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Inverse of α-Hermitian Adjacency Matrix of a Unicyclic Bipartite Graph

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Abstract: Let X be bipartite mixed graph and for a unit complex number α, H_α be its α-hermitian adjacency matrix. If X has a unique perfect matching, then H_α has a hermitian inverse H_α^{-1}. In this paper we give a full description of the entries of H_α^{-1} in terms of the paths between the vertices. Furthermore, for α equals the primitive third root of unity γ and for a unicyclic bipartite graph X with unique perfect matching, we characterize when H_γ^{-1} is ± 1 diagonally similar to γ-hermitian adjacency matrix of a mixed graph. Through our work, we have provided a new construction for the ± 1 diagonal matrix.

1. Introduction

A partially directed graph X is called a mixed graph, the undirected edges in X are called digons and the directed edges are called arcs. Formally, a mixed graph X is a set of vertices $V(X)$ together with a set of undirected edges $E_0(D)$ and a set of directed edges $E_1(X)$. For an arc $xy \in E_1(X)$, x(resp. y) is called initial (resp. terminal) vertex. The graph obtained from the mixed graph X after stripping out the orientation of its arcs is called the underlying graph of X and is denoted by $\Gamma(X)$.

A collection of digons and arcs of a mixed graph X is called a perfect matching if they are vertex disjoint and cover $V(X)$. In other words, perfect matching of a mixed graph is just a perfect matching of its underlying graph. In general, a mixed graph may have more than one perfect matching. We denote the class of bipartite mixed graphs with a unique perfect matching by $\mathcal{H}$. In this class of mixed graphs the unique perfect matching will be denoted by $\mathcal{M}$. For a mixed graph $X \in \mathcal{H}$, an arc e (resp. digon) in $\mathcal{M}$ is called matching arc (resp. matching digon) in X. If D is a mixed subgraph of X, then the mixed graph $X \backslash D$ is the induced mixed graph over $V(X) \backslash V(D)$.

Studying a graph or a digraph structure through properties of a matrix associated with it is an old and rich area of research. For undirected graphs, the most popular and widely investigated matrix in literature is the adjacency matrix. The adjacency matrix of a graph is symmetric, and thus diagonalizable and all of its eigenvalues are real. On the other hand, the adjacency matrix of directed graphs and mixed graphs is not symmetric and its eigenvalues are not all real. Consequently, dealing with such matrix is very challenging. Many researchers have recently proposed other adjacency matrices for digraphs. For instance in [1], the author investigated the spectrum of AA^T, where A is the traditional

adjacency matrix of a digraph. The author called them non negative spectrum of digraphs. In [2], authors proved that the non negative spectrum is totally controlled by a vertex partition called common out neighbor partition. Authors in [3] and in [4] (independently) proposed a new adjacency matrix of mixed graphs as follows:

For a mixed graph X, the hermitian adjacency matrix of X is a $|V| \times |V|$ matrix $H(X) = [h_{uv}]$, where

$$h_{uv} = \begin{cases} 1 & \text{if } uv \in E_0(X), \\ i & \text{if } uv \in E_1(X), \\ -i & \text{if } vu \in E_1(X), \\ 0 & \text{otherwise.} \end{cases}$$

This matrix has many nice properties. It has real spectrum and interlacing theorem holds. Beside investigating basic properties of this hermitian adjacency matrix, authors proved many interesting properties of the spectrum of H. This motivated Mohar in [5] to extend the previously proposed adjacency matrix. The new kind of hermitian adjacency matrices, called α-hermitian adjacency matrices of mixed graphs, are defined as follows: Let X be a mixed graph and α be the primitive n^{th} root of unity $e^{\frac{2\pi}{n}i}$. Then the α hermitian adjacency matrix of X is a $|V| \times |V|$ matrix $H_\alpha(X) = [h_{uv}]$, where

$$h_{uv} = \begin{cases} 1 & \text{if } uv \in E_0(D), \\ \alpha & \text{if } uv \in E_1(D), \\ \overline{\alpha} & \text{if } vu \in E_1(D), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the new kind of hermitian adjacency matrices of mixed graphs is a natural generalization of the old one for mixed graphs and even for the graphs. As we mentioned before these adjacency matrices ($H_i(X)$ and $H_\alpha(X)$) are hermitian and have interesting properties. This paved the way to more a facinating research topic much needed nowadays.

For simplicity when dealing with one mixed graph X, then we write H_α instead of $H_\alpha(X)$.

The smallest positive eigenvalue of a graph plays an important role in quantum chemistry. Motivated by this application, Godsil in [6] investigated the inverse of the adjacency matrix of a bipartite graph. He proved that if T is a tree graph with perfect matching and $A(T)$ is its adjacency matrix then, $A(T)$ is invertabile and there is $\{1, -1\}$ diagonal matrix D such that $DA^{-1}D$ is an adjacency matrix of another graph. Many of the problems mentioned in [6] are still open. Further research appeared after this paper that continued on Godsil's work see [7], [8] and [9].

In this paper we study the inverse of α-hermitian adjacency matrix H_α of unicyclic bipartite mixed graphs with unique perfect matching X. Since undirected graphs can be considered as a special case of mixed graphs, the out comes in this paper are broader than the work done previously in this area. We examine the inverse of α-hermitian adjacency matricies of bipartite mixed graphs and unicyclic bipartite mixed graphs. Also, for $\alpha = \gamma$, the primative third root of unity, we answer the traditional question, when H_α^{-1} is $\{\pm 1\}$ diagonally similar to an α-hermitian adjacency matrix of mixed graph. To be more precise, for a unicyclic bipartite mixed graph X with unique perfect matching we give full characterization when there is a $\{\pm 1\}$ diagonal matrix D such that $DH_\gamma^{-1}D$ is an γ-hermitian adjacency matrix of a mixed graph. Furthermore, through our work we introduce a construction of such diagonal matrix D. In order to do this, we need the following definitions and theorems:

Definition 1. [10] *Let X be a mixed graph and $H_\alpha = [h_{uv}]$ be its α-hermitian adjacency matrix.*

- *X is called elementary mixed graph if for every component X' of X, $\Gamma(X')$ is either an edge or a cycle C_k (for some $k \geq 3$).*
- *For an elementary mixed graph X, the rank of X is defined as $r(X) = n - c$, where $n = |V(X)|$ and c is the number of its components. The co-rank of X is defined as $s(X) = m - r(X)$, where $m = |E_0(X) \cup E_1(X)|$.*

- *For a mixed walk W in X, where $\Gamma(W) = r_1, r_2, \ldots r_k$, the value $h_\alpha(W)$ is defined as*

$$h_\alpha(W) = h_{r_1 r_2} h_{r_2 r_3} h_{r_3 r_4} \ldots h_{r_{k-1} r_k} \in \{\alpha^n\}_{n \in \mathbb{Z}}$$

Recall that a bijective function η from a set V to itself is called permutation. The set of all permutations of a set V, denoted by S_V, together with functions composition form a group. Finally recall that for $\eta \in S_V$, η can be written as composition of transpositions. In fact the number of transpositions is not unique. But this number is either odd or even and cannot be both. Now, we define $sgn(\eta)$ as $(-1)^k$, where k is the number of transposition when η is decomposed as a product of transpositions. The following theorem is well known as a classical result in linear algebra.

Theorem 1. *If $A = [a_{ij}]$ is an $n \times n$ matrix then*

$$det(A) = \sum_{\eta \in S_n} sgn(\eta) a_{1,\eta(1)} a_{2,\eta(2)} a_{3,\eta(3)} \ldots a_{n,\eta(n)}.$$

2. Inverse of α-Hermitian Adjacency Matrix of a Bipartite Mixed Graph

In this section, we investigate the invertibility of the α-hermitian adjacency matrix of a bipartite mixed graph X. Then we find a formula for the entries of its inverse based on elementary mixed subgraphs. This will lead to a formula for the entries based on the type of the paths between vertices. Using Theorem 1, authors in [10] proved the following theorem:

Theorem 2. *(Determinant expansion for H_α) [10] Let X be a mixed graph and H_α its α-hermitian adjacency matrix, then*

$$det(H_\alpha) = \sum_{X'} (-1)^{r(X')} 2^{s(X')} Re\left(\prod_C h_\alpha(\vec{C})\right),$$

where the sum ranges over all spanning elementary mixed subgraphs X' of X, the product ranges over all mixed cycles C in X', and $\vec{C}$ is any mixed closed walk traversing C.

Now, let $X \in \mathcal{H}$ and $\mathcal{M}$ is the unique perfect matching in X. Then since X is bipartite graph, X contains no odd cycles. Now, let C_k be a cycle in X, then if $C_k \cap \mathcal{M}$ is a perfect matching of C_k then, $\mathcal{M} \Delta C_k = \mathcal{M} \backslash C_k \cup C_k \backslash \mathcal{M}$ is another perfect matching in X which is a contradiction. Therefore there is at least one vertex of C_k that is matched by a matching edge not in C_k. This means if $X \in \mathcal{H}$, then X has exactly one spanning elementary mixed subgraph that consist of only K_2 components. Therefore, Using the above discussion together with Theorem 2 we get the following theorem:

Theorem 3. *If $X \in \mathcal{H}$ and H_α is its α-hermitian adjacency matrix then H_α is non singular.*

Now, Let X be a mixed graph and H_α be its α-hermitian adjacency matrix. Then, for invertible H_α, the following theorem finds a formula for the entries of H_α^{-1} based on elementary mixed subgraphs and paths between vertices. The proof can be found in [11].

Theorem 4. *Let X be a mixed graph, H_α be its α-hermitian adjacency matrix and for $i \neq j$, $\rho_{i \to j} = \{P_{i \to j} : P_{i \to j}$ is a mixed path from the vertex i to the vertex $j\}$. If $\det(H_\alpha) \neq 0$, then*

$$[H_\alpha^{-1}]_{ij} = \frac{1}{\det(H_\alpha)} \sum_{P_{i \to j} \in \rho_{i \to j}} (-1)^{|E(P_{i \to j})|} h_\alpha(P_{i \to j}) \sum_{X'} (-1)^{r(X')} 2^{s(X')} Re\left(\prod_C h_\alpha(\vec{C})\right),$$

where the second sum ranges over all spanning elementary mixed subgraphs X' of $X \backslash P_{i \to j}$, the product is being taken over all mixed cycles C in X' and $\vec{C}$ is any mixed closed walk traversing C.

This theorem describes how to find the non diagonal entries of H_α^{-1}. In fact, the diagonal entries may or may not equal to zero. To observe this, lets consider the following example:

Example 1. *Consider the mixed graph X shown in Figure 1 and let $\alpha = e^{\frac{\pi}{5}i}$. The mixed graph X has a unique perfect matching, say M, and this matching consists of the set of unbroken arcs and digons. Further M is the unique spanning elementary mixed subgraph of X. Therefore, using Theorem 2*

$$det[H_\alpha] = (-1)^{8-4}2^{4-4} = 1.$$

So, H_α is invertible. To calculate $[H_\alpha^{-1}]_{ii}$, we observe that

$$[H_\alpha^{-1}]_{ii} = \frac{det((H_\alpha)_{(i,i)})}{det(H_\alpha)} = det((H_\alpha)_{(i,i)}),$$

where $(H_\alpha)_{(i,i)}$ is the matrix obtained from H_α by deleting the ith row and i^{th} column, which is exactly the α-hermitian adjacency matrix of $X\backslash\{i\}$. Applying this on the mixed graph, one can deduce that the diagonal entries of H_α^{-1} are all zeros except the entry $(H_\alpha^{-1})_{11}$. In fact it can be easily seen that the mixed graph $X\backslash\{1\}$ has only one spanning elementary mixed subgraph. Therefore,

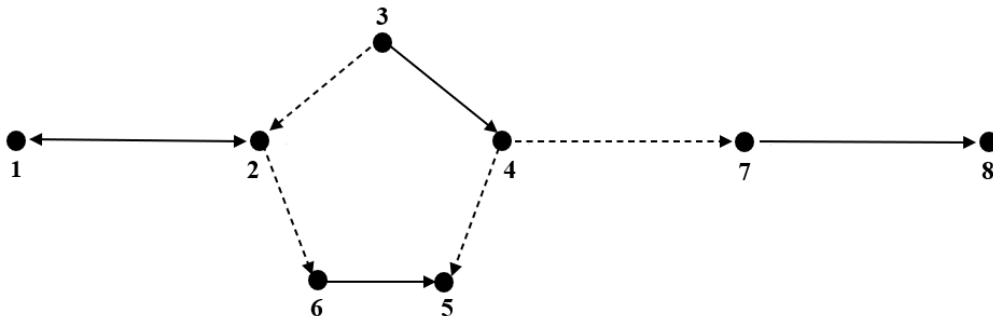$$[H_\alpha^{-1}]_{11} = det((H_\alpha)_{(1,1)}) = (-1)^{7-2}2^{6-5}Re(\alpha) = -2Re(\alpha).$$

Figure 1. Mixed Graph X where H_α^{-1} has Nonzero Diagonal Entry

The following theorem shows that if X is a bipartite mixed graph with unique perfect matching, then the diagonal entries of H_α^{-1} should be all zeros.

Theorem 5. *Let $X \in \mathcal{H}$ and H_α be its α-hermitian adjacency matrix. Then, for every vertex $i \in V(X)$, $(H_\alpha^{-1})_{ii} = 0$.*

Proof. Observing that X is a bipartite mixed graph with a unique perfect matching, and using Theorem 3, we have H_α is invertable. Furthermore,

$$(H_\alpha^{-1})_{ii} = \frac{det((H_\alpha)_{(i,i)})}{det(H_\alpha)}.$$

Note that $(H_\alpha)_{(i,i)}$ is the α-hermitian adjacency matrix of the mixed graph $X\backslash\{i\}$. However X has a unique perfect matching, therefore $X\backslash\{i\}$ has an odd number of vertices. Hence $X\backslash\{i\}$ has neither a perfect matching nor an elementary mixed subgraph and thus $det((H_\alpha)_{(i,i)}) = 0$. $\square$

Now, we investigate the non diagonal entries of the inverse of the α-hermitian adjacency matrix of a bipartite mixed graph, $X \in \mathcal{H}$. In order to do that we need to characterize the structure of the mixed graph $X\backslash P$ for every mixed path P in X. To this end, consider the following theorems:

Theorem 6. *[12] Let M and M′ be two matchings in a graph G. Let H be the subgraph of G induced by the set of edges*

$$M\Delta M' = (M\backslash M') \cup (M'\backslash M).$$

Then, the components of H are either cycles of even number of vertices whose edges alternate in M and M′ or a path whose edges alternate in M and M′ and end vertices unsaturated in one of the two matchings.

Corollary 1. *For any graph G, if G has a unique perfect matching then G does not contain alternating cycle.*

Definition 2. *Let X be a mixed graph with unique perfect matching. A path P between two vertices u and v in X is called co-augmenting path if the edges of the underlying path of P alternates between matching edges and non-matching edges where both first and last edges of P are matching edges.*

Corollary 2. *Let G be a bipartite graph with unique perfect matching $\mathcal{M}$, u and v are two vertices of G. If P_{uv} is a co-augmenting path between u and v, then $G\backslash P_{uv}$ is a bipartite graph with unique perfect matching $\mathcal{M}\backslash P_{uv}$.*

Proof. The part that $\mathcal{M}\backslash P_{uv}$ is being a perfect matching of $G\backslash P_{uv}$ is obvious. Suppose that $M' \neq \mathcal{M}\backslash P_{uv}$ is another perfect matching of $G\backslash P_{uv}$. Using Theorem 6, $G\backslash P_{uv}$ consists of an alternating cycles or an alternating paths, where its edges alternate between $\mathcal{M}\backslash P_{uv}$ and M'. If all $G\backslash P_{uv}$ components are paths, then $G\backslash P_{uv}$ has exactly one perfect matching, which is a contradiction. Therefore, $G\backslash P_{uv}$ contains an alternating cycle say C. Since P_{uv} is a co-augmenting path, we have $M' \cup (P_{uv} \cap \mathcal{M})$ is a perfect matching of G. Therefore G has more than one perfect matching, which is a contradiction. $\square$

Theorem 7. *Let G be a bipartite graph with unique perfect matching $\mathcal{M}$, u and v are two vertices of G. If P_{uv} is not a co-augmenting path between u and v, then $G\backslash P_{uv}$ does not have a perfect matching.*

Proof. Since G has a perfect matching, then G has even number of vertices. Therefore, when P_{uv} has an odd number of vertices, $G\backslash P_{uv}$ does not have a perfect matching.

Suppose that P_{uv} has an even number of vertices. Then, P_{uv} has a perfect matching M. Therefore if $G\backslash P_{uv}$ has a perfect matching M', then $M \cup M'$ will form a new perfect matching of G. This contradicts the fact that G has a unique perfect matching. $\square$

Now, we are ready to give a formula for the entries of the inverse of α-hermitian adjacency matrix of bipartite mixed graph X that has a unique perfect matching. This characterizing is based on the co-augmenting paths between vertices of X.

Theorem 8. *Let X be a bipartite mixed graph with unique perfect matching $\mathcal{M}$, H_α be its α-hermitian adjacency matrix and*

$$\mathfrak{I}_{i\to j} = \{P_{i\to j} : P_{i\to j} \text{ is a co-augmenting mixed path from the vertex i to the vertex j}\}.$$

Then

$$(H_\alpha^{-1})_{ij} = \begin{cases} \displaystyle\sum_{P_{i\to j}\in\mathfrak{I}_{i\to j}} (-1)^{\frac{|E(P_{i\to j})|-1}{2}} h_\alpha(P_{i\to j}) & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Proof. Using Theorem 4,

$$[H_\alpha^{-1}]_{ij} = \frac{1}{\det(H_\alpha)} \sum_{P_{i\to j}\in\rho_{i\to j}} \left[(-1)^{|E(P_{i\to j})|} h_\alpha(P_{i\to j}) \sum_{X'} (-1)^{r(X')} 2^{s(X')} Re(\prod_C h_\alpha(\vec{C})) \right],$$

where the second sum ranges over all spanning elementary mixed subgraphs of $X\backslash P_{i\to j}$. The product is being taken over all mixed cycles C of X' and $\vec{C}$ is any mixed closed walk traversing C.

First, using Theorem 7 we observe that if $P_{i\to j}$ is not a co-augmenting path then $X\backslash P_{i\to j}$ does not have a perfect matching. Therefore, the term corresponds to $P_{i\to j}$ contributes zero. Thus we only care about the co-augmenting paths. According to Corollary 2, for any co-augmenting path $P_{i\to j}$ from the vertex i to the vertex j we get $X\backslash P_{i\to j}$ has a unique perfect matching, namely $\mathcal{M} \cap E(X\backslash P_{i\to j})$. Using Corollary 1, $X\backslash P_{i\to j}$ does not contain an alternating cycle. Thus $X\backslash P_{i\to j}$ contains only one spanning elementary mixed subgraph which is $\mathcal{M}\backslash P_{i\to j}$. So,

$$[H_\alpha^{-1}]_{ij} = \frac{1}{\det(H_\alpha)} \sum_{P_{i \to j} \in \mathfrak{I}_{i \to j}} (-1)^{|E(P_{i \to j})|} h_\alpha(P_{i \to j})(-1)^{V(X \backslash P_{i \to j})-k},$$

where k is the number of components of the spanning elementary mixed subgraph of $X \backslash P_{i \to j}$. Observe that $|V(X \backslash P_{i \to j})| = n - (|E(P_{i \to j})| + 1)$, $k = \frac{n-(|E(P_{i \to j})|+1)}{2}$ and $\det(H_\alpha) = (-1)^{\frac{n}{2}}$, we get the result.   □

3. Inverse of γ-hermitian Adjacency Matrix of a Unicyclic Bipartite Mixed Graph

Let γ be the third root of unity $e^{\frac{2\pi}{3}i}$. Using Theorem 8, $h_\alpha(P_{i \to j}) \in \{\alpha^i\}_{i=1}^n$ plays a central rule in finding the entries of H_α^{-1} and since the third root of unity has the property $\gamma^i \in \{1, \gamma, \overline{\gamma}\}$ we focus our study in this section on $\alpha = \gamma$. The property that $\alpha^i \in \{\pm 1, \pm \alpha, \pm \overline{\alpha}\}$ is not true in general. To illustrate, consider the mixed graph shown in Figure 2 and let $\alpha = e^{\frac{\pi}{5}i}$. Using Theorem 8 we get $H_{05}^{-1} = e^{\frac{3\pi}{5}i}$ which is not from the set $\{\pm 1, \pm \alpha, \pm \overline{\alpha}\}$.

In this section, we are going to answer the classical question whether the inverse of γ-hermitian adjacency matrix of a unicyclic bipartite graph is $\{1, -1\}$ diagonally similar to a hermitian adjacency matrix of another mixed graph or not. Consider the mixed graph shown in Figure 2. Then, obviously entries of H_γ^{-1} are from the set $\{0, \pm 1, \pm \gamma, \pm \overline{\gamma}\}$.
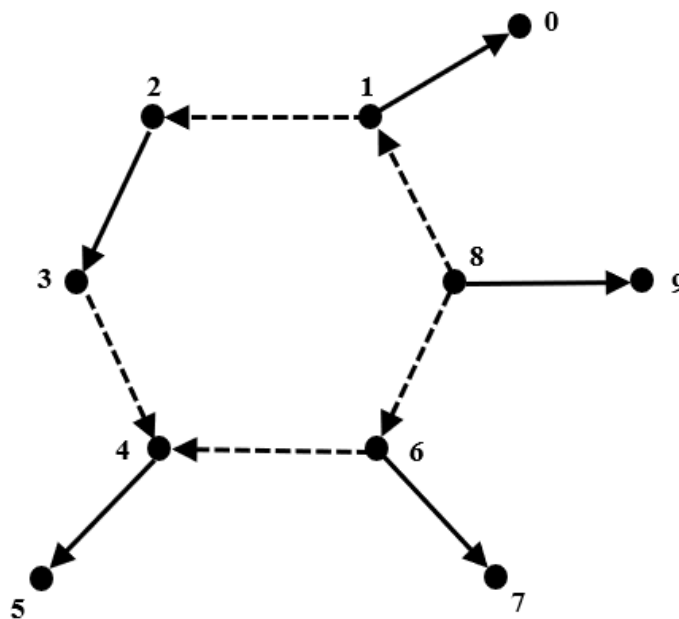

Figure 2. Unicycle Bipartite Mixed Graph with Unique Perfect Matching and 4 Pegs

Another thing we should bear in mind is the existence of $\{1, -1\}$ diagonal matrix D such that $DH_\gamma D$ is γ-adjacency matrix of another mixed graph. In the mixed graph X in Figure 2, suppose that $D = diag\{d_0, d_1, \ldots, d_9\}$ is $\{1, -1\}$ diagonal matrix with the property $DH_\gamma D$ has all entries from the set $\{0, \gamma, \overline{\gamma}\}$. Then,

$$d_0 d_5 = 1,$$
$$d_0 d_9 = -1,$$
$$d_9 d_7 = -1,$$
$$d_5 d_7 = -1,$$

which is impossible. Therefore, such diagonal matrix D does not exist. To discuss the existence of the diagonal matrix D further, let G be a bipartite graph with unique perfect matching. Define X_G to be the mixed graph obtained from G by orienting all non matching edges. Clearly for $\alpha \neq 1$ and $\alpha \neq -1$ changing the orientation of the non matching edges will change the α-hermitian adjacency

matrix. For now lets restrict our study on $\alpha = -1$. Using Theorem 8 one can easily get the following observation.

Observation 1. *Let G be a bipartite mixed graph with unique perfect matching M, H_{-1} be the -1-hermitian adjacency matrix of X_G and*

$$\mathfrak{I}_{i \to j} = \{P_{i \to j} : P_{i \to j} \text{ is a co-augmenting mixed path from the vertex } i \text{ to the vertex } j \text{ in } X_G\}.$$

One can use Theorem 8 to get

$$(H_{-1}^{-1})_{ij} = \begin{cases} |\mathfrak{I}_{i \to j}| & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

So, the question we need to answer now is when $A(G)$ and $H_{-1}(X_G)$ are diagonally similar. To this end, let G be a bipartite graph with a unique perfect matching and $u \in V(G)$. Then for a walk $W = u = r_1, r_2, r_3, \ldots, r_k$ in G, define a function that assign the value $f_W(j)$ for the jth vertex of W as follows:

$$f_W(1) = 1$$

and

$$f_W(j+1) = \begin{cases} -f_W(j) & \text{if } r_j r_{j+1} \text{is unmatching edge in } G, \\ f_W(j) & \text{if } r_j r_{j+1} \text{is matching edge in } G, \end{cases}$$

See Figure 3. Since any path from a vertex u to itself consist of pairs of identical paths and cycle,
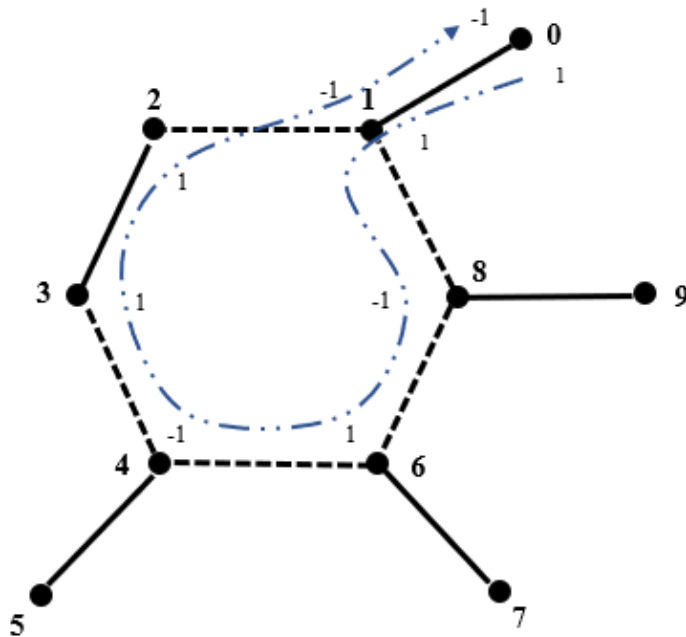

Figure 3. The Values of f_W where W is the Closed Walk Starting from 0

we get the following remark:

Remark 1. *Let G be bipartite graph with unique perfect matching and $F(u) = \{f_W(u) : W$ is a closed walk in G starting at u\}. then, $|F(u)| = 1$ if and only if the number of unmatching edges in each cycle of G is even.*

Finally, let G be a bipartite graph with unique perfect matching and suppose that each cycle of G has an even number of unmatched edges. For a vertex $u \in V(G)$ define the function $w : V(G) \to \{1, -1\}$ by

$$w(v) = f_W(v), \text{ where } W \text{ is a path from } u \text{ to } v.$$

Definition 3. *Suppose that G is bipartite graph with unique perfect matching and every cycle of G has even number of unmatched edges. Suppose further $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $u \in V(G)$. Define the matrix D_u by $D_u = diag\{w(v_1), w(v_2), \ldots, w(v_n)\}$.*

Theorem 9. *Suppose G is a bipartite graph with unique perfect matching and every cycle of G has an even number of unmatched edges. Then for every $u \in V(G)$, we get $D_u A(G) D_u = H_{-1}(X_G)$.*

Proof. Note that, for $x, y \in V(G)$, we have $(D_u A(G) D_u)_{xy} = d_x a_{xy} d_y$. Using the definition of D_u we get,

$$d_x d_y = \begin{cases} -1 & \text{if } xy \text{ is an unmatching edge in } G, \\ 1 & \text{if } xy \text{ is a matching edge in } G, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $(D_u A(G) D_u)_{xy} = (H_{-1})_{xy}$.

$\square$

Now we are ready to discuss the inverse of γ-hermitian adjacency matrix of unicyclic mixed graph. Let X be a unicyclic bipartite graph with unique perfect matching. An arc or digon of X is called a peg if it is a matching arc or digon and incident to a vertex of the cycle in X. Since X is unicyclic bipartite graph with unique perfect matching, X has at least one peg. Otherwise the cycle in X will be alternate cycle, and thus X has more than one perfect matching which contradicts the assumption. Since each vertex of the cycle incident to a matching edge and $|V(X)|$ is even, X should contain at least two pegs. The following theorem will deal with unicyclic bipartite mixed graphs $X \in \mathcal{H}$ with more than two pegs.

Theorem 10. *Let X be a unicyclic bipartite graph with unique perfect matching. If X has more than two pegs, then between any two vertices of X there is at most one co-augmenting path.*

Proof. Let ρ_1, ρ_2 and ρ_3 be three pegs in X, $u, v \in V(D)$, C is the unique cycle in X and suppose there are two co-augmenting paths between u and v, say P and P'. Since X is unicyclic, we have $V(C) \subset P \cup P'$,

Case 1: $E(P) \cup E(P')$ does not contain any of the pegs. Then, if v is the X cycle vertex incident to ρ_1 then, v is not matched by an edge in the cycle, which means one of P or P' is not co-augmenting path, which contradicts the assumption.

Case 2: $E(P) \cup E(P')$ contain pegs. Then, $(E(P) \cup E(P'))$ should contain at most two pegs, suppose that ρ_1 and v is the vertex of X cycle that incident to ρ_1. Then, v belongs to either P or P', again since ρ_1 is a matched edge, v is not matched by the cycle edges which means one of P or P' is not co-augmenting path. which contradicts the assumption.

$\square$

Corollary 3. *Let X be a unicycle bipartite mixed graph with unique perfect matching. If X has more than two pegs, then*

1. $(H_\alpha^{-1})_{ij} = \begin{cases} (-1)^{\frac{|E(P_{i \to j})|-1}{2}} h_\alpha(P_{i \to j}) & \text{if } P_{i \to j} \text{ is a co-augmenting path from } i \text{ to } j, \\ 0 & \text{Otherwise.} \end{cases}$

2. If the cycle of X contains even number of unmatching edges, then for any vertex $u \in V(X)$, $D_u H_\gamma^{-1}(X) D_u$ is γ-hermitian adjacency matrix of a mixed graph.

Proof. Part one is obvious using Theorem 8 together with Theorem 10.

For part two, we observe that $\gamma^i \in \{1, \gamma, \overline{\gamma}\}$. Therefore all $H_\gamma^{-1}(X)$ entries are from the set $\{\pm 1, \pm \gamma, \pm \overline{\gamma}\}$. Also the negative signs in $A(\Gamma(X))^{-1}$ and in H_γ^{-1} appear at the same position. Which means $D_u H_\gamma^{-1} D_u$ is γ-hermitian adjacency matrix of a mixed graph if and only if $D_u A(\Gamma(X)) D_u$ is adjacency matrix of a graph. Finally, Theorem 9 ends the proof.

$\square$

Now we will take care of unicycle graph with exactly two pegs. Using the same technique of the proof of Theorem 10, one can show the following:

Theorem 11. *Let D be a unicyclic bipartite graph with unique perfect matching and exactly two pegs ρ_1 and ρ_2. Then for any two vertices of D, u and v, if there are two co-augmenting paths from the vertex u to the vertex v, then ρ_1 and ρ_2 are edges of the two paths.*

Let X be a unicyclic bipartite mixed graph with unique perfect matching and exactly two pegs, and let uv and $u'v'$ be the two pegs of X where v and v' are vertices of the cycle of X. We, denote the two paths between v and v' by $\mathcal{F}_{v\to v'}$ and $\mathcal{F}^c_{v\to v'}$.

Theorem 12. *Let X be a unicyclic bipartite mixed graph with unique perfect matching and exactly two pegs and let C be the cycle of X. If there are two coaugmenting paths between the vertex x and the vertex y, then*

$$(H_\alpha^{-1})_{xy} = (-1)^{\frac{|E(P_{x\to y})|-1}{2}} \frac{h_\alpha(P_{x\to v})h_\alpha(P_{y\to v'})}{h_\alpha(\mathcal{F}_{v\to v'})}\left[(-1)^{m+1}h_\alpha(C) + 1\right],$$

where $\mathcal{F}_{v\to v'}$ is chosen to be the part of the path $P_{x\to y}$ in the cycle C and C is of size 2m.

Proof. Suppose that $P_{x\to y}$ and $Q_{x\to y}$ are the paths between the vertices x and y, using theorem 8 we have

$$(H_\alpha^{-1})_{xy} = (-1)^{\frac{|E(P_{x\to y})|-1}{2}}h_\alpha(P_{x\to y}) + (-1)^{\frac{|E(Q_{x\to y})|-1}{2}}h_\alpha(Q_{x\to y}).$$

Now, using Theorem 11, $P_{x\to y}$ ($Q_{x\to y}$) can be divided into three parts $P_{x\to v}$, $\mathcal{F}_{v\to v'}$ and $P_{v'\to y}$ (resp. $Q_{x\to v} = P_{x\to v}$, $\mathcal{F}^c_{v\to v'}$ and $Q_{v'\to y} = P_{v'\to y}$).
Observe that the number of unmatched edges in $\mathcal{F}_{v\to v'}$ is $k_1 = \frac{|E(\mathcal{F}_{v\to v'})|+1}{2}$ and the number of unmatched edges in $\mathcal{F}^c_{v\to v'}$ is $k_2 = m - \frac{|E(\mathcal{F}_{v\to v'})|+1}{2} + 1$ we get

$$(H_\alpha^{-1})_{xy} = (-1)^k h_\alpha(P_{x\to v})h_\alpha(P_{v\to y})\left((-1)^{k_1}h_\alpha(\mathcal{F}_{v\to v'}) + (-1)^{k_2}h_\alpha(\mathcal{F}^c_{v\to v'})\right),$$

where $k = \frac{|E(P_{x\to v})|+|E(P_{v'\to y})|}{2} - 1$.
Note here $\overline{h_\alpha(\mathcal{F}_{v\to v'})}h_\alpha(\mathcal{F}^c_{v\to v'}) = h_\alpha(C)$, therefore,

$$(H_\alpha^{-1})_{xy} = (-1)^{\frac{|E(P_{x\to y})|-1}{2}} \frac{h_\alpha(P_{x\to v})h_\alpha(P_{y\to v'})}{h_\alpha(\mathcal{F}_{v\to v'})}\left[(-1)^{m+1}h_\alpha(C) + 1\right].$$

$\square$

Theorem 13. *Let X be a unicyclic bipartite mixed graph with unique perfect matching and H_γ be its γ-hermitian adjacency matrix. If X has exactly two pegs, then H_γ^{-1} is not ± 1 diagonally similar to γ-hermitian adjacency matrix of a mixed graph.*

Proof. Let xx' and yy' be the two pegs of X, where x' and y' are vertices of the cycle C of X. Then, using Theorem 12 we have

$$(H_\gamma^{-1})_{xy} = (-1)^{\frac{|E(P_{x\to y})|-1}{2}} \frac{h_\gamma(P_{x\to x'})h_\gamma(P_{y\to y'})}{h_\gamma(\mathcal{F}_{x'\to y'})}\left[(-1)^{m+1}h_\gamma(C) + 1\right],$$

where $\mathcal{F}_{x'\to y'}$ is chosen to be the part of the path $P_{x\to y}$ in the cycle C and C is of size $2m$. Suppose that $D = diag\{d_v : v \in V(X)\}$ is a $\{\pm 1\}$ diagonal matrix with the property that $DH_\gamma^{-1}D$ is γ-hermitian adjacency matrix of a mixed graph.

Case 1: Suppose m is even say $m = 2r$.

Observe that $(-1)^{m+1} h_\gamma(C) + 1 = 1 - h_\gamma(C)$. If $h_\gamma(C) \in \{1, \gamma, \gamma^2\}$, then $1 - h_\gamma(C) \notin \{\pm 1, \pm \gamma, \pm \gamma^2\}$ and so H_γ^{-1} is not ± 1 diagonally similar to γ-hermitian adjacency matrix of a mixed graph. Thus we only need to discuss the case when $h_\gamma(C) = 1$. To this end, suppose that $h_\gamma(C) = 1$. Then $(H_\gamma^{-1})_{xy} = 0$. Since the length of C is $4r$, we have the number of unmatching edges (number of matching edges) in C is $\frac{4r+2}{2}$ (resp. $\frac{4r-2}{2}$). Since the number of unmatching edges in C is odd, there is a coaugmenting path $\mathcal{F}_{x \to y}$ from x to y that contains odd number of unmatching edges and another coaugmenting path $\mathcal{F}_{x \to y}^c$ with even number of unmatching edges. Now, let $o'o(e'e$) be any matching edges in the path $\mathcal{F}_{x \to y}$ (resp. $\mathcal{F}_{x \to y}^c$). Then, without loss of generality we may assume that there is a coaugmenting path between x and e, x and o (and hence there is a co-augmenting path between y and o', y and e'). Now, if $d_x d_y = 1$ then

- $(DH_\gamma^{-1}D)_{xo} = (-1)^k d_x h_\gamma(P_{x \to o}) d_o$
- $(DH_\gamma^{-1}D)_{yo'} = (-1)^{k'} d_y h_\gamma(P_{y \to o'}) d_{o'}$

Observe that $k + k'$ is odd number, we have $d_o d_{o'} = -1$. This contradict the fact that for every matching edge gg', $d_g d_{g'} = 1$. The case when $d_x d_y = -1$ is similar to the above case but with considering the path $\mathcal{F}_{x \to y}^c$ instead of $\mathcal{F}_{x \to y}$ and the vertex e instead of o.

Case 2: Suppose m is odd say $2r + 1$. Then

$$(H_\gamma^{-1})_{xy} = (-1)^{\frac{|E(P_{x \to y})|-1}{2}} \frac{h_\alpha(P_{x \to v}) h_\alpha(P_{y \to v'})}{h_\alpha(\mathcal{F}_{v \to v'})} [h_\alpha(C) + 1].$$

Therefore,

$$(H_\gamma^{-1})_{xy} = (-1)^{\frac{|E(P_{x \to y})|-1}{2}} \frac{h_\alpha(P_{x \to v}) h_\alpha(P_{y \to v'})}{h_\alpha(\mathcal{F}_{v \to v'})} \begin{cases} -\gamma & \text{if } h_\alpha(C) = \gamma^2, \\ -\gamma^2 & \text{if } h_\alpha(C) = \gamma, \\ 2 & \text{if } h_\alpha(C) = 1. \end{cases}$$

Obviously, when $h_\alpha(C) = 1$, H_γ^{-1} is not ± 1 diagonally similar to γ-hermitian adjacency matrix of a mixed graph. Thus, the cases we need to discuss here are when $h_\alpha(C) = \gamma$ and $h_\alpha(C) = \gamma^2$. Since m is odd, then C contains an even number of unmatched edges. Therefore, either both paths between x and y, $\mathcal{F}_{x \to y}$ and $\mathcal{F}_{x \to y}^c$, contain odd number of unmatching edges or both of them contains even number of unmatching edges.

To this end, suppose that both of the paths $\mathcal{F}_{x \to y}$ and $\mathcal{F}_{x \to y}^c$ contain odd number of unmatched edges. Then, $(H_\gamma^{-1})_{xy} \in \{\gamma^i\}_{i=0}^2$, which means $d_x d_y = 1$. Finally, let $v'v$ be any matching edge in $\mathcal{F}_{x \to y}$ where $P_{x \to v}$ and $P_{v' \to y}$ are coaugmenting paths, then obviously $d_v d_{v'} = 1$. But one of the coaugmenting paths $P_{x \to v}$ and $P_{v' \to y}$ should contain odd number of unmatching edges and the other one should contain even number of unmatched edges. Which means $d_x d_v d_{v'} d_y = -1$. This contradicts the fact that $d_v d_{v'} = 1$.

In the other case, when both $\mathcal{F}_{x \to y}$ and $\mathcal{F}_{x \to y}^c$ contain even number of unmatching edges, one can easily deduce that $d_x d_y = -1$ and using same technique we can get another contradiction.

$\square$

Note that Corollary 3 and Theorem 13 give a full characterization of a unicyclic bipartite mixed graph with unique perfect matching where the inverse of its γ-hermitian adjacency matrix is $\{\pm 1\}$ diagonally similar to γ-hermitian adjacency matrix of a mixed graph. We summarize this characterization in the following corollary.

Theorem 14. *Let X be a unicyclic bipartite mixed graph with unique perfect matching and H_γ its γ-hermitian adjacency matrix. Then, H_γ^{-1} is ± 1 diagonally similar to γ-hermitian adjacency matrix if and only if X has more than two pegs and the cycle of X contains even number of unmatching edges.*
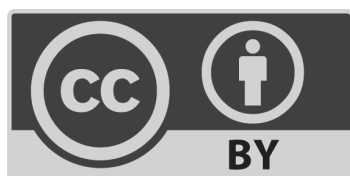
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Conflict of interest

The authors declare no conflict of interest.

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