colors is $(\lceil \log_2 n \rceil + 1) - 1 = \lceil \log_2 n \rceil$ (Recall that $n > 1$.) (Note that this number has not yet been shown to equal min $\mathrm{MRS}(K_n; \{\text{odd cycles}\}, \{\text{cycles}\})$; that will be established in Theorem 3.8.)

Observe that for any color appearing in any colorings obtained above, the subgraph of K_n induced by the set of edges bearing that color is union of vertex-disjoint complete bipartite graphs. Therefore, there are no monochromatic odd cycles in K_n with any of these colorings.

It remains to be seen that none of the colorings described allow a rainbow cycle in K_n. Let C be a cycle in K_n. Let X be a vertex of T (therefore, a subset of $V(K_n)$) such that $V(C) \subseteq X$ and, if Y and Z are the children of X, then $V(C) \cap Y \neq \emptyset \neq V(C) \cap Z$. In other words, X is the vertex of T on the lowest level of T, among vertices of T containing $V(C)$. Since C is a cycle, $E(C) \cap [Y, Z]$ must contain at least two edges. Therefore C is not rainbow. □

Theorem 3.6. $R_2(K_3) = 6$ [2].

Lemma 3.7. $\min \mathrm{MRS}(K_4; \{\textit{odd cycles}\}, \{\textit{cycles}\}) = 2$ *and*

$$\min \mathrm{MRS}(K_5; \{\textit{odd cycles}\}, \{\textit{cycles}\}) = 3.$$

Proof. Clearly $\min \mathrm{MRS}(K_4; \{\text{odd cycles}\}, \{\text{cycles}\}) > 1$, and Figure 2 gives two different edge colorings of K_4 with two colors that admit no monochromatic K_3's and no rainbow cycles.

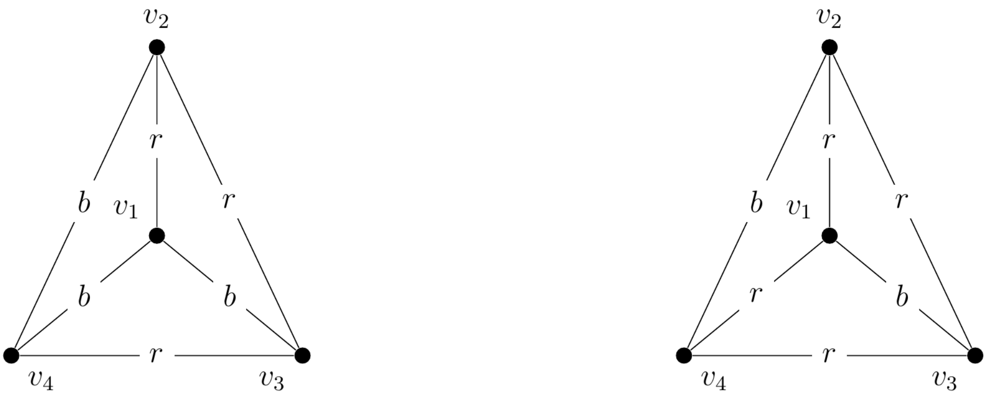

Fig. 2. Two essentially different 2-edge colorings of K_4 that forbid monochromatic odd cycles.

Now suppose that the edges of K_5 are colored with red and blue so that no odd cycle in K_5 is monochromatic. Suppose that K_5 contains a monochromatic $K_{1,3}$ – suppose edges vx, vy, vz are colored red. If any of xy, xz, yz were red then there would be a red C_3 in the edge colored K_5. Therefore all three of those edges are blue, so we have a monochromatic C_3 anyway. So no monochromatic K_3 exists.

It follows that every vertex of K_5 is incident to two red and to two blue edges. The subgraph induced by the blue edges is therefore regular of degree two, so there must be a blue cycle in K_5. It must be a C_4, say on vertices v, w, x, y. Let z be the vertex of K_5 not in this C_4. Of the four edges incident to z, two are blue, so there must be vertices of K_5 incident to three blue edges, a possibility that has already been ruled out. Thus no such coloring exists.

Since $3 = \lceil \log_2 5 \rceil$ there is an edge coloring of K_5 with 3 colors which forbids monochromatic odd cycles and rainbow cycles by Theorem 3.5. □

Theorem 3.8. $\min \mathrm{MRS}(K_n; \{\textit{odd cycles}\}, \{\textit{cycles}\}) = \lceil \log_2 n \rceil$.

Proof. The proof will be by induction on n. Assume that $n \geq 6$ and K_n is edge colored with k colors appearing so that rainbow cycles and monochromatic odd cycles are forbidden. Since there are no rainbow K_3's in K_n, the coloring must be a Gallai coloring as described in Gallai's Theorem. The number of parts t in the partition of $V(K_n)$ in that description must be less than 6 as we know $R(K_3, K_3) = 6$ from Theorem 3.6. By Lemma 3.7 we know $t \neq 5$. So $t \in \{2, 3, 4\}$.

We lifted the following argument from Magnant and Salehi Nowbandegani [5]. Let $t = 3$. Since we are forbidding monochromatic odd cycles, we must use two colors among the three partitions V_1, V_2, and V_3 as depicted in Figure 3. This puts us in the $t = 2$ case; see Figure 4.

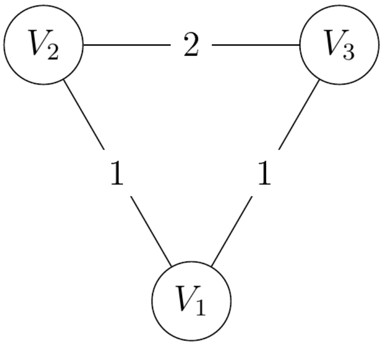

Fig. 3. A Gallai partition with three parts and two interpart colors

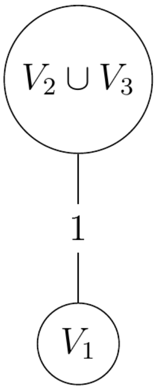

Fig. 4. A Gallai partition with two parts and one interpart color

We are left with cases $t \in \{2, 4\}$. Suppose $t = 2$ with V partitioned by V_1 and V_2. Let $n_1 = |V_1|$, $n_2 = |V_2|$, and $n_1 \leq n_2$. Let c_1 and c_2 be the number of colors in V_1 and V_2 respectively with $c_1 \geq \lceil \log_2 n_1 \rceil$ and $c_2 \geq \lceil \log_2 n_2 \rceil$. Now $n_2 \geq \frac{n}{2}$, so we have

$$k \geq c_2 + 1 \geq \lceil \log_2 n_2 \rceil + 1 \geq \left\lceil \log_2 \frac{n}{2} \right\rceil + 1 = \lceil \log_2 n \rceil - \lceil \log_2 2 \rceil + 1 = \lceil \log_2 n \rceil.$$

Now suppose $t = 4$ with a partition by vertex sets V_1, V_2, V_3, and V_4. Let $n_1 = |V_1|$, $n_2 = |V_2|$, $n_3 = |V_3|$, $n_4 = |V_4|$, and $n_1 \leq n_2 \leq n_3 \leq n_4$. Let c_1, c_2, c_3, and c_4 be the number of colors in V_1, V_2, V_3, and V_4 respectively with $c_1 \geq \lceil \log_2 n_1 \rceil$, $c_2 \geq \lceil \log_2 n_2 \rceil$, $c_3 \geq \lceil \log_2 n_3 \rceil$, and $c_4 \geq \lceil \log_2 n_4 \rceil$. Now $n_4 \geq \frac{n}{4}$, and

$$k \geq c_4 + 2 \geq \lceil \log_2 n_4 \rceil + 2 \geq \left\lceil \log_2 \frac{n}{4} \right\rceil + 2 = \lceil \log_2 n \rceil - \lceil \log_2 4 \rceil + 2 = \lceil \log_2 n \rceil.$$

□

Corollary 3.9. $\mathrm{MRS}(K_n; \{\textit{odd cycles}\}, \{\textit{cycles}\}) = \{\lceil \log_2 n \rceil, \dots, n-1\}$.

Proof. This follows from Theorems 3.5 and 3.8. □

Corollary 3.10. *If* $k \geq 3$, $gr_k(\{\textit{odd cycles}\} : \{\textit{cycles}\}) \leq 2^k + 1$.

Proof. If $n \geq 2^k+1$ then $\lceil \log_2 n \rceil \geq \log_2 n > k$ so, by Corollary 3.9, every k-edge coloring of K_n must either contain a rainbow cycle or a monochromatic odd cycle.

By previous arguments (see Lemma 3.7 it can be seen that Corollary 3.10 holds with equality when $k = 2$. □

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Derrick DeMars
Auburn University, Alabama 36849, USA
E-mail dad0044@auburn.edu

Peter Johnson
Auburn University, Alabama 36849, USA
E-mail johnspd@auburn.edu