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Weak 2-reconstruction in hyper-connected topological spaces

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ABSTRACT

The *deck* of a topological space X, denoted $\mathcal{D}(X) = \{[X_x] : x \in X\}$, comprises the homeomorphism classes of subspaces obtained by removing a single point x from X. A space X is *topologically reconstructible* if $\mathcal{D}(X) = \mathcal{D}(Y)$ implies that X is homeomorphic to Y. The n-*deck* of X, defined as $\mathcal{D}_n(X) = \{[X - \{x_1, x_2, \ldots, x_n\}] : x_1, x_2, \ldots, x_n \in X\}$, generalizes this concept to the removal of n points. A space X is *topologically* n-*reconstructible* if $\mathcal{D}_n(X) = \mathcal{D}_n(Y)$ implies $X \cong Y$. The *multi* n-*deck* of X includes all n-cards, representing every possible homeomorphism type. A space is *weakly* n-*reconstructible* if it can be uniquely determined from its multi n-deck. This paper establishes that certain hyper-connected topological spaces are weakly 2-reconstructible.

Keywords: reconstruction, card, deck, homeomorphism, hyper-connected

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1. Introduction

In graph theory, a *vertex-deleted subgraph*, or *card* $G - v$, of a graph G is formed by removing a vertex v along with all edges incident to it. The *deck* of G is the collection of all such cards. A graph H is a *reconstruction* of G if H shares the same deck as G, and G is *reconstructible* if it is isomorphic to every graph with the same deck. A graph parameter p is reconstructible if it remains consistent across all reconstructions of G. The well-known graph reconstruction conjecture, proposed by Kelly and Ulam [12] in 1941, asserts that every graph G with at least three vertices is reconstructible. Formally, if G and H are finite graphs with $|V(G)| \geq 3$ and $\mathcal{D}(H) = \mathcal{D}(G)$, then $G \cong H$. For reconstructible graphs, Harary and Plantholt [6] defined the reconstruction number

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$rn(G)$ as the minimum number of cards required to uniquely identify G among all non-isomorphic graphs.

In 2016, Pitz and Suabedissen [11] extended the reconstruction concept to topological spaces. For a topological space X, the subspace $X - \{x\}$, denoted X_x, is a *card* of X, and the deck is $\mathcal{D}(X) = \{[X_x] : x \in X\}$, where $[X_x]$ represents the homeomorphism class of X_x. A space Y is a *reconstruction* of X if $\mathcal{D}(X) = \mathcal{D}(Y)$, and X is *reconstructible* if $\mathcal{D}(X) = \mathcal{D}(Y)$ implies $X \cong Y$. A topological property $\mathcal{P}$ is reconstructible if $\mathcal{D}(X) = \mathcal{D}(Y)$ implies that X satisfies $\mathcal{P}$ if and only if Y does.

Kelly [9] generalized the graph reconstruction conjecture to multiple vertex deletions:

Conjecture 1.1 (Kelly)**.** *For each $k \in \mathbb{N}$, there exists an integer M_k such that any graph with at least M_k vertices is reconstructible from its deck of cards obtained by deleting k vertices. The original conjecture posits $M_1 = 3$. See [2] for further details.*

For a topological space X, an *n-card* is a subspace $X - \{x_1, x_2, \ldots, x_n\}$, and the *$n$-deck* is $\mathcal{D}_n(X) = \{[X - \{x_1, x_2, \ldots, x_n\}] : x_1, x_2, \ldots, x_n \in X\}$. A space X is *n-reconstructible* if $\mathcal{D}_n(X) = \mathcal{D}_n(Y)$ implies $X \cong Y$, and a property P is n-reconstructible if $\mathcal{D}_n(X) = \mathcal{D}_n(Y)$ implies that X possesses P if and only if Y does. Reconstructible spaces are also termed 1-reconstructible. The *multi n-deck* of a space consists of all n-cards of X, encompassing every homeomorphism type. A space X is *weakly n-reconstructible* if it can be uniquely reconstructed from its multi n-deck.

Gartside et al. [5, 4, 11] proved that spaces such as the real numbers, rational numbers, irrational numbers, compact Hausdorff spaces with a card having a maximal finite compactification, and Hausdorff continua X with weight $w(X) < |X|$ are reconstructible. They also established that properties like hereditary separation axioms and cardinal invariants are reconstructible. Similarly, Manvel et al. [10] showed that finite sequences are reconstructible from their subsequences. Recently, Jini and Monikandan [1] demonstrated that all finite topological spaces are weakly reconstructible.

In contrast, Pitz and Suabedissen [11] proved that the Cantor set is not reconstructible and that properties such as connectedness, compactness, Lindelöfness, countable compactness, and pseudocompactness are not reconstructible.

Unless otherwise specified, terminology in this paper adheres to [3]. A space is *hyper-connected* if no two non-empty open sets are disjoint. We previously showed that hyper-connected spaces with an isolated point, as well as those without isolated points but containing a finite open set, are reconstructible [8]. This paper builds on these findings by proving that hyper-connected spaces, regardless of whether they have an isolated point or a 2-point open set, are weakly 2-reconstructible. Furthermore, we demonstrate that hyper-connected spaces without isolated points but possessing either a 2-point open set or a finite open set of at least three points are weakly 2-reconstructible.

2. Weak 2-reconstruction

Any space X with $|X|= n+1$ is not n-reconstructible. Consider a topological space $X = \{a_1, a_2, \ldots, a_{n+1}\}$ with topologies $\tau_X^1 = \{U_1, U_2, \ldots, U_k\}$ and $\tau_X^2 = \{V_1, V_2, \ldots, V_l\}$ for some $k, l \in \mathbb{N}$. All n-cards are homeomorphic, as removing n points leaves a single-point space with the indiscrete topology. Thus, distinct non-homeomorphic topologies on X may yield identical n-decks. For example, consider $\tau_X^1 = \{\emptyset, \{a_1\}, \{a_1, a_2\}, \ldots, \{a_1, a_2, \ldots, a_{n+1}\}\}$ and $\tau_X^2 = \{\emptyset, \{a_1, a_2\}, X\}$. Here, $\mathcal{D}_n(\tau_X^1) = \mathcal{D}_n(\tau_X^2)$, yet $\tau_X^1 \not\cong \tau_X^2$. Consequently, we limit our analysis to spaces X with $|X|\geq n+2$.

This paper focuses exclusively on hyper-connected spaces with a cardinality of at least five.

Theorem 2.1 ([7]). *A space X with at least five elements is hyper-connected if and only if at least six of its 2-cards are hyper-connected. Thus, the property of hyper-connectedness in X is weakly 2-reconstructible.*

Lemma 2.2. *Let X be a hyper-connected space with $|X|\geq 5$. Then X has no isolated points if and only if one of the following holds:*

(a) *No 2-card has isolated points.*

(b) *Exactly one 2-card has more than one isolated point.*

(c) *Exactly three 2-cards have exactly one isolated point.*

Proof. *Necessity:* Suppose X has no isolated points. We examine three cases:

Case 1. If every non-empty open set in X contains at least four points, then every 2-card has open sets with at least two points, implying no 2-card has isolated points. Thus, (a) holds.

Case 2. If X has a 3-point open set but no 2-point open set, hyper-connectedness ensures this 3-point set is unique. Removing points from this set yields exactly three 2-cards, each with one isolated point, satisfying (c).

Case 3. If X has both a 2-point and a 3-point open set, the 2-point set is unique due to hyper-connectedness, though multiple 3-point sets may exist. This splits into two subcases:

(3.1) If X has a single 3-point open set, removing points from it produces three 2-cards, each with one isolated point. The 2-card obtained by removing the 2-point set coincides with one of these, fulfilling (c).

(3.2) If X has multiple 3-point sets, the 2-card formed by removing the 2-point set has more than one isolated point, while others have at most one, satisfying (b).

Sufficiency: Assume (a) holds, but X has an isolated point. Since X is hyper-connected, this isolated point is unique. With $|X|\geq 5$, some 2-card includes this point, contradicting (a).

If (b) holds, suppose X has an isolated point (unique due to hyper-connectedness). Consider four cases:

Case 1. If X has no 2-point or 3-point open sets, every 2-card has at most one isolated point, contradicting (b).

Case 2. Suppose X has 3-point open sets but no 2-point open sets. As X is hyper-connected, each 3-point open set must contain the unique isolated point of X. Since X has no 2-point open sets, no two 3-point open sets can have two points in common. Therefore, the intersection of any two 3-point open sets is precisely a singleton set, which corresponds to the isolated point of X. Thus, every 2-card can have at most one isolated point, contradicting (b).

Case 3. Suppose X admits 2-point open sets but no 3-point open sets. Then X can have only one 2-point open set. Indeed, if X has two distinct 2-point open sets in X, they would necessarily share a common isolated point. Their union would then be a 3-point open set, contradicting our assumption that X has no 3-point open sets. Let $\{a\}$ be the isolated point and $\{a, b\}$ be the unique 2-point open set in X. Now every 2-card other than $X - \{a, b\}$, must contains only one isolated point a and $X - \{a, b\}$ has no isolated points. Hence, every 2-card can contain at most one isolated point, contradicting (b).

Case 4. If X has both 2-point and 3-point sets, a unique 2-point set exists. Subcases arise:

(1) If a 3-point set includes the isolated point and two others, multiple 2-cards have more than one isolated point, contradicting (b).

(2) If a 3-point set comprises the 2-point set plus one point, every 2-card has one isolated point, contradicting (b).

If (c) holds, but X has an isolated point, with $|X| \geq 5$, at least six 2-cards include it, contradicting (c). Thus, X has no isolated points. □

Theorem 2.3. *Let X be a hyper-connected space with $|X| \geq 5$. The property of whether X has isolated points is weakly 2-reconstructible.*

Proof. This follows directly from Lemma 2.2. □

Lemma 2.4. *Let X be a hyper-connected space without isolated points. Then X has no 2-point open sets if and only if one of the following holds:*

(a) *No 2-card has isolated points.*

(b) *Exactly three 2-cards have exactly one isolated point.*

Proof. *Necessity:* If X has no 2-point open sets, every open set has at least three points. If the smallest open set has four or more points, every 2-cardrs open sets have at least two points, satisfying (a). If X has a unique 3-point open set, (b) holds.

Sufficiency: If (a) holds, but X has a 2-point set $\{a, b\}$, with $|X| \geq 5$, consider $X - \{a, c\}$ (where $c \neq a, b$). Point b is isolated, contradicting (a). Thus, no 2-point set exists.

If (b) holds, but X has $\{a, b\}$, with points c, d, e distinct, each 2-card $X - \{x_1, x_2\}$ (where $x_1 \in \{a, b\}$, $x_2 \in \{c, d, e\}$) has an isolated point, yielding at least six such 2-cards, contradicting (b). Hence, no 2-point set exists. □

Theorem 2.5. *Let X be a hyper-connected space without isolated points and $|X|\geq 5$. The property of whether X has a 2-point open set is weakly 2-reconstructible.*

Proof. This follows from Lemma 2.4. □

Theorem 2.6. *Let X be a hyper-connected space without isolated points and $|X|\geq 5$. If X contains a 2-point open set, then it is weakly 2-reconstructible.*

Proof. Since X is hyper-connected with a 2-point open set $\{a, b\}$, this set is unique and contained in every open set. The 2-card $X - \{a, b\}$ in the multi 2-deck enables reconstruction of the topology: $\tau_X = \{\emptyset\} \cup \{U \cup \{a, b\} : U \in \tau_{X-\{a,b\}}\}$. □

Theorem 2.7. *Let X be a hyper-connected space with no isolated points or 2-point open sets, and $|X|\geq 5$. If X contains a finite open set with at least three points, then it is weakly 2-reconstructible.*

Proof. Previous theorems establish that hyper-connectedness, absence of isolated points, absence of 2-point sets, and presence of a finite open set with at least three points are weakly 2-reconstructible. Let U be the unique smallest open set with $|U|= n \geq 3$, contained in all open sets. In the multi 2-deck, 2-cards with $(n - 2)$-point open sets are minimal; others have at least $(n - 1)$-point sets. Adding two points to each open set of a 2-card with an $(n - 2)$-point set, plus $\emptyset$, reconstructs τ_X. □

Lemma 2.8. *Let X be a finite space. Then X contains a point not in any proper open set if and only if at least $\binom{|X|}{2} - |X|+1$ 2-cards have a point not in any proper open set.*

Proof. *Necessity:* Suppose X contains a point a that is not part of any proper open set. Now, consider a 2-card $X - \{b, c\}$, where b and c are distinct from a. This 2-card contains a, but no proper open set of $X - \{b, c\}$ includes a, according to the subspace topology. Additionally, since $(|X|-1)$ 2-cards do not contain a, there are $(|X|C_2) - (|X|-1)$ 2-cards with this property.

Sufficiency: Suppose that there exists a 2-card with a point not contained in any proper open set, and the number of such 2-cards is at least $|X|C_2 - |X|+1$. Now, suppose that every point in X is contained in a proper open set. Let $X = \{x_1, x_2, ..., x_n\}$. For any two points x_i and x_j, where $i, j \in 1, 2, ..., n$, the 2-card $X - \{x_i, x_j\}$ contains $n - 2$ points and its open sets are formed according to the subspace topology. Therefore, every point in every 2-card would be contained in at least one proper open set, which contradicts our assumption. Thus, X must contain a point that is not part of any proper open set. □

Theorem 2.9. *Let X be a finite space. The property that X contains a point not in any proper open set is weakly 2-reconstructible.*

Proof. This follows directly from Lemma 2.8. □

Theorem 2.10. *Let X be a finite hyper-connected space with an isolated point. If X contains a point not in any proper open set, then it is weakly 2-reconstructible.*

Proof. The properties of hyper-connectedness, having an isolated point, and containing a point not in any proper open set are weakly 2-reconstructible (Theorems 2.1, 2.3, 2.9). For a 2-card $X - \{x, y\}$, where x is the isolated point and y is not in any proper open set, the topology is $\tau_X = \{\emptyset, U \cup \{x\}, X - \{x, y\} \cup \{y\} : U \in \tau_{X-\{x,y\}}\}$. □

3. Conclusion

In this study, we have explored the weak 2-reconstruction of hyper-connected topological spaces, extending the theoretical framework established by Pitz and Suabedissen [11]. Our primary contribution lies in proving that hyper-connected spaces, under specific conditions such as the absence of isolated points or the inclusion of finite open sets of certain cardinalities, are weakly 2-reconstructible from their multi 2-decks. These findings enrich the discourse on topological reconstruction by applying reconstructibility principles to a class of spaces previously underexplored in this context.

Looking ahead, our research opens several promising directions for future investigation. One immediate extension would be to examine the weak n-reconstructibility of hyper-connected spaces for $n \geq 3$, which could reveal whether higher-order decks provide additional discriminatory power or introduce new challenges. Similarly, exploring hyper-connected spaces with different separation properties or degrees of connectedness might uncover further classes amenable to reconstruction, broadening the scope of our results.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability

This study is theoretical and does not involve the generation or analysis of datasets.

Author Contributions

All authors contributed equally to the preparation of this manuscript. All authors have read and approved the final version of the manuscript for publication.

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