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Article

Block Structured Hadamard matrices from certain arrays

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Abstract: We have constructed Block structured Hadamard matrices in which odd number of blocks are used in a row (column). These matrices are different than those introduced by Agaian. Generalised forms of arrays developed by Goethals-Seidel, Wallis-Whiteman and Seberry-Balonin heve been employed. Such types of matrices are applicable in the constructions of nested group divisible designs.

Keywords: Hadamard matrix, Kronecker product, block structure, group divisible design
Mathematics Subject Classification: 05B20, 62K10, 05B05.

1. Introduction

Hadamard matrices discovered by Sylvester in 1867 [1] have profound structural properties. These kind of matrices have Hadamard matrices as their sub blocks. In his matrices even number of Hadamard blocks were arranged in a row (or column). However in 1985 Agaian [2] introduced Hadamard matrices with odd number of Hadamard blocks in a row. These matrices are named Block Structured Hadamard matrices. He has demonstrated their application in signal processing as well [3]. The matrices constructed by him are in correspondence to the Williamson matrices arranged in his array.

Williamson's array has been further generalized by Goethals and Seidel [4]. Matrices suitable for Goethals- Seidel arrays have been constructed by several authors [5–7]. In 2015, Seberry and Balonin [8] introduced what is known as the "propus array" and its generalizations. This was accomplished by imposing certain restrictions on Williamson's array and applying the arrangement principles from the Goethals-Seidel array.

The result presented in this paper involves the construction of block-structured Hadamard matrices using the generalizations of Goethals-Seidel array and Seberry-Balonin array. Similar to that of Agaian, these matrices also have odd number of blocks arranged in a row. However these are different from Agaian's matrices, as these do not correspond to Williamson matrices. Furthermore, their applications in nested group divisible design are also discussed.

The paper is organized as follows: Section 2 contains all the relevant definitions and results. Section 3 comprises of the main result of this paper. Results are presented in three main theorems and two corollaries. Examples of each type are also included here. Section 4 is conclusion in which we have discussed some properties and an application of block structured Hadamard matrices in the construction of nested group divisible designs.

2. Preliminaries

We recall some basic definitions here [4, 9–12]. An *Hadamard matrix* is a square matrix H of order n with entries ± 1 such that $HH^\top = nI_n$. Four matrices A, B, C, D of order n with entries ± 1 are called *Williamson matrices* if (i) A, B, C, D are circulant, symmetric and commuting (ii)$A^2 + B^2 + C^2 + D^2 = 4nI_n$. If A, B, C, D of same order are circulant ± 1 matrices satisfying $AA^\top + BB^\top + CC^\top + DD^\top = 4nI_n$ then these are called Goethals-Seidel (GS) matrices. Dokovic et al. [6] have constructed GS matrices of type $(ssss)$, $(sssk)$, $(sskk)$, $(skkk)$ where 's' stand for symmetric and 'k' for skew type GS matrices. In this article we shall be using $(sssa)$ and $(kkka)$ type GS matrices, where 'a' stands for any (symmetric, skew type or other) type of matrix. An Hadamard matrix H of order $4tn$ will be called *Block Structured* if all its $4t \times 4t$ blocks are Hadamard matrices for a given t. If $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of order m and n respectively then the *Kronecker product* $A \otimes B$ is the matrix of order mn given by $A \otimes B = [a_{ij}B]$. Two matrices A and B are said to be *Amicable* if $AB^\top = BA^\top$.
Let the elements z_i of an additive abelian group G be ordered in a fixed way. Let $X \subset G$. Then the matrix $M = (m_{ij})$ defined by

$$m_{ij} = \psi(z_j - z_i), \;\; where \;\; \psi(z_j - z_i) = \begin{cases} 1 & if \;\; z_j - z_i \in X, \\ 0 & otherwise, \end{cases}$$

is called *type 1* incidence matrix of X in G, and the matrix $N = (n_{ij})$ defined by

$$n_{ij} = \psi(z_j + z_i), \;\; where \;\; \psi(z_j + z_i) = \begin{cases} 1 & if \;\; z_j + z_i \in X, \\ 0 & otherwise, \end{cases}$$

is called *type 2* incidence matrix of X in G. A *Circulant* matrix $M = (m_{ij})$ defined by $m_{ij} = m_{1,j-i+1}$ is a special case of type 1 matrix and a *backcirculant* matrix $N = (n_{ij})$ defined by $n_{ij} = n_{1,i+j-1}$ is a special case of type 2 matrix. Following proposition is useful in development of the results in this paper.

Proposition 1. *[Seberry] If X and Y are type 1 matrices and Z is type 2 matrix then*
$XY = YX$, $X^\top Y = YX^\top$, $XY^\top = Y^\top X$, $X^\top Y^\top = Y^\top X^\top$ *and*
$XZ^\top = ZX^\top$, $XZ = Z^\top X^\top$, $X^\top Z^\top = ZX$, $X^\top Z = Z^\top X$.

Notation: Throughout this paper I_n denotes the identity matrix of order n and J_n denotes the all-1 matrix of order n. Wherever I and J are used without subscript, they have the same meaning and their orders can be determined from the context.

3. Main Result

Lemma 1. *Let $A_i, i = 0, 1, 2, 3$ be circulant matrices of order n and let $R = [r_{ij}]$ be defined as $r_{i,n-i+1} = 1, r_{ij} = 0$ otherwise. Then*

1. $(A_iR)^\top = (A_iR)$ & $(A_i^\top R)^\top = (A_i^\top R), i = 1, 2, 3$
2. $A_0(A_iR)^\top = (A_iR)A_0^\top$, *i=1, 2, 3*
3. $A_0(A_i^\top R)^\top = (A_i^\top R)A_0^\top$, *i=1, 2, 3*
4. $(A_iR)(A_j^\top R)^\top = (A_j^\top R)(A_iR)^\top, i, j = 1, 2, 3$
5. $(A_iR)(A_{i+j}R)^\top = (A_{i+j}^\top R)(A_i^\top R)^\top$ & $(A_{i+j}^\top R)(A_i^\top R)^\top = (A_i^\top R)(A_{i+j}^\top R)^\top, i, j = 1, 2,\ i + j \le 3$

Proof. Using Proposition (1). □

Using this lemma following Theorem can be proven easily.

Theorem 1. *Let there exist* $(0,1,-1)$*-matrices* $X_i, Y_j; 0 \le i \le 3, 1 \le j \le 3$ *of order* $4t; (t \in \mathbb{N})$ *such that*
(i) $X_0X_0^\top = X_iX_i^\top + Y_iY_i^\top = tI_{4t}; 1 \le i \le 3$
(ii) $X_0X_i^\top + X_iX_0^\top = 0 = X_0Y_i^\top + Y_iX_0^\top, i = 1,2,3$
(iii) $X_iY_i^\top = 0 = Y_iX_i^\top, \forall i = 1,2,3$
(iv) $X_iY_j^\top + X_jY_i^\top = 0 = Y_jX_i^\top + Y_iX_j^\top, 1 \le i \ne j \le 3$
(v) $X_iX_{i+j}^\top + Y_{i+j}Y_i^\top = 0 = X_{i+j}X_i^\top + Y_iY_{i+j}^\top, i, j = 1,2, i+j \le 3.$
Then we have
(1) If there exist Goethals-Seidel matrices $A_i, i = 0,1,2,3$ *of order n of the type (kkka) or (sssa) then there exists a block structured Hadamard matrix of order* $4nt$ *with Hadamard blocks of order* $4t$*.*
(2) If there exist type 2 matrix A_0 *and three type 1 matrices* A_i*;* $1 \le i \le 3$ *defined on the same Abelian group of order n such that* $\sum_{i=0}^{3} A_iA_i^\top = 4nI_n$ *then there exists a block structured Hadamard matrix of order* $4nt$ *with Hadamard blocks of order* $4t$*.*

Proof. (1) Define a matrix H by

$$H = A_0 \otimes X_0 + A_1R \otimes X_1 + A_1^\top R \otimes Y_1 + A_2R \otimes X_2 + A_2^\top R \otimes Y_2 + A_3R \otimes X_3 + A_3^\top R \otimes Y_3,$$

where A_is correspond to 's' or 'k' type for $i = 1,2,3$ and A_0 corresponds to 'a' type. Then

$$\begin{aligned}
HH^\top =& A_0A_0^\top \otimes X_0X_0^\top + \sum_{i=1}^{3}\{(A_iR)(A_iR)^\top \otimes X_iX_i^\top + (A_i^\top R)(A_i^\top R)^\top \otimes Y_iY_i^\top\}\\
&+ [\sum_{i=1}^{3}\{(A_0(A_iR)^\top \otimes X_0X_i^\top + (A_iR)A_0^\top \otimes X_iX_0^\top + A_0(A_i^\top R)^\top \otimes X_0Y_i^\top + (A_i^\top R)A_0^\top \otimes Y_iX_0^\top)\}\\
&+ \sum_{j=1}^{3}\sum_{i=1}^{3}\{(A_iR)(A_j^\top R)^\top \otimes X_iY_j^\top + (A_j^\top R)(A_iR)^\top \otimes Y_jX_i^\top\}\\
&+ \sum_{i=1}^{2}\{(A_iR)(A_{i+1}R)^\top \otimes X_iX_{i+1}^\top + (A_{i+1}R)(A_iR)^\top \otimes X_{i+1}X_i^\top\\
&\qquad + (A_i^\top R)(A_{i+1}^\top R)^\top \otimes Y_iY_{i+1}^\top + (A_{i+1}^\top R)(A_i^\top R)^\top \otimes Y_{i+1}Y_i^\top)\}\\
&+ \sum_{i=1}\{(A_iR)(A_{i+2}R)^\top \otimes X_iX_{i+2}^\top + (A_{i+2}R)(A_iR)^\top \otimes X_{i+2}X_i^\top\\
&\qquad + (A_i^\top R)(A_{i+2}^\top R)^\top \otimes Y_iY_{i+2}^\top + (A_{i+2}^\top R)(A_i^\top R)^\top \otimes Y_{i+2}Y_i^\top)\}]
\end{aligned}$$

$$\begin{aligned}
=& A_0A_0^\top \otimes X_0X_0^\top + \sum_{i=1}^{3} A_iA_i^\top \otimes (X_iX_i^\top + Y_iY_i^\top)\\
&+ [\sum_{i=1}^{3}\{A_0(A_iR)^\top \otimes (X_0X_i^\top + X_iX_0^\top) + A_0(A_i^\top R)^\top \otimes (X_0Y_i^\top + Y_iX_0^\top)\}\\
&+ \sum_{j=1}^{3}\sum_{i=1}^{3}(A_iR)(A_j^\top R)^\top \otimes (X_iY_j^\top + Y_jX_i^\top)\\
&+ \sum_{i=1}^{2}\{(A_iR)(A_{i+1}R)^\top \otimes (X_iX_{i+1}^\top + Y_{i+1}Y_i^\top) + (A_{i+1}R)(A_iR)^\top \otimes (X_{i+1}X_i^\top + Y_iY_{i+1}^\top)\}\\
&+ \sum_{i=1}\{(A_iR)(A_{i+2}R)^\top \otimes (X_iX_{i+2}^\top + Y_{i+2}Y_i^\top) + (A_{i+2}R)(A_iR)^\top \otimes (X_{i+2}X_i^\top + Y_iY_{i+2}^\top)\}]
\end{aligned}$$

$$
\begin{aligned}
=&A_0A_0^\top \otimes X_0X_0^\top + \sum_{i=1}^{3} A_iA_i^\top \otimes (X_iX_i^\top + Y_iY_i^\top) \\
&+ [0 + \{\sum_{i=j} A_i^2 \otimes (X_iY_i^\top + Y_iX_i^\top) + \sum_{i\neq j} A_iA_j \otimes (X_iY_j^\top + Y_jX_i^\top + X_jY_i^\top + Y_iX_j^\top)\} + 0 + 0]
\end{aligned}
$$

$$
\begin{aligned}
=&\{\sum_{i=0}^{4} A_iA_i^\top\} \otimes tI_{4t} \\
=&4ntI_{4nt}.
\end{aligned}
$$

Hence H is an Hadamard matrix. Now each $4t \otimes 4t$ block (say H_{ij}) is a linear combination of $X_i's$ and $Y_i's$ of the form
$H_{ij} = \sum_{i=0}^{3} \rho_i X_i + \sum_{j=1}^{3} \sigma_j Y_j, \rho_i, \sigma_j \in \{1, -1\}$.

Case 1: A_is are of the type $(kkka)$
Then $A_i^\top = -A_i$, $i = 1, 2, 3$
$\Rightarrow A_i^\top R = -A_iR$, $i = 1, 2, 3$
$\Rightarrow \rho_i = -\sigma_i$, $i = 1, 2, 3$.

Case 2: A_is are of the type $(sssa)$
Then $A_i^\top = A_i$, $i = 1, 2, 3$
$\Rightarrow A_i^\top R = A_iR$, $i = 1, 2, 3$
$\Rightarrow \rho_i = \sigma_i$, $i = 1, 2, 3$.
Now

$$
\begin{aligned}
H_{ij}H_{ij}^\top =& \sum_{i=0}^{3} X_iX_i^\top + \sum_{i=1}^{3} Y_iY_i^\top \pm [\sum_{i=1}^{3}(X_0X_i^\top + X_iX_0^\top + X_0Y_i^\top + Y_iX_0^\top) \\
&\pm \sum_j \sum_i (X_iY_j^\top + Y_jX_i^\top) \pm \sum (X_iX_{i+1}^\top + X_{i+1}X_i^\top + Y_iY_{i+1}^\top + Y_{i+1}Y_i^\top)] \\
=&4tI_{4t}.
\end{aligned}
$$

$\Rightarrow$ each $4t \otimes 4t$ block of H is an Hadamard matrix.
(2) Proof is similar to case (1), just define H by
$H = A_0 \otimes X_0 + \sum_{i=1}^{3}\{A_i \otimes X_i + A_i^\top \otimes Y_i\}$ $\square$

Example 1. *The matrices*

$$
X_0 = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix},
X_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
X_2 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
X_3 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix},
Y_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},
Y_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0
\end{bmatrix},
$$

$$Y_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

together with (sssa) or (kkka) type Goethals-Seidel matrices of order n taken as A_i^s give the Block Structured Hadamard matrix of order $8n$ with Hadamard blocks of order 8.

Theorem 2. *Let there exist three $(0, 1, -1)$-matrices of order $4t$ ($t \in \mathbb{N}$) such that*
(i) $X_iX_i^\top = tI_{4t}$; $i = 1, 3$ and $X_2X_2^\top = 2tI_{4t}$
(ii) $X_iX_j^\top + X_jX_i^\top = 0$; $1 \le i \ne j \le 3$.
Then we have
(1) If there exist four Williamson matrices $A_i, 1 \le i \le 4$ such that $A_4 = A_2$ of order n then there exists a block structured Hadamard matrix of order $4nt$ with Hadamard blocks of order $4t$.
(2)If there exist three pairwise amicable ± 1 matrices A_i; $1 \le i \le 3$ of order n such that $A_1A_1^\top + 2A_2A_2^\top + A_3A_3^\top = 4nI_n$ then there exists a block structured Hadamard matrix of order $4nt$ with Hadamard blocks of order $4t$.

Proof. (1) Define H by $H = \sum_{i=1}^{3} A_i \otimes X_i$ then

$$\begin{aligned} HH^\top &= \sum_{i=1}^{3} A_iA_i^\top \otimes X_iX_i^\top + [\sum_{i=1}^{2}(A_iA_{i+1}^\top \otimes X_iX_{i+1}^\top + A_{i+1}A_i^\top \otimes X_{i+1}X_i^\top) \\ &\quad + \sum_{i=1}(A_iA_{i+2}^\top \otimes X_iX_{i+2}^\top + A_{i+2}A_i^\top \otimes X_{i+2}X_i^\top)] \\ &= \sum_{i=1}^{3} A_iA_i^\top \otimes X_iX_i^\top + [\sum_{i=1}^{2}(A_iA_{i+1}^\top \otimes \{X_iX_{i+1}^\top + X_{i+1}X_i^\top\}) \\ &\quad + \sum_{i=1}(A_iA_{i+2}^\top \otimes \{X_iX_{i+2}^\top + X_{i+2}X_i^\top\})] \end{aligned}$$

$$\begin{aligned} &=\{A_1A_1^\top + 2A_2A_2^\top + A_3A_3^\top\} \otimes tI_{4t} \\ &=4nI_n \otimes tI_{4t} \\ &=4ntI_{4nt}. \end{aligned}$$

Now, each $m \otimes m$ blocks of H_{ij} is a linear combination of $X_i's$ i.e., $H_{ij} = \sum_{i=1}^{3} \rho_i X_i$; $\rho_i \in \{1, -1\}$ therefore,

$$\begin{aligned} H_{ij}H_{ij}^\top &= \sum_{i=1}^{3} X_iX_i^\top \pm \sum_{1 \le i \ne j \le 3} (X_iX_j^\top + X_jX_i^\top) \\ &=4tI_{4t}. \end{aligned}$$

(2) Can be proven similarly. □

Example 2. *Take*

$$X_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, X_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Corollary 1. *If there exist matrices X_i, $i = 1, 2, 3$ of order $4t$, $t \in \mathbb{N}$ satisfying conditions of Theorem (2), the matrices X_i, $i = 1, 2, 3$ of order $8t$, $t \in \mathbb{N}$ also exist.*

Proof. Define new X_i as $X_i \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes X_i, i = 1, 2, 3.$ □

Corollary 2. *If there exist matrices X_i, $i = 1, 2, 3$ of order $4t$, $t \in \mathbb{N}$ satisfying conditions of Theorem (2), and there exists three Amicable Hadamard matrices of order r then there exist X_i matrices of order $4rt$.*

Proof. Let $H_i, i = 1, 2, 3$ be Amicable Hadamard matrix of order r. Define new X_i as $X_i \Rightarrow X_i \otimes H_i, i = 1, 2, 3.$ □

Theorem 3. *Let there exist $(0, 1, -1)$-matrices X_i, Y_j; $0 \le i \le 2, 1 \le j \le 2$ of order $4t$; $t \in \mathbb{N}$ such that*
(i) $X_0X_0^\top = X_2X_2^\top + Y_2Y_2^\top = tI_{4t}$ *and* $X_1X_1^\top + Y_1Y_1^\top = 2tI_{4t}$
(ii) $X_0X_i^\top + X_iX_0^\top = 0 = X_0Y_i^\top + Y_iX_0^\top, i = 1, 2$
(iii) $X_iY_i^\top = 0 = Y_iX_i^\top, \forall i = 1, 2$
(iv) $X_iY_j^\top + X_jY_i^\top = 0 = Y_jX_i^\top + Y_iX_j^\top, 1 \le i \ne j \le 2$
(v) $X_iX_{i+j}^\top + Y_{i+j}Y_i^\top = 0 = X_{i+j}X_i^\top + Y_iY_{i+j}^\top, i, j = 1, i + j \le 2.$
Then we have, if there exist three pairwise commutative ± 1 matrices $A_i; 0 \le i \le 2$ of order n such that $A_0A_0^\top + 2A_1A_1^\top + A_2A_2^\top = 4nI_n$ then there exists a block structured Hadamard matrix of order $4nt$ with Hadamard blocks of order $4t$.

Proof. Proof is simillar to Theorem (1) and Theorem (2). Just take
$H = A_0 \otimes X_0 + A_1R \otimes X_1 + A_1^\top R \otimes Y_1 + A_2R \otimes X_2 + A_2^\top R \otimes Y_2.$ □

Example 3. *From Example (1) take $X_i's$ and $Y_i's$ and set $X_0 = X_0, X_1 = X_1 + X_2, X_2 = X_3, Y_1 = Y_1 + Y_2, Y_2 = Y_3$ to get the desired result.*

4. Conclusion

In this paper we have constructed block structured Hadamard matrices different from those of Sylvester and Agaian. Matrices $X_i's$ and $Y_i's$ of Theorem (1) can be obtained from Goethals-Seidel array and its generalization. In this method block structured Hadamard matrix of order $4nt$ is constructed using the blocks of a set containing at the most 16 distinct Hadamard blocks of order t. (Here two Hadamard blocks H_i and H_j are considered to be distinct if $H_i \ne \pm H_j$.) Matrices $A_i's$ can be found in the works of several authors specially in [4, 11, 12].

Matrices $X_i's$ and $Y_i's$ of Theorem (2) can be obtained from Propus array and Propus type arrays. Seberry and Balonin have constructed required $A_i's$ matrices in abundance [8]. Methods of construction of these matrices are not proposed here. In this construction there are precisely four distinct Hadamard blocks viz. $\pm(X_1 + X_2 + X_3)$, $\pm(X_1 - X_2 + X_3)$, $\pm(X_1 - X_2 - X_3)$ and $\pm(X_1 + X_2 - X_3)$. If the matrices $A_i's$ are constructed from Turyn's method then there are only three distinct Hadamard blocks viz. $\pm(X_1 + X_2 + X_3)$, $\pm(X_1 - X_2 + X_3)$, $\pm(X_1 - X_2 - X_3)$.

Theorem (3) is a product of the Theorems (1) and (2). In this construction number of distinct Hadamard blocks are 16. Efforts could be made to find a method to construct matrices $X_i's$ and $Y_i's$ in general, which could result in some Hadamard matrices of new orders. Block structured Hadamard matrices may be used in nested group divisible (GD) designs as follows:

It is well known that replacing 1 by I_2 and -1 by $(J - I)_2$ in a Hadamard matrix H of order $4nt$ we obtain a series of resolvable semi-regular GD designs with parameters (see Saurabh and Sinha [13]):

$$v = b = m^*n^* = 8nt, r = k = 4nt, \lambda_1 = 0, \lambda_2 = 2nt, m^* = 4nt, n^* = 2. \quad (1)$$

Further since H contains Hadamard blocks each of order $4t$, removing s rows of blocks of incidence matrix of (1) we obtain a series of GD designs with parameters:

$$v = 8t(n-s), b = 8nt, r = 4nt, k = 4t(n-s), \lambda_1 = 0, \lambda_2 = 2nt, m^* = 4t(n-s), n^* = 2; s < n. \quad (2)$$

Hence a GD design with parameters (2) is nested within a GD design with parameters (1). For details on GD designs we refer to Raghvarao and Padgett [14], Saurabh and Sinha [13] and Saurabh and Prasad [15].

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