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Article

Gregarious Y_5-Tree Decompositions of Tensor Product of Complete Graphs

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Abstract: Y_k-tree is defined as $(v_1, v_2, \ldots, v_{k-1}; v_{k-2}v_k)$ by taking their vertices as $(v_1, v_2, \ldots, v_k)$ and edges as $\{(v_1v_2, v_2v_3, \ldots, v_{k-2}v_{k-1}) \cup (v_{k-2}v_k)\}$. It is also represented as $(P_{k-1} + e)$. One can obtain the necessary condition as $mn(m-1)(n-1) \equiv 0 \pmod{2(k-1)}$, for $k \geq 5$ to establish a Y_k-tree decomposition in $K_m \times K_n$. Here the tensor product is denoted by $\times$. In this manuscript, it is shown that a Y_5-tree (gregarious Y_5-tree) decomposition exists in $K_m \times K_n$, if and only if, $mn(m-1)(n-1) \equiv 0 \pmod{8}$.

Keywords: Decomposition, Y_5-Tree, Gregarious Y_5-Tree, Tensor product
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1. Introduction

A *decomposition of graph G* is a collection of subgraphs $\{G_i, 1 \leq i \leq n\}$, in which each subgraph, is mutually - distinct by their edges along with $E(G) = \cup_{i=1}^n E(G_i)$. Let K_n and P_n respectively denotes complete graph and path, on n vertices. Y_k-*tree* is defined as $(v_1, v_2, \ldots, v_{k-1}; v_{k-2}v_k)$ by taking their vertices as $(v_1, v_2, \ldots, v_k)$ and edges as $\{(v_1v_2, v_2v_3, \ldots, v_{k-2}v_{k-1}) \cup (v_{k-2}v_k)\}$. It is also represented as $(P_{k-1} + e)$. Y_5-tree or $(P_4 + e)$ graph, defined as $\{v_1, v_2, v_3, v_4; v_3v_5\}$, which is depicted in Figure 1. The
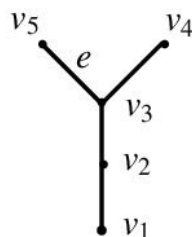

Figure 1. $Y_5 - tree$

tensor product($\times$) of K_m and K_n be defined in this way: $V(K_m \times K_n) = \{(p,q) \mid p \in V(K_m), q \in V(K_n)\}$ and $E(K_m \times K_n) = \{(p,q)(r,s) \mid pr \in E(K_m) \text{ and } qs \in E(K_n)\}$. Let it be taken as $V(K_m \times K_n) = \{\bigcup_{i=1}^m i_j, 1 \leq j \leq n\}$, by considering $V(K_m) = \{1, 2, \ldots, m\}$ and $V(K_n) = \{1, 2, \ldots, n\}$. For instance, see Figure 2.

As all are aware of that tensor product has commutative and distributive properties. So we write as $K_m \times K_n \simeq K_n \times K_m$ (commutative). If we have a Y_5-tree decomposition for K_m, we then write $K_m = Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5$. By applying the distributive properties, we can write as $K_m \times K_n = [(Y_5 \oplus$
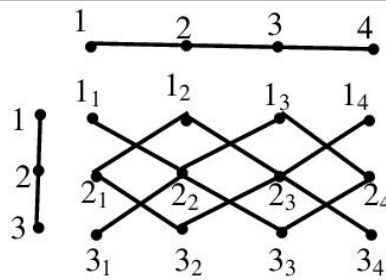
Figure 2. $P_3 \times P_4$

$Y_5 \oplus \ldots \oplus Y_5) \times K_n] \simeq [(Y_5 \times K_n) \oplus (Y_5 \times K_n) \oplus \ldots \oplus (Y_5 \times K_n)]$. If Y_5-tree decomposition arises in $K_m \times K_n$, in addition to, if every vertex of each Y_5-tree's is placed in different partite sets, then it called as *gregarious Y_5-tree decomposition*.

Tree decomposition and its special properties such as resolvable tree decompositions are considered by many authors. Ringel [1] has conjectured that K_{2m+1} can be decomposed as tree having m edges. C. Huang and A. Rosa [2] has proved that Y_5-tree decomposition for K_m holds if and only if $m \equiv 0, 1 \pmod 8$. A G-decomposition of K_m have been investigated in [3], where $|V(G)| = 5$. Inspired from Ringel conjecture, G. Sethuraman and V. Murugan [4] has conjectured that, for $m \geq 1$, (G, H)-decomposition exists in K_{4m+1}, when the size of tree G and H is m. Followed by these results, many authors have begun to look into, tree decomposition in various graphs. To know more about it, refer [5–11]. As a development of it, gregarious tree decomposition in the tensor product of complete graph has been started to investigate followed by the given results. A. Tamil Elakkiya and A. Muthusamy [12] have been proved a gregarious kite decomposition of $K_m \times K_n$, if and only if, $mn(m-1)(n-1) \equiv 0 \pmod 8$. A. Tamil Elakkiya and A. Muthusamy [13] have been proved a gregarious kite factorization of $K_m \times K_n$, if and only if, $mn \equiv 0 \pmod 4$ and $(m-1)(n-1) \equiv 0 \pmod 2$. In this way, we focus on the Y_5-tree (gregarious Y_5-tree) decomposition of tensor product of complete graphs. In this manuscript, we establish a Y_5-tree decomposition (gregarious Y_5-tree) of $K_m \times K_n$, if and only if, $mn(m-1)(n-1) \equiv 0 \pmod 8$.

Y_5-tree decomposition falls on the cases given below:

(i) $m \equiv 0 \pmod 4$ and for all $n, n \geq 2$.
(ii) $m \equiv 1 \pmod 4$ and for all $n, n \geq 2$.
(iii) $m \equiv 3 \pmod 4$ and $n \equiv 0 \pmod 4$.
(iv) $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$.
(v) $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$.
(vi) $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$.

Also we prove a gregarious Y_5-tree decomposition in $K_m \times K_n$ for the following cases:

(i) $m \equiv 0, 1 \pmod 8$ and for all $n, n \geq 2$.
(ii) $m \equiv 5 \pmod 8$ and for all $n, n \geq 2$.
(iii) $m \equiv 4 \pmod 8$ and for all $n, n \geq 2$.
(iv) $m \equiv 6 \pmod 8$ and $n = 4$.
(v) $m \equiv 7 \pmod 8$ and $n = 4$.
(vi) $m \equiv 10 \pmod 8$ and $n = 4$.
(vii) $m \equiv 11 \pmod 8$ and $n = 4$.

In order to prove our main result, we require the following:

Theorem 1. *[2] Y_5-tree decompositions holds in K_m if and only if $m \equiv 0, 1 \pmod 8$.*

2. Decompositions of $K_m \times K_n$ into Y_5 Tree

Lemma 1. *$K_4 \times K_n$ admits a Y_5-tree decomposition for all $n, n \geq 2$.*

Proof. Since for every K_n has a K_2 decomposition, we then write as
$K_4 \times K_n = K_4 \times [\underset{4}{K_2 \oplus K_2 \oplus \ldots \oplus K_2}] = (K_4 \times K_2) \oplus (K_4 \times K_2) \oplus \ldots \oplus (K_4 \times K_2)$. By taking as $V(K_4 \times K_2) = \{\bigcup_{i=1}^{4} i_j, 1 \leq j \leq 2\}$, the following set $\{(1_1, 2_2, 4_1, 3_2; 4_1 1_2), (1_1, 4_2, 3_1, 1_2; 3_1 2_2), (1_1, 3_2, 2_1, 4_2; 2_1 1_2)\}$, proves a Y_5-tree decomposition exists in $K_4 \times K_2$. Thus, we obtain a Y_5-tree decomposition for $K_4 \times K_n$. $\qquad\square$

Lemma 2. *$Y_5 \times K_n$ admits a Y_5-tree decomposition for all $n, n \geq 2$.*

Proof. As discussed in Lemma 1, we can take $Y_5 \times K_n = Y_5 \times [K_2 \oplus K_2 \oplus \ldots \oplus K_2] = (Y_5 \times K_2) \oplus (Y_5 \times K_2) \oplus \ldots \oplus (Y_5 \times K_2)$. By taking as $V(Y_5 \times K_2) = \{\bigcup_{i=1}^{5} i_j, 1 \leq j \leq 2\}$, the following set $\{(1_1, 2_2, 3_1, 4_2; 3_1 5_2), (1_2, 2_1, 3_2, 4_1; 3_2 5_1)\}$, proves a Y_5-tree decomposition exists in $Y_5 \times K_2$. Thus, we obtain a Y_5-tree decomposition for $Y_5 \times K_n$. $\qquad\square$

Lemma 3. *$K_{4,4} \times K_n$ admits a Y_5-tree decomposition for all $n, n \geq 2$.*

Proof. Let us write as $K_{4,4} \times K_n = K_{4,4} \times [K_2 \oplus K_2 \oplus \ldots \oplus K_2]$.
By taking as $V(K_{4,4}) = \{\bigcup_{i=1}^{2} i_j, 1 \leq j \leq 4\}$, the following set $\{(2_1, 1_2, 2_2, 1_4; 2_2 1_1), (1_1, 2_1, 1_3, 2_4; 1_3 2_2), (2_1, 1_4, 2_3, 1_1; 2_3 1_3), (2_3, 1_2, 2_4, 1_4; 2_4 1_1)\}$, proves a Y_5-tree decomposition exists in $K_{4,4}$. Now $K_{4,4} \times K_n = (Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5) \times (K_2 \oplus K_2 \oplus \ldots \oplus K_2) = (Y_5 \times K_2) \oplus (Y_5 \times K_2) \oplus \ldots \oplus (Y_5 \times K_2)$. A Y_5-tree decomposition for $Y_5 \times K_2$ derives from Lemma 2, which gives a Y_5-tree decomposition for $K_{4,4} \times K_n$. $\qquad\square$

Lemma 4. *$K_m \times K_n$ admits a Y_5-tree decomposition for $m \equiv 0 \pmod 4$ and for all $n, n \geq 2$.*

Proof. Here, $m \equiv 0 \pmod 4$ can be classified into two cases *(i)* $m \equiv 0 \pmod 8$ and *(ii)* $m \equiv 4 \pmod 8$.
Case (i): Let $m = 8s, s \geq 1$. Then $K_m \times K_n = K_{8s} \times K_n$. We have a Y_5-tree decomposition for K_m, according to the theorem 1. Then $K_{8s} \times K_n = [Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5] \times K_n = (Y_5 \times K_n) \oplus (Y_5 \times K_n) \oplus \ldots \oplus (Y_5 \times K_n)$. Lemma 2 which shows that a $Y_5 \times K_n$, has a Y_5-tree decomposition. Thus, we obtain a Y_5-tree decomposition for $K_{8s} \times K_n$.
Case (ii): Let $m = 8s + 4, s \geq 0$. Then $K_m \times K_n = K_{8s+4} \times K_n = [(2s + 1)K_4 \oplus s(2s + 1)K_{4,4}] \times K_n = (2s + 1)[K_4 \times K_n] \oplus s(2s + 1)[K_{4,4} \times K_n]$. Now, the Lemmas 1 and 3, shown that for $K_4 \times K_n$ and $K_{4,4} \times K_n$ has a Y_5-tree decomposition. Thus, we obtain a Y_5-tree decomposition for $K_{8s+1} \times K_n$. From the aforementioned Cases, we can get a Y_5-tree decomposition for $K_m \times K_n$. $\qquad\square$

Lemma 5. *$K_5 \times K_n$ admits a Y_5-tree decomposition for all $n, n \geq 2$.*

Proof. Now $K_5 \times K_n = K_5 \times [K_2 \oplus K_2 \oplus \ldots \oplus K_2] = (K_5 \times K_2) \oplus (K_5 \times K_2) \oplus \ldots \oplus (K_5 \times K_2)$. Let $V(K_5 \times K_2) = \{\bigcup_{i=1}^{5} i_j, 1 \leq j \leq 2\}$. A Y_5-tree decomposition can be derives in $K_5 \times K_2$ as follows: $\{(1_2, 2_1, 4_2, 5_1; 4_2 3_1), (4_1, 5_2, 3_1, 2_2; 3_1 1_2), (2_1, 5_2, 1_1, 2_2; 1_1 4_2), (2_2, 4_1, 3_2, 1_1; 3_2 2_1), (4_1, 1_2, 5_1, 3_2; 5_1 2_2)\}$. Thus, we obtain a Y_5-tree decomposition for $K_5 \times K_n$. $\qquad\square$

Lemma 6. *$K_{5,4} \times K_n$ admits a Y_5-tree decomposition for all $n, n \geq 2$.*

Proof. Let $K_{5,4} \times K_n = K_{5,4} \times [K_2 \oplus K_2 \oplus \ldots \oplus K_2]$. Let $V(K_{5,4}) = \bigcup_{i=1}^{2} V_i$, where $V_1 = \{1_j \mid 1 \leq j \leq 5\}$ and $V_2 = \{2_j \mid 1 \leq j \leq 4\}$. Clearly, the set $\{(1_1, 2_2, 1_3, 2_3; 1_3 2_4), (2_2, 1_2, 2_3, 1_4; 2_3 1_5), (1_5, 2_4, 1_1, 2_1; 1_1 2_3), (1_2, 2_4, 1_4, 2_2; 1_4 2_1), (2_2, 1_5, 2_1, 1_2; 2_1 1_3)\}$, proves a Y_5-tree decomposition exists in $K_{5,4}$. Now $K_{5,4} \times K_n = (Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5) \times (K_2 \oplus K_2 \oplus \ldots \oplus K_2) = (Y_5 \times K_2) \oplus (Y_5 \times K_2) \oplus \ldots \oplus (Y_5 \times K_2)$. From the Lemma 2, we obtain a Y_5-tree decomposition in $Y_5 \times K_2$. Thus, we get a Y_5-tree decomposition for $K_{5,4} \times K_n$. $\qquad\square$

Lemma 7. $K_m \times K_n$ *admits a Y_5-tree decomposition for $m \equiv 1 \pmod 4$ and for all n, $n \geq 2$.*

Proof. Here $m \equiv 1 \pmod 4$ can be classified into two Cases (i) $m \equiv 1 \pmod 8$ and (ii) $m \equiv 5 \pmod 8$.

Case (i): Let $m = 8s + 1$, $s \geq 1$. Then $K_m \times K_n = K_{8s+1} \times K_n$. We have a Y_5-tree decomposition for K_m, according to the theorem 1. Then $K_{8s+1} \times K_n = [Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5] \times K_n = (Y_5 \times K_n) \oplus (Y_5 \times K_n) \oplus \ldots \oplus (Y_5 \times K_n)$. Now, from Lemma 2, which has been shown that a Y_5-tree decomposition holds for $Y_5 \times K_n$. Thus, we obtain a Y_5-tree decomposition for $K_{8s+1} \times K_n$.

Case (ii): Let $m = 8s+5$, $s \geq 0$. Then $K_m \times K_n = K_{8s+5} \times K_n = [K_5 \oplus 2sK_4 \oplus s(2s-1)K_{4,4} \oplus 2sK_{5,4}] \times K_n = [K_5 \times K_n] \oplus 2s[K_4 \times K_n] \oplus s(2s-1)[K_{4,4} \times K_n] \oplus 2s[K_{5,4} \times K_n]$. Now, the Lemmas 5, 1, 3 and 6, show that a Y_5-tree decomposition holds in $K_5 \times K_n$, $K_4 \times K_n$, $K_{4,4} \times K_n$ and $K_{5,4} \times K_n$ respectively. Thus, we obtain a Y_5-tree decomposition for $K_{8s+5} \times K_n$. From the aforementioned Cases, we can get a Y_5-tree decomposition for $K_m \times K_n$. □

Lemma 8. $K_{4,3} \times K_n$ *admits a Y_5-tree decomposition for all n, $n \geq 2$.*

Proof. Let $K_{4,3} \times K_n = K_{4,3} \times [K_2 \oplus K_2 \oplus \ldots \oplus K_2]$. Let $V(K_{4,3}) = \bigcup_{i=1}^{2} V_i$, where $V_1 = \{1_j \mid 1 \leq j \leq 4\}$ and $V_2 = \{2_j \mid 1 \leq j \leq 3\}$. Then the following set $\{(2_2, 1_2, 2_3, 1_1; 2_3 1_3), (2_1, 1_1, 2_2, 1_4; 2_2 1_3), (2_3, 1_4, 2_1, 1_3; 2_1 1_2)\}$ gives a Y_5-tree decomposition for $K_{4,3}$. Now $K_{4,3} \times K_n = (Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5) \times (K_2 \oplus K_2 \oplus \ldots \oplus K_2) = (Y_5 \times K_2) \oplus (Y_5 \times K_2) \oplus \ldots \oplus (Y_5 \times K_2)$. Now, the Lemma 2, has shown that for $Y_5 \times K_2$ has a Y_5-tree decomposition. Thus, we obtain a Y_5-tree decomposition for $K_{4,3} \times K_n$. □

Lemma 9. $Y_5 \times Y_5$ *admits a Y_5-tree decomposition.*

Proof. Let us write as $Y_5 \times Y_5 = Y_5 \times [K_2 \oplus K_2 \oplus K_2 \oplus K_2] = (Y_5 \times K_2) \oplus (Y_5 \times K_2) \oplus (Y_5 \times K_2) \oplus (Y_5 \times K_2)$. We have a Y_5-tree decomposition for $Y_5 \times K_2$, according to the Lemma 2. Thus, we obtain a Y_5-tree decomposition for $Y_5 \times Y_5$. □

Lemma 10. $K_m \times K_n$ *admits a Y_5-tree decomposition for $m \equiv 3 \pmod 4$ and $n \equiv 0 \pmod 4$.*

Proof. Let $m = 4s + 3$, $s \geq 0$ and $n = 4t$, $t \geq 1$. We write $K_m \times K_n = K_{4s+3} \times K_{4t} = [sK_4 \oplus K_3 \oplus sK_{4,3} \oplus \frac{s(s-1)}{2}K_{4,4}] \times K_{4t} = s[K_4 \times K_{4t}] \oplus [K_3 \times K_{4t}] \oplus s[K_{4,3} \times K_{4t}] \oplus \frac{s(s-1)}{2}[K_{4,4} \times K_{4t}] = s[K_4 \times K_{4t}] \oplus t[K_3 \times K_4] \oplus \frac{t(t-1)}{2}[K_3 \times K_{4,4}] \oplus s[K_{4,3} \times K_{4t}] \oplus \frac{s(s-1)}{2}[K_{4,4} \times K_{4t}]$. Now, the Lemmas 1, 3 and 8 show that a Y_5-tree decomposition holds in $K_4 \times K_{4t}$, $K_3 \times K_4$, $K_3 \times K_{4,4}$, $K_{4,4} \times K_{4t}$ and $K_{4,3} \times K_{4t}$ respectively. Thus, we obtain a Y_5-tree decomposition for $K_m \times K_n$. □

Lemma 11. $K_m \times K_n$ *admits a Y_5-tree decomposition for $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$.*

Proof. Let $m = 4s + 3$, $s \geq 0$ and $n = 4t + 1$, $t \geq 1$. We write $K_m \times K_n = K_{4s+3} \times K_{4t+1} = [sK_4 \oplus K_3 \oplus sK_{4,3} \oplus \frac{s(s-1)}{2}K_{4,4}] \times [K_5 \oplus (t-1)K_4 \oplus \frac{(t-1)(t-2)}{2}K_{4,4} \oplus (t-1)K_{5,4}] = \{s[K_4 \times K_5] \oplus s(t-1)[K_4 \times K_4] \oplus \frac{s(t-1)(t-2)}{2}[K_4 \times K_{4,4}] \oplus s(t-1)[K_4 \times K_{5,4}]\} \oplus \{[K_3 \times K_5] \oplus (t-1)[K_3 \times K_4] \oplus \frac{(t-1)(t-2)}{2}[K_3 \times K_{4,4}] \oplus (t-1)[K_3 \times K_{5,4}]\} \oplus \{s[K_{4,3} \times K_5] \oplus s(t-1)[K_{4,3} \times K_4] \oplus \frac{s(t-1)(t-2)}{2}[K_{4,3} \times K_{4,4}] \oplus s(t-1)[K_{4,3} \times K_{5,4}]\} \oplus \{\frac{s(s-1)}{2}[K_{4,4} \times K_5] \oplus \frac{s(s-1)(t-1)}{2}[K_{4,4} \times K_4] \oplus \frac{s(s-1)(t-1)(t-2)}{4}[K_{4,4} \times K_{4,4}] \oplus \frac{s(s-1)(t-1)}{2}[K_{4,4} \times K_{5,4}]\}$. Now, the Lemmas 1, 3, 5, 6, 8 and 9 show that a Y_5-tree decomposition holds in $K_4 \times K_n$, $K_{4,4} \times K_n$, $K_5 \times K_n$, $K_{5,4} \times K_n$, $K_{4,3} \times K_n$ and $Y_5 \times Y_5$ respectively. Thus, we obtain a Y_5-tree decomposition for $K_m \times K_n$. □

Lemma 12. $K_{4,2} \times K_n$ *admits a Y_5-tree decomposition for all n, $n \geq 2$.*

Proof. Let $K_{4,2} \times K_n = K_{4,2} \times [K_2 \oplus K_2 \oplus \ldots \oplus K_2]$. Let $V(K_{4,2}) = \bigcup_{i=1}^{2} V_i$, where $V_1 = \{1_j \mid 1 \leq j \leq 4\}$ and $V_2 = \{2_j \mid 1 \leq j \leq 2\}$. Clearly, the set $\{(2_1, 1_3, 2_2, 1_1; 2_2 1_2), (2_2, 1_4, 2_1, 1_2; 2_1 1_1)\}$, proves a Y_5-tree decomposition exists in $K_{4,2}$. Now $K_{4,2} \times K_n = (Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5) \times (K_2 \oplus K_2 \oplus \ldots \oplus K_2) = (Y_5 \times K_2) \oplus (Y_5 \times K_2) \oplus \ldots \oplus (Y_5 \times K_2)$. Now, the Lemma 2, shows that for $Y_5 \times K_2$ has a Y_5-tree decomposition. Thus, we obtain a Y_5-tree decomposition for $K_{4,2} \times K_n$. □

Lemma 13. $K_m \times K_n$ *admits a* Y_5-*tree decomposition for* $m \equiv 2$ (mod 4) *and* $n \equiv 0$ (mod 4).

Proof. Let $m = 4s+2, s \geq 0$ and $n = 4t, t \geq 1$. We write $K_m \times K_n = K_{4s+2} \times K_{4t} = [sK_4 \oplus K_2 \oplus \frac{s(s-1)}{2}K_{4,4} \oplus sK_{4,2}] \times K_{4t} = [sK_4 \oplus K_2 \oplus \frac{s(s-1)}{2}K_{4,4} \oplus sK_{4,2}] \times [tK_4 \oplus \frac{t(t-1)}{2}K_{4,4}] = \{st[K_4 \times K_4] \oplus \frac{st(t-1)}{2}[K_4 \times K_{4,4}]\} \oplus \{t[K_2 \times K_4] \oplus \frac{t(t-1)}{2}[K_2 \times K_{4,4}]\} \oplus \{\frac{s(s-1)}{2}t[K_{4,4} \times K_4] \oplus \frac{s(s-1)t(t-1)}{4}[K_{4,4} \times K_{4,4}]\} \oplus \{st[K_{4,2} \times K_4] \oplus \frac{st(t-1)}{2}[K_{4,2} \times K_{4,4}]\}$. Now, the Lemmas 1, 3, 9 and 12 show that a Y_5-tree decomposition holds in $K_4 \times K_n, K_{4,4} \times K_n, Y_5 \times Y_5$ and $K_{4,2} \times K_n$ respectively. Thus, we obtain a Y_5-tree decomposition for $K_m \times K_n$. $\square$

Lemma 14. $K_m \times K_n$ *admits a* Y_5-*tree decomposition for* $m \equiv 2$ (mod 4) *and* $n \equiv 1$ (mod 4).

Proof. Let $m = 4s + 2, s \geq 0$ and $n = 4t + 1, t \geq 1$. Then $K_m \times K_n = K_{4s+2} \times K_{4t+1} = [sK_4 \oplus K_2 \oplus \frac{s(s-1)}{2}K_{4,4} \oplus sK_{4,2}] \times [K_5 \oplus (t-1)K_4 \oplus \frac{(t-1)(t-2)}{2}K_{4,4} \oplus (t-1)K_{5,4}] = \{\{s[K_4 \times K_5] \oplus s(t-1)[K_4 \times K_4] \oplus \frac{s(t-1)(t-2)}{2}[K_4 \times K_{4,4}] \oplus s(t-1)[K_4 \times K_{5,4}]\} \oplus \{[K_2 \times K_5] \oplus (t-1)[K_2 \times K_4] \oplus \frac{(t-1)(t-2)}{2}[K_2 \times K_{4,4}] \oplus (t-1)[K_2 \times K_{5,4}]\} \oplus \{\frac{s(s-1)}{2}[K_{4,4} \times K_5] \oplus \frac{s(s-1)(t-1)}{2}[K_{4,4} \times K_4] \oplus \frac{s(s-1)(t-1)(t-2)}{4}[K_{4,4} \times K_{4,4}] \oplus \frac{s(s-1)(t-1)}{2}[K_{4,4} \times K_{5,4}]\} \oplus \{s[K_{4,2} \times K_5] \oplus s(t-1)[K_{4,2} \times K_4] \oplus \frac{s(t-1)(t-2)}{2}[K_{4,2} \times K_{4,4}] \oplus s(t-1)[K_{4,2} \times K_{5,4}]\}\}$. Now, the Lemmas 1, 3, 5, 6, 9 and 12 show that a Y_5-tree decomposition holds in $K_4 \times K_n, K_{4,4} \times K_n, K_5 \times K_2, K_{5,4} \times K_n, Y_5 \times Y_5$ and $K_{4,2} \times K_n$ respectively. Thus, we obtain a Y_5-tree decomposition for $K_m \times K_n$. $\square$

Theorem 2. *A* Y_5-*tree decomposition for* $K_m \times K_n$ *holds, if and only if,* $mn(m-1)(n-1) \equiv 0$ (mod 8).

Proof. One can obtain a necessary condition to find a Y_5-tree decomposition for $K_m \times K_n$ as $mn(m-1)(n-1) \equiv 0$ (mod 8). From the Lemmas 4, 7, 10, 11, 13 and 14, we can conclude the aforementioned necessary condition is sufficient. $\square$

3. Gregarious Y_5-tree Decomposition for $K_m \times K_n$

Lemma 15. $Y_5 \times K_n$ *admits a gregarious* Y_5-*tree decomposition for all* n, $n \geq 2$.

Proof. Let us suppose as $Y_5 \times K_n = Y_5 \times [K_2 \oplus K_2 \oplus \ldots \oplus K_2] = (Y_5 \times K_2) \oplus (Y_5 \times K_2) \oplus \ldots \oplus (Y_5 \times K_2)$. Let $V(Y_5 \times K_2) = \{\bigcup_{i=1}^{5} i_j, 1 \leq j \leq 2\}$. A gregarious Y_5-tree decomposition can be derives in $Y_5 \times K_2$ as follows: $\{(1_1, 2_2, 3_1, 4_2; 3_1 5_2), (1_2, 2_1, 3_2, 4_1; 3_2 5_1)\}$. Thus, we obtain a gregarious Y_5-tree decomposition for $Y_5 \times K_n$. $\square$

Lemma 16. $K_m \times K_n$ *admits a gregarious* Y_5-*tree decomposition for* $m \equiv 0, 1$ (mod 8) *and for all* n, $n \geq 2$.

Proof. Since by theorem 1, Y_5-tree decomposition for K_m, $m = 8s$, *or*, $8s + 1$ exists, let us write as $K_m \times K_n = [Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5] \times K_n = [Y_5 \times K_n] \oplus [Y_5 \times K_n] \oplus \ldots \oplus [Y_5 \times K_n]$. Now, the Lemma 15, which shows that $Y_5 \times K_n$ has a gregarious Y_5-tree decomposition. Thus, we obtain a gregarious Y_5-tree decomposition for $K_m \times K_n$. $\square$

Lemma 17. $K_5 \times K_n$ *admits a gregarious* Y_5-*tree decomposition for all* $n, n \geq 2$.

Proof. Let us suppose as $K_5 \times K_n = K_5 \times [K_2 \oplus K_2 \oplus \ldots \oplus K_2] = (K_5 \times K_2) \oplus (K_5 \times K_2) \oplus \ldots \oplus (K_5 \times K_2)$. Let $V(K_5 \times K_2) = \{\bigcup_{i=1}^{5} i_j, 1 \leq j \leq 2\}$. A gregarious Y_5-tree decomposition can be derives in $K_5 \times K_2$ as follows: $\{(1_1, 5_2, 2_1, 4_2; 2_1 3_2), (2_1, 1_2, 3_1, 5_2; 3_1 4_2), (3_1, 2_2, 4_1, 1_2; 4_1 5_2), (4_1, 3_2, 5_1, 2_2; 5_1 1_2), (5_1, 4_2, 1_1, 3_2; 1_1 2_2)\}$. Thus, we obtain a gregarious Y_5-tree decomposition for $K_5 \times K_n$. $\square$

Lemma 18. $K_{8,8} \times K_n$ *admits a gregarious* Y_5-*tree decomposition for all* $n, n \geq 2$.

Proof. Let $V(K_{8,8}) = \{\bigcup_{i=1}^{2} i_j, 1 \leq j \leq 8\}$. For $0 \leq s \leq 1$ and $1 \leq i \leq 8$, $\kappa_s^i = \{1_{i+8s}, 2_{i+(3s+4)}, 1_{i+(4s+3)}, 2_{i+(8s+5)}; 1_{i+(4s+3)} 2_{i+(6s+6)}\}$, by taking the residues of Z_8 as $\{1, 2, 3, \ldots, 8\}$ for subscripts. Let it be clear that, the set $\{\kappa_0^1, \kappa_1^1\}$ provides a base block of Y_5-tree decomposition for $K_{8,8}$.

Thus, the collection $\kappa = \{\kappa_0^i, \kappa_1^i\}$, $1 \le i \le 8$, holds the Y_5-tree decomposition for $K_{8,8}$. Now we write, $K_{8,8} \times K_n = [Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5] \times K_n = (Y_5 \times K_n) \oplus (Y_5 \times K_n) \oplus \ldots \oplus (Y_5 \times K_n)$. Then, the Lemma 15, which shows that $Y_5 \times K_n$ has a gregarious Y_5-tree decomposition. Thus, we obtain a gregarious Y_5-tree decomposition for $K_{8,8} \times K_n$. $\qquad\square$

Lemma 19. $K_{8,5} \times K_n$ *admits a gregarious* Y_5-*tree decomposition for all* $n, n \ge 2$.

Proof. Let $V(K_{8,5}) = \bigcup_{i=1}^{2} V_i$, where $V_1 = \{1_j \mid 1 \le j \le 8\}$ and $V_2 = \{2_j \mid 1 \le j \le 5\}$. Then the following set gives a Y_5-tree decomposition for $K_{8,5}$:
$\{(2_1, 1_1, 2_3, 1_4; 2_3 1_5), (2_5, 1_8, 2_4, 1_3; 2_4 1_2), (2_5, 1_3, 2_2, 1_1; 2_2 1_4), (1_5, 2_1, 1_6, 2_3; 1_6 2_4),$
$(1_6, 2_5, 1_2, 2_1; 1_2 2_2), (2_1, 1_3, 2_3, 1_7; 2_3 1_2), (2_5, 1_7, 2_4, 1_1; 2_4 1_5), (2_5, 1_5, 2_2, 1_6; 2_2 1_7),$
$(1_1, 2_5, 1_4, 2_4; 1_4 2_1), (1_7, 2_1, 1_8, 2_2; 1_8 2_3)\}$. Let $K_{8,5} \times K_n = [Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5] \times K_n = (Y_5 \times K_n) \oplus (Y_5 \times K_n) \oplus \ldots \oplus (Y_5 \times K_n)$. Now, the Lemma 15, which shows that $Y_5 \times K_n$ has a gregarious Y_5-tree decomposition. Thus, we obtain a gregarious Y_5-tree decomposition for $K_{8,5} \times K_n$. $\qquad\square$

Lemma 20. $K_m \times K_n$ *admits a gregarious* Y_5-*tree decomposition for* $m \equiv 5 \pmod 8$ *and for all* n, $n \ge 2$.

Proof. Let $m = 8s + 5$, $s \ge 0$. Then $K_m \times K_n = K_{8s+5} \times K_n = [sK_8 \oplus K_5 \oplus \frac{s(s-1)}{2} K_{8,8} \oplus sK_{8,5}] \times K_n = s[K_8 \times K_n] \oplus [K_5 \times K_n] \oplus \frac{s(s-1)}{2}[K_{8,8} \times K_n] \oplus s[K_{8,5} \times K_n]$. Now, the Lemmas 16, 17, 18 and 19, show that a gregarious Y_5-tree decomposition holds in $K_8 \times K_n$, $K_5 \times K_n$, $K_{8,8} \times K_n$ and $K_{8,5} \times K_n$ respectively. Thus, we obtain a gregarious Y_5-tree decomposition for $K_m \times K_n$. $\qquad\square$

Lemma 21. $K_{12} \times K_n$ *admits a gregarious* Y_5-*tree decomposition.*

Proof. Let $K_{12} \times K_n = [K_5 \oplus (K_{12} \setminus K_5)] \times K_n = [K_5 \times K_n] \oplus [(K_{12} \setminus K_5) \times K_n]$. Let $V(K_{12} \setminus K_5) = \{1, 2, 3, \ldots, 12\}$. Then the following set yields a Y_5-tree decomposition for $K_{12} \setminus K_5$:
$\{(11, 1, 12, 5; 12\ 6), (12, 2, 1, 6; 1\ 7), (1, 3, 2, 7; 2\ 8), (2, 4, 3, 8; 3\ 9), (3, 5, 4, 9; 4\ 10),$
$(4, 8, 5, 10; 5\ 11), (1, 4, 12, 7; 12\ 3), (6, 5, 1, 8; 1\ 9), (3, 6, 2, 9; 2\ 10), (8, 7, 3, 10; 3\ 11),$
$(7, 6, 4, 11; 4\ 7), (5, 7, 10, 6; 10\ 1), (8, 6, 9, 7; 9\ 5), (5, 2, 11, 6; 11\ 7)\}$.
Therefore, $K_{12} \times K_n = [K_5 \times K_n] \oplus [(Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5) \times K_n] = [K_5 \times K_n] \oplus [(Y_5 \times K_n) \oplus (Y_5 \times K_n) \ldots (Y_5 \times K_n)]$. Now, the Lemmas 15 and 17, show that a gregarious Y_5-tree decomposition holds in $Y_5 \times K_n$ and $K_5 \times K_n$ respectively. Thus, we obtain a gregarious Y_5-tree decomposition for $K_{12} \times K_n$. $\qquad\square$

Lemma 22. $K_{12,8} \times K_n$ *admits a gregarious* Y_5-*tree decomposition for all* $n, n \ge 2$.

Proof. Let $V(K_{12,8}) = \bigcup_{i=1}^{2} V_i$, where $V_1 = \{1_j \mid 1 \le j \le 12\}$ and $V_2 = \{2_j \mid 1 \le j \le 8\}$. Then the following set gives a Y_5-tree decomposition of $K_{12,8}$.
$\{(1_7, 2_2, 1_4, 2_8; 1_4 2_5), (1_2, 2_3, 1_5, 2_1; 1_5 2_6), (1_8, 2_7, 1_4, 2_6; 1_4 2_3), (2_8, 1_5, 2_7, 1_{12}; 2_7 1_7),$
$(1_8, 2_8, 1_{11}, 2_7; 1_{11} 2_6), (1_3, 2_1, 1_9, 2_7; 1_9 2_5), (1_8, 2_2, 1_2, 2_4; 1_2 2_6), (1_6, 2_2, 1_1, 2_1; 1_1 2_8),$
$(1_6, 2_8, 1_{12}, 2_5; 1_{12} 2_3), (1_2, 2_8, 1_9, 2_3; 1_9 2_4), (2_1, 1_{11}, 2_3, 1_8; 2_3 1_{10}), (1_9, 2_2, 1_{12}, 2_4; 1_{12} 2_1),$
$(1_6, 2_4, 1_{11}, 2_5; 1_{11} 2_2), (1_2, 2_5, 1_5, 2_2; 1_5 2_4), (2_4, 1_7, 2_5, 1_8; 2_5 1_1), (2_4, 1_4, 2_1, 1_6; 2_1 1_2),$
$(1_1, 2_6, 1_8, 2_4; 1_8 2_1), (1_{12}, 2_6, 1_3, 2_4; 1_3 2_2), (1_1, 2_3, 1_7, 2_6; 1_7 2_1), (2_1, 1_{10}, 2_7, 1_2; 2_7 1_1),$
$(1_1, 2_4, 1_{10}, 2_2; 1_{10} 2_5), (2_8, 1_{10}, 2_6, 1_9; 2_6 1_6), (1_3, 2_5, 1_6, 2_3; 1_6 2_7), (1_7, 2_8, 1_3, 2_3; 1_3 2_7)\}$. Now $K_{12,8} \times K_n = [Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5] \times K_n = (Y_5 \times K_n) \oplus (Y_5 \times K_n) \oplus \ldots \oplus (Y_5 \times K_n)$. Now, the Lemma 15, which shows that a gregarious Y_5-tree decomposition holds in $Y_5 \times K_n$. Thus, $K_{12,8} \times K_n$ has a gregarious Y_5-tree decomposition. This completes the proof. $\qquad\square$

Lemma 23. $K_m \times K_n$ *admits a gregarious* Y_5-*tree decomposition for* $m \equiv 4 \pmod 8$ *and for all* n, $n \ge 2$.

Proof. Let $m = 8s + 4, s \geq 1$. Then $K_m \times K_n = K_{8s+4} \times K_n = [K_{12} \oplus (s-1)K_8 \oplus (s-1)K_{12,8} \oplus \frac{(s-2)(s-1)}{2}K_{8,8}] \times K_n = [K_{12} \times K_n] \oplus [(s-1)K_8 \times K_n] \oplus [(s-1)K_{12,8} \times K_n] \oplus \frac{(s-2)(s-1)}{2}[K_{8,8} \times K_n]$. Now, the Lemmas 21, 16, 22 and 18, shows that a gregarious Y_5-tree decomposition holds in $K_{12} \times K_n$, $K_8 \times K_n$, $K_{12,8} \times K_n$ and $K_{8,8} \times K_n$ respectively. Thus, we obtain a gregarious Y_5-tree decomposition for $K_m \times K_n$. $\square$

Lemma 24. $K_6 \times K_4$ *admits a gregarious Y_5-tree decomposition.*

Proof. Let $V(K_6) = \{1, 2, 3, \dots 6\}$ and $V(K_4) = \{1, 2, 3, 4\}$. Then $V(K_6 \times K_4) = \bigcup_{i=1}^{6} V_i$, $V_i = \{i_j \mid 1 \leq j \leq 4\}$. Let it be clear that the set given below yields a gregarious Y_5-tree decomposition for $K_6 \times K_4$.
$\{(5_1, 3_2, 1_4, 6_1; 1_4 2_3), (6_4, 4_2, 2_4, 1_1; 2_4 5_2), (1_4, 5_2, 3_4, 2_1; 3_4 4_3), (2_4, 6_2, 4_4, 3_1; 4_4 5_3),$
$(3_4, 1_2, 5_4, 4_1; 5_4 6_3), (4_4, 2_2, 6_4, 5_1; 6_4 1_3), (3_3, 6_1, 1_2, 2_3; 1_2 4_1), (4_3, 1_1, 2_2, 3_3; 2_2 5_1),$
$(5_3, 2_1, 3_2, 4_1; 3_2 6_1), (6_3, 3_1, 4_2, 5_3; 4_2 1_1), (1_3, 4_1, 5_2, 6_3; 5_2 2_1), (2_3, 5_1, 6_2, 1_3; 6_2 3_1),$
$(5_2, 1_1, 6_3, 3_2; 6_3 2_4), (6_2, 2_1, 1_3, 4_2; 1_3 5_2), (1_2, 3_1, 2_3, 5_4; 2_3 6_2), (2_2, 4_1, 3_3, 6_2; 3_3 1_2),$
$(3_2, 5_4, 4_3, 1_2; 4_3 6_2), (4_2, 6_1, 5_3, 2_2; 5_3 3_2), (1_2, 2_1, 6_4, 5_2; 6_4 3_2), (2_2, 3_1, 1_4, 6_2; 1_4 4_2),$
$(5_1, 4_3, 2_4, 1_2; 2_4 3_3), (4_2, 5_1, 3_4, 2_2; 3_4 6_2), (3_2, 4_4, 6_1, 5_2; 6_1 1_3), (6_2, 1_1, 5_4, 4_2; 5_4 2_2),$
$(1_1, 6_4, 2_3, 3_4; 2_3 4_2), (2_1, 1_4, 3_3, 6_4; 3_3 5_2), (3_1, 2_4, 4_1, 1_4; 4_1 6_2), (4_3, 6_4, 5_3, 2_4; 5_3 1_2),$
$(5_1, 4_4, 6_3, 3_4, 6_3 2_2), (6_1, 5_4, 1_3, 4_4; 1_3 3_2), (1_2, 5_1, 2_4, 6_1; 2_4 3_2), (2_2, 6_1, 3_4, 1_1; 3_4 4_2),$
$(3_2, 1_1, 4_4, 2_1; 4_4 5_2), (4_2, 2_1, 5_4, 3_1; 5_4 6_2), (5_2, 3_1, 6_4, 4_1; 6_4 1_2), (5_1, 1_4, 4_3, 3_2; 4_3 2_2),$
$(6_2, 5_3, 3_1, 1_3; 3_1 4_3), (1_2, 6_3, 4_1, 2_3; 4_1 5_3), (2_2, 1_3, 5_1, 3_3, 5_1 6_3), (4_3, 6_1, 2_3, 3_2; 2_3 5_2),$
$(4_2, 3_3, 1_1, 5_3; 1_1 2_3), (5_2, 4_3, 2_1, 6_3; 2_1 3_3), (4_2, 6_3, 1_4, 5_3; 1_4 2_2), (2_4, 1_3, 3_4, 5_3; 3_4 4_1),$
$(5_4, 3_3, 4_4, 2_3; 4_4 1_2)\}$. $\square$

Lemma 25. $K_{8,6} \times K_4$ *admits a gregarious Y_5-tree decomposition.*

Proof. Let us write as $V(K_{8,6}) = \bigcup_{i=1}^{2} V_i$, $V_1 = \{1_j \mid 1 \leq j \leq 8\}$ and $V_2 = \{2_j \mid 1 \leq j \leq 6\}$. Then the following set gives a Y_5-tree decomposition of $K_{8,6}$.
$\{(2_6, 1_1, 2_3, 1_4; 2_3 1_5), (2_6, 1_8, 2_4, 1_3; 2_4 1_2), (1_4, 2_6, 1_3, 2_5; 1_3 2_1), (1_5, 2_1, 1_6, 2_3; 1_6 2_4),$
$(1_6, 2_5, 1_2, 2_6; 1_2 2_2), (2_2, 1_3, 2_3, 1_7; 2_3 1_2), (1_5, 2_4, 1_7, 2_6; 1_7 2_5), (1_1, 2_5, 1_4, 2_4; 1_4 2_1),$
$(1_7, 2_1, 1_8, 2_2; 1_8 2_3), (2_6, 1_6, 2_2, 1_7; 2_2 1_4), (1_8, 2_5, 1_5, 2_2; 1_5 2_6), (1_2, 2_1, 1_1, 2_2; 1_1 2_4)\}$. Then $K_{8,6} \times K_4 = [Y_5 \oplus Y_5 \oplus \dots \oplus Y_5] \times K_4 = (Y_5 \times K_4) \oplus (Y_5 \times K_4) \oplus \dots \oplus (Y_5 \times K_4)$. A gregarious Y_5-tree decomposition for $Y_5 \times K_4$ derives from Lemma 15, which provides a gregarious Y_5-tree decomposition for $K_{8,6} \times K_4$. $\square$

Lemma 26. $K_m \times K_n$ *admits a gregarious Y_5-tree decomposition for $m \equiv 6$ (mod 8) and $n = 4$.*

Proof. Let $m = 8s + 6, s \geq 1$. Then $K_m \times K_n = K_{8s+6} \times K_4 = [sK_8 \oplus K_6 \oplus \frac{s(s-1)}{2}K_{8,8} \oplus sK_{8,6}] \times K_4 = s[K_8 \times K_4] \oplus [K_6 \times K_4] \oplus \frac{s(s-1)}{2}[K_{8,8} \times K_4] \oplus s[K_{8,6} \times K_4]$. Now, the Lemmas 16, 18, 24 and 25, show that a gregarious Y_5-tree decomposition holds in $K_8 \times K_4$, $K_{8,8} \times K_4$, $K_6 \times K_4$ and $K_{8,6} \times K_4$ respectively. Thus, we obtain a gregarious Y_5-tree decomposition for $K_m \times K_n$. $\square$

Lemma 27. $K_7 \times K_4$ *admits a gregarious Y_5-tree decomposition.*

Proof. Let us write as $K_7 \times K_4 = [K_5 \oplus (K_7 \setminus K_5)] \times K_4 = [K_5 \times K_4] \oplus [(K_7 \setminus K_5) \times K_4]$. By considering $V(K_7 \setminus K_5) = \{1, 2, 3, \dots, 7\}$ and $V(K_4) = \{1, 2, 3, 4\}$, we write as $V([K_7 \setminus K_5] \times K_4) = \bigcup_{i=1}^{7} V_i$, $V_i = \{i_j \mid 1 \leq j \leq 4\}$. Therefore, the set given below yields a gregarious Y_5-tree decomposition for $[K_7 \setminus K_5] \times K_4$.
$\{(2_2, 3_3, 1_2, 7_4; 1_2 6_4), (2_3, 3_4, 1_3, 7_1; 1_3 4_2), (2_4, 3_1, 1_4, 7_2; 1_4 6_1), (2_1, 3_2, 1_1, 7_3; 1_1 6_3),$
$(1_2, 7_3, 2_4, 5_1; 2_4 6_1), (1_3, 7_4, 2_1, 5_2; 2_1 6_2), (7_1, 1_4, 2_2, 5_3; 2_2 6_3), (1_1, 7_2, 2_3, 5_4; 2_3 6_4),$
$(7_2, 2_4, 1_1, 6_2; 1_1 3_4), (7_3, 2_1, 1_2, 6_3; 1_2 3_1), (1_3, 3_1, 2_2, 6_4; 2_2 7_4), (3_3, 1_4, 2_3, 6_1; 2_3 7_1),$
$(1_1, 5_3, 2_4, 4_1; 2_4 3_2), (1_2, 5_4, 2_1, 4_2; 2_1 3_4), (1_3, 5_1, 2_2, 4_3; 2_2 7_1), (1_4, 5_2, 2_3, 4_4; 2_3 3_2),$
$(1_1, 5_2, 2_4, 4_3; 2_4 3_3), (1_2, 5_3, 2_1, 4_4; 2_1 7_2), (1_3, 6_1, 2_2, 4_1; 2_2 7_3), (1_4, 5_1, 2_3, 4_2; 2_3 7_4),$

$(3_4, 2_2, 1_1, 6_4; 1_1 4_2), (3_1, 2_3, 1_2, 6_1; 1_2 4_3), (3_2, 1_3, 2_4, 6_2; 2_4 7_1), (3_3, 2_1, 1_4, 6_2; 1_4 4_1),$
$(2_1, 4_3, 1_1, 5_4; 1_1 3_3), (2_2, 4_4, 1_2, 5_1; 1_2 3_4), (2_3, 4_1, 1_3, 5_2; 1_3 6_4), (2_4, 4_2, 1_4, 5_3; 1_4 3_2),$
$(5_4, 2_2, 1_3, 4_4; 1_3 6_2), (6_3, 2_4, 1_2, 7_1; 1_2 4_1), (6_4, 2_1, 1_3, 7_2; 1_3 5_4), (2_1, 6_3, 1_4, 7_3; 1_4 4_3),$
$(6_2, 2_3, 1_1, 7_4; 1_1 4_4)\}$. Now, the Lemmas 17, which shows that a gregarious Y_5-tree decomposition holds in $K_5 \times K_4$. Thus, we obtain a gregarious Y_5-tree decomposition for $K_7 \times K_4$. $\square$

Lemma 28. $K_{8,7} \times K_4$ *admits a gregarious Y_5-tree decomposition.*

Proof. Let $V(K_{8,7}) = \bigcup_{i=1}^{2} V_i$, where $V_1 = \{1_j \mid 1 \le j \le 8\}$ and $V_2 = \{2_j \mid 1 \le j \le 7\}$. Then the following set gives a Y_5-tree decomposition of $K_{8,7}$.
$\{(2_6, 1_1, 2_3, 1_4; 2_3 1_5), (2_6, 1_8, 2_7, 1_3; 2_7 1_2), (1_4, 2_6, 1_3, 2_5; 1_3 2_1), (1_5, 2_1, 1_6, 2_3; 1_6 2_7),$
$(1_6, 2_5, 1_2, 2_6; 1_2 2_2), (2_2, 1_3, 2_3, 1_7; 2_3 1_2), (1_3, 2_4, 1_7, 2_6; 1_7 2_5), (1_2, 2_4, 1_4, 2_5; 1_4 2_1),$
$(1_7, 2_1, 1_8, 2_2; 1_8 2_3), (2_6, 1_6, 2_2, 1_7; 2_2 1_4), (1_8, 2_5, 1_5, 2_2; 1_5 2_7), (1_2, 2_1, 1_1, 2_2; 1_1 2_4),$
$(2_6, 1_5, 2_4, 1_6; 2_4 1_8), (2_5, 1_1, 2_7, 1_7; 2_7 1_4)\}$. Then $K_{8,7} \times K_4 = [Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5] \times K_4 = (Y_5 \times K_4) \oplus (Y_5 \times K_4) \oplus \ldots \oplus (Y_5 \times K_4)$. A gregarious Y_5-tree decomposition for $Y_5 \times K_4$ derives from Lemma 15, which provides a gregarious Y_5-tree decomposition for $K_{8,7} \times K_4$. $\square$

Lemma 29. $K_m \times K_n$ *admits a gregarious Y_5-tree decomposition for $m \equiv 7 \pmod 8$ and $n = 4$.*

Proof. Let $m = 8s + 7$, $s \ge 1$. Then $K_m \times K_n = K_{8s+7} \times K_4 = [sK_8 \oplus K_7 \oplus \frac{s(s-1)}{2}K_{8,8} \oplus sK_{8,7}] \times K_4 = s[K_8 \times K_4] \oplus [K_7 \times K_4] \oplus \frac{s(s-1)}{2}[K_{8,8} \times K_4] \oplus s[K_{8,7} \times K_4]$. Now, the Lemmas 16, 18, 27 and 28, show that a gregarious Y_5-tree decomposition holds in $K_8 \times K_4$, $K_{8,8} \times K_4$, $K_7 \times K_4$ and $K_{8,7} \times K_4$ respectively. Thus, we obtain a gregarious Y_5-tree decomposition for $K_m \times K_n$. $\square$

Lemma 30. $K_{10} \times K_4$ *admits a gregarious Y_5-tree decomposition.*

Proof. Let $K_{10} \times K_4 = [K_7 \oplus (K_{10} \setminus K_7)] \times K_4 = [K_7 \times K_4] \oplus [(K_{10} \setminus K_7) \times K_4]$. Let $V(K_{10} \setminus K_7) = \{1, 2, 3, \ldots, 10\}$. Clearly, the set $\{(1, 4, 3, 6; 3\ 10), (2, 6, 1, 8; 1\ 5), (2, 7, 3, 1; 3\ 8), (7, 1, 2, 4; 2\ 5), (5, 3, 2, 8; 2\ 10),$
$(10, 1, 9, 2; 9\ 3)\}$ gives a Y_5-tree decomposition for $K_{10} \setminus K_7$. Now $K_{10} \times K_4 = [K_7 \times K_4] \oplus [(Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5) \times K_4] = [K_7 \times K_4] \oplus [(Y_5 \times K_4) \oplus (Y_5 \times K_4) \oplus \ldots \oplus (Y_5 \times K_4)]$. Now, the Lemmas 15 and 27, which shows that a gregarious Y_5-tree decomposition holds in $Y_5 \times K_4$ and $K_7 \times K_4$ respectively. Thus, we obtain a gregarious Y_5-tree decomposition for $K_{10} \times K_4$. $\square$

Lemma 31. $K_{10,8} \times K_4$ *admits a gregarious Y_5-tree decomposition.*

Proof. Let $V(K_{10,8}) = \bigcup_{i=1}^{2} V_i$, where $V_1 = \{1_j \mid 1 \le j \le 10\}$ and $V_2 = \{2_j \mid 1 \le j \le 8\}$. Then the following set gives a Y_5-tree decomposition for $K_{10,8}$.
$\{(1_7, 2_2, 1_4, 2_8; 1_4 2_5), (1_2, 2_3, 1_5, 2_1; 1_5 2_6), (1_8, 2_7, 1_4, 2_6; 1_4 2_3), (2_8, 1_5, 2_7, 1_{10}; 2_7 1_7),$
$(1_3, 2_1, 1_{10}, 2_2; 1_{10} 2_5), (1_8, 2_2, 1_2, 2_1; 1_2 2_8), (1_6, 2_2, 1_1, 2_1; 1_1 2_8), (1_8, 2_8, 1_{10}, 2_3; 1_{10} 2_4),$
$(2_1, 1_6, 2_3, 1_8; 2_3 1_9), (1_2, 2_5, 1_5, 2_2; 1_5 2_4), (2_4, 1_7, 2_5, 1_8; 2_5 1_1), (2_1, 1_4, 2_4, 1_6; 2_4 1_2),$
$(1_{10}, 2_6, 1_8, 2_4; 1_8 2_1), (1_2, 2_6, 1_3, 2_4; 1_3 2_2), (1_1, 2_3, 1_7, 2_6; 1_7 2_1), (2_1, 1_9, 2_7, 1_2; 2_7 1_1),$
$(1_1, 2_4, 1_9, 2_2; 1_9 2_5), (2_8, 1_9, 2_6 1_1; 2_6 1_6), (1_3, 2_5, 1_6, 2_8; 1_6 2_7), (1_7, 2_8, 1_3, 2_3; 1_3 2_7)\}$. Then $K_{10,8} \times K_4 = [Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5] \times K_4 = (Y_5 \times K_4) \oplus (Y_5 \times K_4) \oplus \ldots \oplus (Y_5 \times K_4)$. A gregarious Y_5-tree decomposition of $Y_5 \times K_4$ derives from Lemma 15, which provides a gregarious Y_5-tree decomposition for $K_{10,8} \times K_4$. $\square$

Lemma 32. $K_m \times K_n$ *admits a gregarious Y_5-tree decomposition for $m \equiv 10 \pmod 8$ and $n = 4$.*

Proof. Let $m = 8s + 10$, $s \ge 1$. Then $K_m \times K_n = K_{8s+10} \times K_4 = [K_{10} \oplus sK_8 \oplus \frac{s(s-1)}{2}K_{8,8} \oplus sK_{10,8}] \times K_4 = [K_{10} \times K_4] \oplus s[K_8 \times K_4] \oplus \frac{s(s-1)}{2}[K_{8,8} \times K_4] \oplus s[K_{10,8} \times K_4]$. Now, the Lemmas 16, 18, 30 and 31, show that a gregarious Y_5-tree decomposition holds in $K_8 \times K_4$, $K_{8,8} \times K_4$, $K_{10} \times K_4$ and $K_{10,8} \times K_4$ respectively. Thus, we obtain a gregarious Y_5-tree decomposition for $K_m \times K_n$. $\square$

Lemma 33. $K_{11} \times K_4$ *admits a gregarious Y_5-tree decomposition.*

Proof. Let $V(K_{11} \setminus K_6) = \{1, 2, 3, \ldots, 11\}$. Clearly the set $\{(1, 2, 8, 4; 8\ 5), (2, 5, 1, 6; 1\ 7), (3, 4, 9, 5; 9\ 2), (2, 3, 1, 8; 1\ 11), (2, 6, 4, 10; 4\ 7),$ $(5, 6, 3, 8; 3\ 11), (7, 2, 10, 5; 10\ 1), (7, 5, 11, 4; 11\ 2), (3, 5, 4, 2; 4\ 1), (1, 9, 3, 7; 3\ 10)\}$ gives a Y_5-tree decomposition for $K_{11} \setminus K_6$. Now $K_{11} \times K_4 = [(K_{11} \setminus K_6) \oplus K_6] \times K_4 = [(K_{11} \setminus K_6) \times K_4] \oplus [K_6 \times K_4]$ $= [(Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5) \times K_4] \oplus [K_6 \times K_4] = [(Y_5 \times K_4) \oplus (Y_5 \times K_4) \ldots (Y_5 \times K_4)] \oplus [K_6 \times K_4]$. Now, the Lemmas 15 and 24, show that a gregarious Y_5-tree decomposition holds in $Y_5 \times K_4$ and $K_6 \times K_4$ respectively. Thus, we obtain a gregarious Y_5-tree decomposition for $K_{11} \times K_4$. $\square$

Lemma 34. $K_{11,8} \times K_4$ *admits a gregarious Y_5-tree decomposition.*

Proof. Let $V(K_{11,8}) = \bigcup_{i=1}^{2} V_i$, where $V_1 = \{1_j \mid 1 \leq j \leq 11\}$ and $V_2 = \{2_j \mid 1 \leq j \leq 8\}$. Then the following set gives a Y_5-tree decomposition for $K_{11,8}$.
$\{(1_7, 2_2, 1_4, 2_8; 1_4 2_5), (1_2, 2_3, 1_5, 2_1; 1_5 2_6), (1_8, 2_7, 1_4, 2_6; 1_4 2_3), (2_8, 1_5, 2_7, 1_{10}; 2_7 1_7),$
$(1_8, 2_8, 1_{11}, 2_7; 1_{11} 2_6), (1_3, 2_1, 1_{10}, 2_2; 1_{10} 2_5), (1_8, 2_2, 1_2, 2_4; 1_2 2_8), (1_6, 2_2, 1_1, 2_1; 1_1 2_8),$
$(1_6, 2_8, 1_{10}, 2_3; 1_{10} 2_4), (2_1, 1_{11}, 2_3, 1_8; 2_3 1_9), (1_6, 2_4, 1_{11}, 2_5; 1_{11} 2_2), (1_2, 2_5, 1_5, 2_2; 1_5 2_4),$
$(2_4, 1_7, 2_5, 1_8; 2_5 1_1), (2_4, 1_4, 2_1, 1_6; 2_1 1_2), (1_{10}, 2_6, 1_8, 2_4; 1_8 2_1), (1_2, 2_6, 1_3, 2_4; 1_3 2_2),$
$(1_1, 2_3, 1_7, 2_6; 1_7 2_1), (2_1, 1_9, 2_7, 1_2; 2_7 1_1), (1_1, 2_4, 1_9, 2_2; 1_9 2_5), (2_8, 1_9, 2_6 1_1; 2_6 1_6),$
$(1_3, 2_5, 1_6, 2_3; 1_6 2_7), (1_7, 2_8, 1_3, 2_3; 1_3 2_7)\}$. Then $K_{11,8} \times K_4 = [Y_5 \oplus Y_5 \oplus \ldots \oplus Y_5] \times K_4 = (Y_5 \times K_4) \oplus (Y_5 \times K_4) \oplus \ldots \oplus (Y_5 \times K_4)$. Now, the Lemma 15, which shows that $Y_5 \times K_4$ has a gregarious Y_5-tree decomposition. Thus, we obtain a gregarious Y_5-tree decomposition for $K_{11,8} \times K_4$. $\square$

Lemma 35. $K_m \times K_n$ *admits a gregarious Y_5-tree decomposition for $m \equiv 11 \pmod 8$ and $n = 4$.*

Proof. Let $m = 8s + 11$, $s \geq 1$. Then $K_m \times K_n = K_{8s+11} \times K_4 = [K_{11} \oplus sK_8 \oplus \frac{s(s-1)}{2} K_{8,8} \oplus sK_{11,8}] \times K_4 = [K_{11} \times K_4] \oplus s[K_8 \times K_4] \oplus \frac{s(s-1)}{2}[K_{8,8} \times K_4] \oplus s[K_{11,8} \times K_4]$. Now, the Lemmas 16, 18, 33 and 34, show that a gregarious Y_5-tree decomposition holds in $K_8 \times K_4$, $K_{8,8} \times K_4$, $K_{11} \times K_4$ and $K_{11,8} \times K_4$ respectively. Thus, we obtain a gregarious Y_5-tree decomposition for $K_m \times K_n$. $\square$

4. Main Result

Theorem 3. *There exists a gregarious Y_5-tree decomposition for $K_m \times K_n$, if and only if, $mn(m-1)(n-1) \equiv 0 \pmod 8$.*

Proof. **Necessity:** Since $\mid E(K_m \times K_n) \mid = \frac{mn(m-1)(n-1)}{2}$ and $\mid E(Y_5) \mid = 4$, the required edge divisibility condition to find a gregarious Y_5-tree decomposition in $K_m \times K_n$ is $\frac{|E(K_m \times K_n)|}{|E(Y_5)|} = \frac{mn(m-1)(n-1)}{2*4}$. That is, $8 \mid mn(m-1)(n-1)$. It can be written as $mn(m-1)(n-1) \equiv 0 \pmod 8$.
Sufficiency: Follows from Lemmas 16, 20, 23, 26, 29, 32 and 35. $\square$

5. Conclusion

In this manuscript, we give a complete solution to the problem of finding the existence of a Y_5-tree (gregarious Y_5-tree) decomposition in $K_m \times K_n$. In general, the problem of decomposing $K_m \times K_n$ into Y_k-tree (gregarious Y_k-tree) is yet to solve for $k \geq 6$.

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Conflict of Interest

The authors have no relavant financial or non-financial interests to disclose.

Author Contributions

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