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Article

Cycles and Paths Related Vertex-Equitable Graphs

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Abstract: A vertex labeling ξ of a graph χ is referred to as a 'vertex equitable labeling (VEq.)' if the induced edge weights, obtained by summing the labels of the end vertices, satisfy the following condition: the absolute difference in the number of vertices v and u with labels $\xi(v) = a$ and $\xi(u) = b$ (where $a,\ b \in Z$) is approximately 1, considering a given set A that consists of the first $\lceil \frac{q}{2} \rceil$ non-negative integers. A graph χ that admits a vertex equitable labeling (VEq.) is termed a 'vertex equitable' graph. In this manuscript, we have demonstrated that graphs related to cycles and paths are examples of vertex-equitable graphs.

Keywords: vertex equitable labeling, cycle, complete graph, path, corona product of graphs.
Mathematics Subject Classification: 05C15, 05C05.

1. Introduction

All the considered graphs are finite, connected, simple, and undirected. The basic terminologies follow from [1]. Let χ be a (p, q) graph with p vertices and q edges. In this context, a vertex labeling ξ assigns non-negative integers to the vertices of the graph. Several types of labelings have been extensively studied in the literature, such as graceful, multiplicative, vertex equitable (VEq.), harmonious, cordial, and prime labelings, among others, with detailed results available in [2]. In some labelings, vertex labels induce edge labels, and a vertex labeling ξ is termed a 'vertex-difference labeling' if it induces the label $|\xi(x) - \xi(y)|$ for all edges $xy \in E(\chi)$, which we call 'edge-weight' $wt_{xy}(\xi)$.

A 'κ-equitable' labeling is a vertex-difference labeling where the appearance of edge-weights induced by ξ occurs exactly κ times. Bloom and Ruiz in [3] described the idea of extending vertex-difference labeling.

Seenivasan and Lourdusamy, [4] introduced the concept of 'vertex equitable labeling (VEq.)' using the first $\lceil \frac{q}{2} \rceil$ non-negative integers, i.e., $Z = 0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil$ for vertices. In this labeling, the induced edge-weights obtained by summing the end vertex labels satisfy the condition: the absolute difference in the number of vertices v and u with labels $\xi(v) = a$ and $\xi(u) = b$ for $a,\ b \in Z$ is approximately 1. The proved results regarding vertex equitable graphs include paths P_n, bi-star $B(n, n)$, combs $P_n \odot K_1$, bipartite complete $K_{2,n}$, friendship graph $C_3^{(t)},\ t \geq 2$, quadrilateral snake, $K_2 + mK_1, K_{1,p} \cup K_{1,p+k}$, ladder graph L_n, uniform subdivision of a path, and cycle $C_m,\ m \equiv 0$ or 3 (mod 4). On the other hand, the proved results about graphs that are not vertex-equitable include $K_{1,p},\ p \geq 4$, Eulerian graphs with

p edges $p \equiv 1$ or 2 (mod 4), the wheel graph W_p, the complete graph K_p, $p > 3$, and triangular cacti with q edges $q \equiv 0$ or 6 or 9.

Lourdusamy and Patrick [5] observed that the graph TL_n, K_1, $L_n \bigodot mK_1$, $Q_n \bigodot K_1$, and $A(T_n)$ admit VEq labeling. Acharya, Jain and Kansal [6] introduced the concept of VEq labeling for signed graphs and observed the VEq behavior of signed paths, stars, and complete bipartite graphs $K_{2,n}$. Jeyanthi and Maheswari provided vertex equitable labeling of $T\hat{o}P_n, T\hat{o}2P_n, T\hat{o}C_n, T\tilde{o}C_n$. Jeyanthi *et al.* showed that $KY(m,n), P(2, QS_n), P(m, QS_n), C(n, QS_n), NQ(m), K_{1,n} \times P_2$ admit VEq labeling. Additionally, Jeyanthi *et al.* [7] demonstrated that $(D(T_n))$ (where $S(D(T_n))$ represents the subdivision of the double triangular snake), $S(D(Q_n))$ (where $S(D(Q_n))$ denotes the subdivision of double quadrilateral snakes), $S(DA(T_n))$ (where $S(DA(T_n))$ denotes the subdivision of the double alternate triangular snake), $DA(T_m) \bigodot nK_1$, and $DA(Q_m) \bigodot nK_1$ are VEq. Furthermore, [4] clearly shows that Eulerian graphs with n edges, triangular cacti under certain conditions are VEq. They also proved that $K_2 + mK_1 + K_{1,n}UK_{1,n+k}$ are vertex equitable if and only if $k = 1, 2, 3$. They derived the following inequality as a criteria for determining not vertex equitable graphs:

$$p < \left\lceil \frac{q}{2} \right\rceil + 2,$$

where p and q define the elements of the graph. Jeyanthi, Maheswar and Vijayalakshm [8] provided VEq labeling for J_n, $(JF)_n$, $BL(n, 2, m)$, the corona product of ladder and the complement of the complete graph, and $< L_n\hat{o}K_{1,m} >$, while in [9], it was proven that $P_m\tilde{o}nC_4$, $C_m\tilde{o}nC_4$, and $L_m\tilde{o}nC_4$ are VEq graphs. Jeyanthi and Maheswari [10] gave VEq labeling for ${B_{n,n}}^2$, $C^2_{n1}, C^2_{n2}, \ldots, C^2_{nk}$, where each $n_i \equiv 0(mod 4)$. They also proved that kC_4 snakes, where k is greater than or equal to 1, are VEq, and kC_n snakes are vertex equitable under certain conditions.

Definition 1. [11] *For an (m,n) graph χ, a vertex labeling $\xi : V(\chi) \to Z = \{0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil\}$ with induced edge-weights ξ^* such that $\xi^*(uv) = \xi(u) + \xi(v)$, $\forall uv \in E(\chi)$ is vertex-equitable labeling if for $v_\xi(a) = \#\{v \in V(\chi) : \xi(v) = a\}$ $|v_\xi(a) - v_\xi(b)| \leq 1$, $\forall a, b \in Z$ and the induced edge-weights are* $\{1, 2, 3, \ldots, q\}$.

Frucht and Harary [12] defined the corona product of graphs. The *corona product* $G \bigodot H$ of two graphs G and H, with $|V(G)| = r$, is defined as:

$$V(G \bigodot H) = V(G) \cup \bigcup_{j=1}^{r} V(H_j)$$

and

$$E(G \bigodot H) = E(G) \cup \bigcup_{j=1}^{r} (E(H_j) \cup \{(v_j, u) : v_j \in V(G) \text{and } u \in V(H_j)\}.$$

In other words, a corona graph is obtained from two graphs, G of order r and H, taking one copy of G and r copies of H and joining by an edge the jth vertex of G to every vertex in the jth copy of H. Chang, DF, and DG [13] gave a harmonious labeling of the join of the star S_n and K_1.

2. Cycle related vertex equitable graphs

Theorem 1. *The graph $G = C_6 \bigodot K^c_p$ is vertex equitable.*

Proof. Consider the vertex set $\{a, b, c, d, e, f\}$ of a cycle and let a_i, b_i, c_i, d_i, e_i, f_i be the vertices adjacent to a, b, c, d, e, f, respectively, where $i = \{1, 2, 3, \ldots, p\}$, to obtain $G = C_6 \bigodot K^c_p$. The vertices and edges of G are given by $\{a, b, c, d, e, f, a_i, b_i, c_i, d_i, e_i, f_i : 1 \leq i \leq p\}$ and $\{ab, ac, bd, ce, df, ef, aa_i, bb_i, cc_i, dd_i, ee_i, ff_i : i = \{1, 2, 3, \ldots, p\}\}$. Thus, G has $6p + 6$ vertices and $6p + 6$ edges.

Let F be a mapping from $V(G)$ to $A = \left\{0, 1, 2, \ldots, \left\lceil \frac{6p+6}{2} \right\rceil\right\}$ defined as follows:

$$
\begin{aligned}
F(a) &= 0, \\
F(b) &= p + 1, \\
F(c) &= 3(p + 1), \\
F(d) &= p + 1, \\
F(e) &= 2(p + 1) + 1, \\
F(f) &= 2(p + 1) + 1, \\
F(a_i) &= i, && i = 1, 2, 3, \ldots, p, \\
F(b_i) &= i, && i = 1, 2, 3, \ldots, p. \\
F(c_i) &= 2(p + 1) + 1 + i, && i = 1, 2, 3, \ldots, p, \\
F(d_i) &= p + 1 + i, && i = 1, 2, 3, \ldots, p, \\
F(e_i) &= 2(p + 1), && i = 1, \\
F(e_i) &= 2(p + 1) + i, && i = 1, 2, 3, \ldots, p, \\
F(f_i) &= p + 1 + i, && i = 1, 2, 3, \ldots, p.
\end{aligned}
$$

The constructed labeling above satisfies the condition $|v_F(a) - v_F(b)| \leq 1$, thereby completing the proof of our theorem. Figure 1 below demonstrates the vertex equitable (VEq) labeling of the graph $G = C_6 \bigodot K_4^c$. □

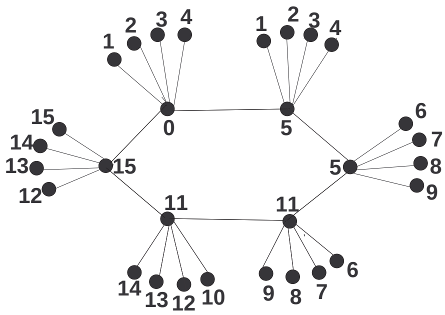

Figure 1. VEq labeling of $G = C_6 \bigodot K_4^c$.

Theorem 2. *The graphs $G = C_3(2K_n^c)$ and $G = C_6(2K_p^c)$ admits vertex equitable labeling.*

Proof. Consider x, y, z be the vertex set of a cycle C_3 and $x_i's$and $y_i's$, $1 \leq i \leq n$ are the vertices that are adjacent to x and y respectively. In order to obtain the graph $G = C_3(2K_n^c)$, the vertex and edge set of G are $\{x, y, z, x_i, y_i : i = \{1, 2, 3, \ldots, n\}\}$ and $\{xy, xz, yz, xx_i, yy_i : i = \{1, 2, 3, \ldots, n\}\}$. Now it is clear that the graph G has $2n + 3$, $2n + 3$ vertices and edges respectively. Define a vertex labeling $F : V(G) \to A = \left\{0, 1, 2, \ldots, \left\lceil \frac{2n+3}{2} \right\rceil\right\}$ as:

$$
\begin{aligned}
F(x) &= 0, \\
F(y) &= n + 2, \\
F(z) &= n + 1, \\
F(x_i) &= i, && i = 1, 2, 3, \ldots, n, \\
f(y_i) &= i, && i = 1, 2, 3, \ldots, n.
\end{aligned}
$$

The above construction fulfills the requirement of VEq labeling, hence $G = C_3(2K_n^c)$ admits vertex equitable labeling. Figure 2 given below shows VEq labeling of the graph $G = C_3(2K_n^c)$.

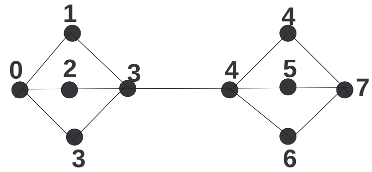

Figure 2. VEq labeling of $G = C_3(2K_n^c)$.

Now, consider a, b, c, d, e, f be the vertex set of a cycle C_6 and a, d be the vertices joined to $a_k's$ and $d_k's$ respectively and $k = \{1, 2, 3, \dots, p\}$ in order to construct G. Assume that the vertex and edge set of G are V and G. $V = \{a, b, c, d, e, f, a_k, d_k : k = 1, 2, 3, \dots, p\}$ and $E = \{ab, af, bc, cd, de, ef, aa_k, dd_k : k = \{1, 2, 3, \dots, p\}\}$. Thus G has $2p + 6$ vertices and $2p + 7$ edges. Consider a mapping F from vertex set of G to $A = \left\{0, 1, 2, \dots, \left\lceil \frac{2p+7}{2} \right\rceil\right\}$ as follows:

$$
\begin{aligned}
F(a) &= 0, \\
F(b) &= n + 2, \\
F(c) &= n + 4, \\
F(d) &= n + 3, \\
F(e) &= n + 2, \\
F(f) &= n + 1, \\
F(a_k) &= k, && k = 1, 2, 3, \dots, p, \\
F(d_k) &= k, && k = 1, 2, 3, \dots, p, \\
F(d_k) &= p + 1, && k = p.
\end{aligned}
$$

Clearly the graph $G = C_6(2K_p^c)$ fulfills the requirement of VEq labeling. Figure 3 given below shows VEq labeling of the graph $G = C_6(2K_p^c)$. □

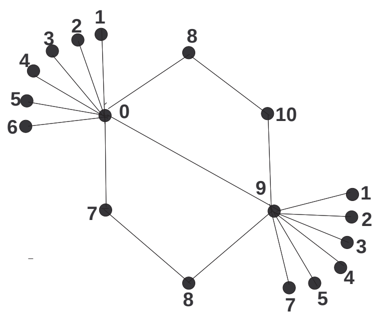

Figure 3. VEq labeling of $G = C_6(2K_p^c)$.

Theorem 3. *Let G be a graph obtained by connecting two cycles of length 3 and adding p vertices to a single vertex, each having degree 2 in its corresponding triangle. The graph G admits a vertex equitable labeling.*

Proof. Consider a, b, c, d, e, f be the vertices of two triangles. To obtain G join b to $b_k's$, e to $e_k's$ and c to d. The graph G has vertex and edge set as follows $\{a, b, c, d, e, f, b_k, e_k : k = 1, 2, 3, \dots, p\}$ and $\{ab, ac, bc, cd, de, df, ef, bb_k, ee_k : k = 1, 2, 3, \dots, p\}$. Thus G consists of $2p + 6$ vertices and $2p + 7$ edges. Consider $F : V(G) \to A = \left\{0, 1, 2, ..., \left\lceil \frac{2p+7}{2} \right\rceil\right\}$ defined as follows:

$$
\begin{aligned}
F(a) &= 0, \\
F(b) &= 1, \\
F(c) &= 2,
\end{aligned}
$$

$$F(d) = n + 2,$$
$$F(e) = n + 3,$$
$$F(f) = 3,$$
$$F(b_k) = k + 2, \qquad k = 1, 2, 3, \ldots, p,$$
$$F(e_k) = k + 3, \qquad k = 1, 2, 3, \ldots, p,$$
$$F(e_k) = k + 4, \qquad p - 3 \leq k \leq p.$$

The above vertex labeling induces edge labeling that satisfy $|v_F(a) - v_F(b)| \leq 1$. Figure 4 given below shows VEq labeling of the graph G. □

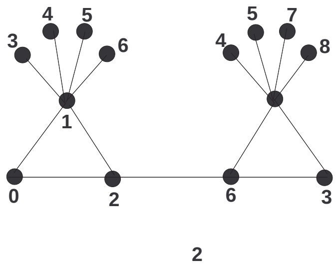

Figure 4. VEq labeling of graph G obtained by connecting two cycles of length 3 and adding p vertices to a single vertex, each having degree 2 in its corresponding triangle.

Theorem 4. *The graph G obtained by joining p cycles of length* 4 *is vertex equitable.*

Proof. Let a_l, b_l, c_l, d_l, where $l = 1, 2, 3, \ldots, p$ be the vertex set of the graph G and each d_l is adjacent to a_{l+1} to achieve the graph G. Let $V = \{a_l, b_l, c_l, d_l : l = 1, 2, 3, \ldots, p\}$ and $E\{a_lb_l, a_lc_l, b_ld_l, c_ld_l, d_la_{l+1} : l = 1, 2, 3, \ldots, p\}$ be the vertices and edges of G. It shows that G has order $4p$ and size $5p - 1$.

Define a mapping F from $V(G)$ to $A = \left\{0, 1, 2, \ldots, \left\lceil \frac{5p-1}{2} \right\rceil\right\}$ as follows:

$$F(a_{2l-1}) = 5l - 5, \qquad 1 \leq l \leq \left\lfloor \frac{p}{2} \right\rfloor,$$
$$F(a_2) = 3,$$
$$F(a_{2l}) = f(s_{2l-2}) + 5, \qquad l = 2, 3, \ldots, p,$$
$$F(b_1) = 2,$$
$$F(b_{2l-1}) = F(b_{2l-3}) + 5, \qquad l = 2, 3, \ldots, p,$$
$$F(b_2) = 4,$$
$$F(b_{2l}) = F(b_{2l-2}) + 5, \qquad l = 2, 3, \ldots, p,$$
$$F(c_l) = 1,$$
$$F(c_{2l-1}) = F(c_{2l-3}) + 5, \qquad l = 2, 3, \ldots, p,$$
$$F(c_2) = 3,$$
$$F(c_{2l}) = F(c_{2l-2}) + 5, \qquad l = 2, 3, \ldots, p,$$
$$F(d_1) = 2,$$
$$F(d_{2l-1}) = F(d_{2l-3}) + 5, \qquad l = 2, 3, \ldots, p,$$
$$F(d_2) = 5,$$
$$F(d_{2l}) = F(d_{2l-2}) + 5, \qquad l = 2, 3, \ldots, p.$$

Thus the constructed labeling satisfy the condition $|v_F(a) - v_F(b)| \leq 1$. Figure 5 given below verifies VEq labeling of G. □

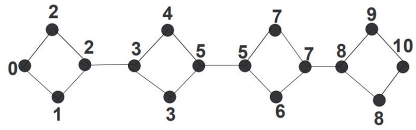

Figure 5. VEq labeling of graph G obtained by joining p cycles of length 4.

Theorem 5. *The graph G obtained by joining p cycles of length 4 where in each cycle there is a chord is vertex equitable.*

Proof. Consider a_l, b_l, c_l, d_l where $l = 1, 2, 3, \ldots, p$ be the vertex set of G and d_l is joined with a_{l+1} to obtain G. In each cycle a_l is adjacent to d_l. The vertex set and edge set of G are $\{a_l, b_l, c_l, d_l : l = 1, 2, 3, \ldots, p\}\}$ and $\{a_lb_l, a_lc_l, b_ld_l, c_ld_l, d_lc_{l+1}, a_ld_l : l = 1, 2, 3, \ldots, p\}$.

It shows that G has $4p$ vertices and $6p - 1$ edges. Consider a mapping F from $V(G)$ to $A = \left\{0, 1, 2, \ldots, \left\lceil \frac{6p-1}{2} \right\rceil\right\}$ of the form:

$$
\begin{aligned}
&F(a_l) = 3l - 3, && \\
&F(b_1) = 1, && \\
&F(b_l) = F(b_{l-1}) + 3, && l = 2, 3, \ldots, p, \\
&F(c_1) = 2, && \\
&F(c_l) = F(c_{l-1}) + 3, && l = 2, 3, \ldots, p, \\
&F(d_l) = 3l. &&
\end{aligned}
$$

The vertex labeling defined above induces an edge labeling that satisfies $|v_F(a) - v_F(b)|$ less than or equal to 1. Figure 6 given below verifies VEq labeling of G. □

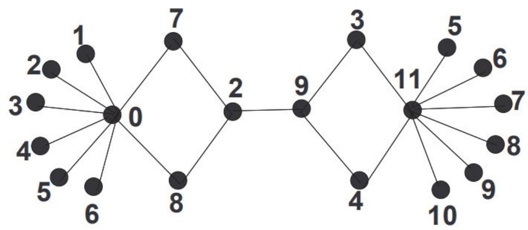

Figure 6. VEq labeling graph of G obtained by joining p cycles of length 4 where in each cycle there is a chord.

Theorem 6. *The graph G obtained by joining two cycles C_4 where in each cycle K_p^c vertices are adjacent to the vertices whose labels are 0 and 11 fulfills the requirement of vertex equitable labeling.*

Proof. Let a, b, c, d be the vertices of one cycle C_4 and e, f, g, h be the vertex set of other cycle C_4. Suppose that a_k, h_k where $k = \{1, 2, 3, \ldots, p\}$ are adjacent to a and h respectively. The vertices d and e are joined to obtain the graph G. The vertex and edge set of G are $\{a, b, c, d, e, f, g, h, a_k, h_k : k = 1, 2, 3, \ldots, p\}$ and $\{ab, ac, cd, bd, de, ef, eg, fh, gh, aa_k, hh_k : k = 1, 2, 3, \ldots, p\}$. It shows that G has $2p + 8$ vertices and $2p + 9$ edges.

Consider $F : V(G) \to A = \left\{0, 1, 2, \ldots, \left\lceil \frac{2p+9}{2} \right\rceil\right\}$ as:

$$
\begin{aligned}
&F(a) = 0, \\
&F(b) = p + 1, \\
&F(c) = p + 2, \\
&F(d) = 2, \\
&F(e) = p + 3,
\end{aligned}
$$

$$F(f) = 3,$$
$$F(g) = 4,$$
$$F(h) = \left\lceil \frac{2p+9}{2} \right\rceil,$$
$$F(a_k) = k, \qquad k = 1, 2, 3, \dots, p,$$
$$F(d_k) = p - 2 + k, \qquad k = 1, 2, 3, \dots, p.$$

Clearly G performs VEq labeling. Figure 7 given below verifies VEq labeling of G. □

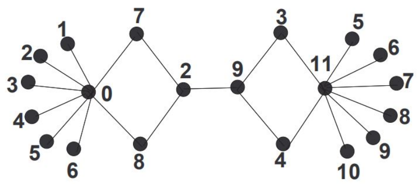

Figure 7. VEq labeling of graph G obtained by joining two cycles C_4 where in each cycle K_p^c vertices are adjacent to the vertices whose labels are 0 and 11.

3. Path related vertex equitable graphs

Theorem 7. *The graph $G = P_3(2K_m^c)$ is vertex equitable.*

Proof. Let p, q, r be the vertex set of a path. Consider $p_l's$ and $r_l's$ are joined to p and r respectively, where $1 \leq l \leq m$ to obtain $G = P_3(2K_m)^c$. Consider the vertex and edge set of G as $\{p, q, r, p_l, r_l : 1 \leq l \leq m\}$ and $\{pq, qr, pp_l, rr_l : 1 \leq l \leq m\}$. It is clear that G has $2m + 3$ vertices and $2m + 2$ edges.

Consider a mapping F from $V(G)$ to $A = \left\{0, 1, 2, \dots, \left\lceil \frac{2m+2}{2} \right\rceil\right\}$ as:

$$F(p) = 0,$$
$$F(q) = m + 1,$$
$$F(r) = m + 1,$$
$$F(p_l) = l, \qquad l = 1, 2, 3, , m,$$
$$F(r_l) = l, \qquad l = 1, 2, 3, \dots, m.$$

The verification of the above vertex labeling gives that the induced edge labels are $1, 2, 3, \dots, 2m + 2$. Figure 8 given below verifies VEq labeling of $G = P_3(2K_m^c)$. □

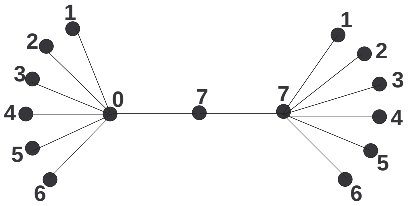

Figure 8. VEq labeling of $G = P_3(2K_m^c)$.

Theorem 8. *The graph $G = P_5(2K_q^c)$ is vertex equitable.*

Proof. The vertex set of path is a, b, c, d, e and $a_j's$ and $d_j's$ where $1 \leq j \leq q$ are linked with a and d to construct $G = P_5(2K_q)^c$. We obtain $\{a, b, c, d, e, a_j, d_j : 1 \leq j \leq q\}$ and $\{ab, bc, cd, de, aa_j, dd_j : 1 \leq j \leq q\}$ as vertices and edges for G. It is clear that G is a graph having $2q + 5$ vertices and $2q + 4$ edges.

Consider $F : V(G) \to A = \left\{0, 1, 2, \ldots, \left\lceil \frac{2q+4}{2} \right\rceil\right\}$ as defined below:

$$
\begin{aligned}
F(a) &= 0, \\
F(b) &= q+1, \\
F(c) &= 1, \\
F(d) &= q+2, \\
F(e) &= q+2, \\
F(a_j) &= j, && j = 1, 2, 3, \ldots, q, \\
F(d_j) &= j+1, && j = 1, 2, 3, \ldots, q.
\end{aligned}
$$

Hence G satisfies all the axioms of VEq labeling i.e., the edge labels satisfies $|v_F(a) - v_F(b)| \leq 1$. The verification of VEq labeling of the graph $G = P_5(2K_q^c)$ is given in figure 9. □

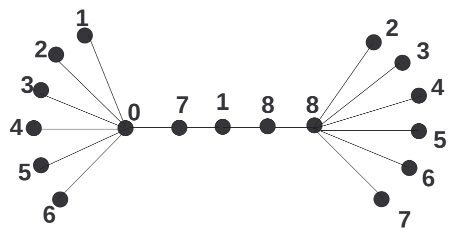

Figure 9. VEq labeling of $G = P_5(2K_q^c)$.

Theorem 9. *The graph $G = P_4 \bigodot K_n^c$ is vertex equitable.*

Proof. Consider the vertex set of path P_4 as a, b, c, d and suppose that $a_k's, b_k's, c_k's$ and $d_k's$ are joined with a, b, c, d to achieve a graph $G = P_4 \bigodot K_n^c$. Suppose that the vertex and edge set of G are $\{a, b, c, d, a_k, b_k, c_k, d_k : 1 \leq k \leq n\}$ and $\{ab, bc, cd, aa_k, bb_k, cc_k, dd_k : 1 \leq k \leq n\}$. Thus it is clear that G has $4n + 4$ nodes and $4n + 3$ lines.

Suppose a mapping $F : V(G) \to A = \left\{0, 1, 2, \ldots, \left\lceil \frac{4n+3}{2} \right\rceil\right\}$ of the form:

$$
\begin{aligned}
F(a) &= 0, \\
F(b) &= n+1, \\
F(c) &= n+1, \\
F(d) &= 2(n+1).
\end{aligned}
$$

For $1 \leq i \leq n$:

$$
\begin{aligned}
F(a_k) &= k, \\
F(b_k) &= k && F(c_k) = n+1+k, \\
F(d_k) &= n+1+k.
\end{aligned}
$$

The above labeling clears that the induced edge labels are $1, 2, 3, \ldots, 4n+3$. So G is VEq graph. The verification of VEq labeling of the graph $G = P_4 \bigodot K_n^c$ is given in figure 10. □

Theorem 10. *Let P_2 be a path graph with 2 vertices and 1 edge. If we join $q-1$ copies of the path P_2 together using 2 edges each, then the resulting graph K is vertex equitable (VEq).*

Proof. Consider the vertex set of path P_2 as r_k, s_k and the graph K is acquired by attaching r_k and s_k where $1 \leq k \leq q$, $r_k s_{k+1}$ and $s_k r_{k+1}$, where $1 \leq k \leq q$. The vertices and edges of K are $\{r_k, s_k : 1 \leq$

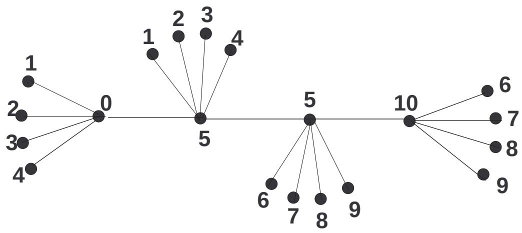

Figure 10. VEq labeling of $G = P_4 \odot K_n^c$.

$k \leq q\}$ and $\{r_k s_k : 1 \leq k \leq q\} \cup \{r_k s_{k+1}, s_k r_{k+1} : 1 \leq k \leq q-1\}$. Thus K has order $2q$ and size $3q-2$. Observe a mapping F from $V(K)$ to $A = \left\{0, 1, 2, \ldots, \left\lceil \frac{3q-2}{2} \right\rceil\right\}$ as follows:

$$
\begin{aligned}
F(r_{2k-1}) &= 3k-3, && 1 \leq k \leq \left\lceil \frac{q}{2} \right\rceil,\\
F(r_{2k}) &= 3k-1, && 1 \leq k \leq \left\lfloor \frac{q}{2} \right\rfloor,\\
F(s_{2k-1}) &= 3k-2, && 1 \leq k \leq \left\lceil \frac{q}{2} \right\rceil,\\
F(s_{2k}) &= 3k-1, && 1 \leq k \leq \left\lfloor \frac{q}{2} \right\rfloor.
\end{aligned}
$$

Now it is obvious that the vertex labeling constructed above induces an edge labeling that shows that K admits VEq labeling. The verification of VEq labeling of the graph K is given in figure 11. □

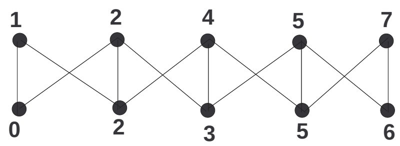

Figure 11. VEq labeling of K defined in Theorem 10.

Conflict of interest

The authors declare no conflict of interests.

References

1. Harary, F., 1972. Graph theory. Reading, Massachusetts. Addison-Wesley.
2. Gallian, J.A., 2018. A dynamic survey of graph labeling. *Electronic Journal of Combinatorics*, 1(DynamicSurveys), p.DS6.
3. Bloom, G.S. and Ruiz, S., 1994. Decompositions into linear forests and difference labelings of graphs. *Discrete Applied Mathematics, 49*(1-3), pp.61-75.
4. Seenivasan, M. and Lourdusamy, A., 2008. Vertex equitable labeling of graphs. *Journal of Discrete Mathematical Sciences and Cryptography, 11*(6), pp.727-735.
5. Lourdusamy, A. and Patrick, F., 2017. New construction on SD-prime cordial labeling. In *Theoretical Computer Science and Discrete Mathematics: First International Conference, ICTCSDM 2016, Krishnankoil, India, December 19-21, 2016, Revised Selected Papers 1* (pp. 134-143). Springer International Publishing.
6. Acharya, M., Jain, R. and Kansal, S., 2017. Vertex equitable labeling of signed graphs. *Electronic Notes in Discrete Mathematics, 63*, pp.461-468.

7. Jeyanthi, P., Maheswar, A. and Vijayalakshm, M., 2016. New results on vertex equitable labeling. *Journal of Algebra Combinatorics Discrete Structures and Applications*, pp.97-104.

8. Jeyanthi, P. and Maheswari, A., 2015. Vertex equitable labeling of cycle and star related graphs. *Journal of Scientific Research, 7*(3), pp.33-42.

9. Jeyanthi, P. and Maheswari, A., 2014. Vertex Equitable Labeling of Cyclic Snakes and Bistar Graphs. *Journal of Scientific Research, 6*(1), pp. 79-85.

10. Jeyanthi, P., Maheswari, A. and Vijayalakshmi, M., 2016. Vertex equitable labeling of union of cyclic snake related graphs. *Proyecciones (Antofagasta), 35*(2), pp.177-186.

11. Lourdusamy, A. and Patrick, F., 2017. Vertex equitable labeling for ladder and snake related graphs. *Journal of Prime Research in Mathematics, 13*, pp.01-07.

12. Harary, F. and Frucht, R., 1970. On the Corona of Two Graphs. *Aequationes Mathematicae, 5*, pp.322-325.

13. Chang, G.J., DF, H. and DG, R., 1981. Additive variations on a graceful theme: some results on harmonious and other related graphs. *Congressus Numerantium, 32*, pp. 181–197.