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Article

NDC Pebbling Number for Some Class of Graphs

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Abstract: Let G be a connected graph. A pebbling move is defined as taking two pebbles from one vertex and the placing one pebble to an adjacent vertex and throwing away the another pebble. A dominating set D of a graph $G = (V, E)$ is a non-split dominating set if the induced graph $< V - D >$ is connected. The Non-split Domination Cover(NDC) pebbling number, $\psi_{ns}(G)$, of a graph G is the minimum of pebbles that must be placed on $V(G)$ such that after a sequence of pebbling moves, the set of vertices with a pebble forms a non-split dominating set of G, regardless of the initial configuration of pebbles. We discuss some basic results and determine ψ_{ns} for some families of standard graphs.

Keywords: Non-split dominating set, NDC pebbling number, Cover pebbling number

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1. Introduction

Lagarias and Saks were the first one to introduce the concept of pebbling and Chung [1] used the concept in pebbling to solve a number theoretic conjecture. Then many others followed suit including Glenn Hulbert who published a survey of pebbling variants [2]. The subject to graph pebbling has seen a massive growth after Hulbert's survey. The past 30 years so many new variants in graph pebbling have been developed which can be applied to the filed of transportation, computer memory allocation, game theory and the installation of mobile towers.

Let us denote G's vertex and edge sets as $V(G)$ and $E(G)$, respectively. Consider a graph with a fixed number of pebbles at each vertex. One pebble is thrown away and the other is placed on an adjacent vertex when two pebbles are removed from a vertex. This process is known as a pebble move. The pebbling number of a vertex v in a graph G is the smallest number $\pi(G, v)$ that allows us to shift a pebble to v using a sequence of pebbling move, regardless of how these pebbles are located on G's vertices. The pebbling number, $\pi(G)$, of a graph G is the maximum of $\pi(G, v)$ over all the vertices v of a graph. Considering the concept of cover pebbling [3] and non-split domination [4] we develop a new concept, called the non-split domination cover pebbling number of a graph, denoted by $\psi_{ns}(G)$. In paper [3] "The cover pebbling number, $\lambda(G)$, is defined as the minimum number of pebbles required such that given any initial configuration of at least $\lambda(G)$ pebbles, it is possible to make a series of pebbling moves to place at least one pebble on every vertex of G" and in [4] The domination cover pebbling number, $\gamma(G)$, is defined as "the minimum number of pebbles required so that any initial configuration of pebbles can be shifted by a sequence of pebbling moves so that the set of vertices that contain pebbles form a dominating set S of G". Lourdusamy et al. [5] proposed the concept of covering cover pebbling number. The covering cover pebbling number of a graph of G, $\sigma(G)$, is the minimum number of pebbles required so that any initial configuration of pebbles can be transformed by a sequence of pebbling moves so that the set of vertices that contain pebbles form a covering set of G. Kulli et al. introduced the non-split domination number in [4]. A dominating set D of a graph $G = (V, E)$ is a non-split dominating set if the induced graph $< V - D >$ is connected. The non-split domination number $\gamma_{ns}(G)$ of G is the minimum cordinality of a non-split dominating set. We develop the concept of non-split domination cover pebbling deriving from concept of cover pebbling and non-split domination in graphs. Thus, we arrived the definition of the non-split domination cover(NDC) pebbling number, $\psi_{ns}(G)$, of a graph G as the minimum of pebbles that must be placed on $V(G)$ such that after a sequence of pebbling moves, the set of vertices with a pebble forms a non-split dominating set of G, regardless of the initial configuration of pebbles. In this paper, we use the 'source vertex' where all the pebbles are stocked or shared to move the pebbles to cover the non-split domination set. Source vertices can be one or more than one. From all the source vertices when we move the pebbles, we should cover all the vertices of non-split domination set. The notation $(x_i)\underset{\rightarrow}{t}(x_l)$ refers taking off at least $2t$ pebbles from (x_i) and placing at least t pebbles on (x_l) and the notation $(x_i)\underset{\leftarrow}{t}(x_l)$ refers taking off at least $2t$ pebbles from (x_l) and placing at least t pebbles on (x_i). We discuss the basic results and determine ψ_{ns} for complete graphs, path graphs, wheel graphs, cycle graphs, friendship graphs, comb graphs, banana tree and fire cracker tree.

2. Preliminaries

For graph-theoretic terminologies, the reader can refer to [6, 7].

Theorem 1. [4]

1). The non-split domination number of a complete graph K_n is $\gamma_{ns}(K_n) = 1$.
2). The non-split domination number of a Wheel graph is $\gamma_{ns}(W_n) = 1$.
3). The non-split domination number of a path is $\gamma_{ns}(P_n) = n - 2$.
4). The non-split domination number of Cycle is $\gamma_{ns}(C_n) = n - 2$.

Theorem 2. [8] *The domination cover pebbling number of the wheel graph is $\psi(W_n) = n - 2$.*

Theorem 3. [3] *The cover pebbling number of path P_n is $\gamma(P_n) = 2^n - 1$.*

Conjecture 1. *For a simple connected graph G, $\psi(G) \leq \psi_{ns}(G) \leq \sigma(G)$.*

3. Main Results

Theorem 4. *The non-split domination cover pebbling number of a graph G, $\psi_{ns}(G) = 1$ iff G is a complete graph.*

Proof. The non-split domination number for a complete graph is 1 and hence by placing one pebble on any vertex, we are done. Conversely, if placing a pebble on any vertex produces a non-split domination cover solution, then it implies that the non-split domination number is 1 and hence the graph G is complete. □

Theorem 5. *The non-split domination cover pebbling number of a wheel graph W_n is, $\psi_{ns}(W_n) = n-2$*

Proof. Let $V(W_n) = \{v_0, v_1, v_2, v_3, \cdots, v_{n-1}\}$ where v_0 is called the hub of W_n. Then $E(W_n) = \{v_0v_i, v_jv_{j+1}, v_{n-1}v_1\}$ where $1 \leq i \leq n-1$ and $1 \leq\leq n-2$. Consider the nonsplit dominating set $D = \{v_0\}$. Placing one pebble on each $n - 3$ consecutive outer vertices leaves a vertex of W_n undominated. If there are 2 pebbles on any vertex, shift it to the center. Thus, we cover D. Similarly, if there is a pebble on v_0, we can cover dominate D. Thus, consider all distribution containing pebbled vertices that each contain a pebble. If there are $n-2$ vertices having a pebble each, the 2 outer vertices of W_n are dominated since there are only 3 vertices in all W_n that do not have pebbles. Hence, $\psi_{ns}(W_n) = n-2$. □

Theorem 6. *The NDC pebbling number of path graphs is,* $\psi_{ns}(P_n) = 2^{n-1} + 2^{n-3} - 1$.

Proof. Let $V(P_n) = \{v_1, v_2, v_3, \cdots, v_n\}$. Consider the non-split dominating set $D = \{v_1, v_2, \cdots, v_{n-3}, v_n\}$ or $\{v_1, v_4, \cdots, v_n\}$. In both the non-split dominating set D there will be $(n-2)$ vertices. Then $V(< V-D >) = \{v_{n-2}, v_{n-1}\}$ or $\{v_2, v_3\}$ and $< V-D >$ is connected. Though we have many ways to construct the non-split dominating set, we consider these two because they require minimum number of pebbles to cover the non-split dominating sets.

Now we prove the necessary condition. We place all the pebbles either on v_1 or v_n. Without loss of generality, place $2^{n-1} + 2^{n-3} - 2$ pebbles on v_1. To cover v_n we need 2^{n-1} pebbles. Then with the remaining $2^{n-3} - 2$ pebbles, it is not possible to cover v_1. Hence, $\psi_{ns}(P_n) \geq 2^{n-1} + 2^{n-3} - 1$.

Now we prove the sufficient condition. Consider a configuration C with $2^{n-1} + 2^{n-3} - 1$ pebbles on the vertices of P_n.

Case 1: Let the source vertex be v_n.

If we place all the pebbles on v_n, to cover v_1 we need 2^{n-1} pebbles. The set $\{v_2, v_3\}$ is connected but not a subset of D. Next we need to place a pebble each on $v_4, v_5, \cdots, v_n$. Now using the remaining $2^{n-3} - 1$ pebbles we cover the remaining vertices in the non-split dominating set D. We can shift pebbles as follows: $v_n \xrightarrow{2^{n-4}-1} v_{n-1} \xrightarrow{2^{n-)}-1} \cdots, \xrightarrow{3} v_5 \xrightarrow{1} v_4$ using $2^{n-)} + 2^{n-)} + 2^{n-6} + \cdots + 2^2 + 2^1 + 1 = 2^{n-3} - 1$ pebbles. Hence with the configuration C we cover all the vertices in D.

Case 2: Let the source vertex be either v_4 or v_{n-3}.

Let us place $2^{n-1} + 2^{n-3} - 1$ pebbles on v_4. By Theorem 3, we can cover a path of length $n-3$ from v_4 to v_n using $2^{n-3} - 1$ pebbles. Hence, we are left with 2^{n-1} pebbles. To cover v_1 we require only 8 pebbles. Thus, with a Configuration of at most $2^{(n-3)} + 7$ pebbles we cover the vertices of D. By symmetry, the proof follows for the source vertex v_{n-3}.

Case 3: Let the source vertex be v_k ,$5 \leq k \leq n-4$.

Consider $V(< V-D >) = \{v_2, v_3\}$. When n is even either $v_{\lfloor \frac{n}{2} \rfloor}$ or $v_{\lfloor \frac{n}{2} \rfloor + 1}$ can be taken as a source vertex. If $v_{\lfloor \frac{n}{2} \rfloor}$ is the source vertex, then to the right of $v_{\lfloor \frac{n}{2} \rfloor}$ we have to cover the vertices in a path of length $\lfloor \frac{n}{2} \rfloor + 1$ and to the left of $v_{\lfloor \frac{n}{2} \rfloor}$ we have to cover the vertices in a path of length $\lfloor \frac{n}{2} \rfloor - 3$ excluding the vertex v_1 and the vertices in the graph $< V-D >$. By Theorem 3 we require at most $2^{\lfloor \frac{n}{2} \rfloor + 1} - 1 + 2^{\lfloor \frac{n}{2} \rfloor - 3} - 2$ pebbles to cover the vertices in the above path of length $\lfloor \frac{n}{2} \rfloor + 1$ and $\lfloor \frac{n}{2} \rfloor - 3$. Now to cover v_1 we need $2^{\lfloor \frac{n}{2} \rfloor - 1}$ pebbles. The number of pebbles used in this process is $2^{\lfloor \frac{n}{2} \rfloor + 1} - 1 + 2^{\lfloor \frac{n}{2} \rfloor - 3} - 2 + 2^{\lfloor \frac{n}{2} \rfloor - 1} < 2^{n-1} + 2^{n-3} - 1$ pebbles.

If $v_{\lfloor \frac{n}{2} \rfloor + 1}$ is the source vertex, then to the right of $v_{\lfloor \frac{n}{2} \rfloor + 1}$ we have to cover the vertices in a path of length $\lfloor \frac{n}{2} \rfloor$ and to the left of $v_{\lfloor \frac{n}{2} \rfloor}$ we have to cover the vertices in a path of length $\lfloor \frac{n}{2} \rfloor - 2$ excluding the vertex v_1 and the vertices in the graph $< V-D >$. By Theorem 3 we require at most $2^{\lfloor \frac{n}{2} \rfloor} - 1 + 2^{\lfloor \frac{n}{2} \rfloor - 2} - 2$ pebbles to cover the vertices in the above path of length $\lfloor \frac{n}{2} \rfloor$ and $\lfloor \frac{n}{2} \rfloor - 2$. Now to cover v_1 we need $2^{\lfloor \frac{n}{2} \rfloor}$ pebbles. The number of pebbles used in this process is $2^{\lfloor \frac{n}{2} \rfloor} - 1 + 2^{\lfloor \frac{n}{2} \rfloor - 2} - 2 + 2^{\lfloor \frac{n}{2} \rfloor} < 2^{n-1} + 2^{(n-3)} - 1$ pebbles.

If v_k $(k < \lfloor \frac{n}{2} \rfloor \; or \; k > \lfloor \frac{n}{2} \rfloor + 1)$ is the source vertex, then to the right of v_k we have to cover the the vertices in a path of length $n-k$ and to the left of v_k we have to cover the vertices in a path of length $k-3$ excluding the vertex v_1 and the vertices in the graph $< V-D >$. By Theorem 3 we require at most $2^{n-k} - 1 + 2^{k-3} - 2$ pebbles to cover the vertices in the above path of length $n-k$ and $k-3$. Now to cover v_1 we need 2^k pebbles. The number of pebbles used in this process is $2^{n-k} - 1 + 2^{k-3} - 2 + 2^k < 2^{n-1} + 2^{n-3} - 1$ pebbles.

When n is odd $v_{\lfloor \frac{n}{2} \rfloor+1}$ can be taken as a source vertex. If $v_{\lfloor \frac{n}{2} \rfloor+1}$ is the source vertex, then to the right of $v_{\lfloor \frac{n}{2} \rfloor+1}$ we have to cover the the vertices in a path of length $\lfloor \frac{n}{2} \rfloor$ and to the left of $v_{\lfloor \frac{n}{2} \rfloor+1}$ we have to cover the vertices in a path of length $\lfloor \frac{n}{2} \rfloor - 2$ excluding the vertex v_1 and the vertices in the graph $< V - D >$. By Theorem 3 we require at most $2^{\lfloor \frac{n}{2} \rfloor} - 1 + 2^{\lfloor \frac{n}{2} \rfloor - 2} - 2$ pebbles to cover the vertices in the above path of length $\lfloor \frac{n}{2} \rfloor$ and $\lfloor \frac{n}{2} \rfloor - 2$. Now to cover v_1 we need $2^{\lfloor \frac{n}{2} \rfloor}$ pebbles. The number of pebbles used in this process is $2^{\lfloor \frac{n}{2} \rfloor} - 1 + 2^{\lfloor \frac{n}{2} \rfloor - 2} - 2 + 2^{\lfloor \frac{n}{2} \rfloor} < 2^{n-1} + 2^{n-3} - 1$ pebbles. If v_k ($k < \lfloor \frac{n}{2} \rfloor + 1$ *or* $k > \lfloor \frac{n}{2} \rfloor + 1$) is the source vertex, then to the right of v_k we have to cover the the vertices in a path of length $n - k$ and to the left of v_k we have to cover the vertices in a path of length $k - 3$ excluding the vertex v_1 and the vertices in the graph $< V - D >$. By Theorem 3 we require at most $2^{n-k} - 1 + 2^{k-3} - 2$ pebbles to cover the vertices in the above path of length $n - k$ and $k - 3$. Now to cover v_1 we need 2^k pebbles. The number of pebbles used in this process is $2^{n-k} - 1 + 2^{k-3} - 2 + 2^k < 2^{n-1} + 2^{n-3} - 1$. Hence, $\psi_{ns}(P_n) = 2^{n-1} + 2^{n-3} - 1$.

□

3.1. NDC Pebbling Number for Cycle Related Graphs

Theorem 7. *The NDC pebbling number of cycle graphs is,*

$$\psi_{ns}(C_n) = \begin{cases} 2^{\lfloor \frac{n-1}{2} \rfloor} - 1 + 2^{\lfloor \frac{n}{2} \rfloor} - 2 & n \text{ is even}, \\ 2^{\lfloor \frac{n}{2} \rfloor + 1} - 3 & n \text{ is odd}. \end{cases}$$

Proof. Let $V(C_n) = \{v_1, v_2, v_3, \cdots, v_n\}$. When n is even, the non-split dominating set $D_1 = \{v_1, v_2, \cdots, v_{\lfloor \frac{n}{2} \rfloor - 1}, v_{\lfloor \frac{n}{2} \rfloor + 2}, v_{\lfloor \frac{n}{2} \rfloor + 3}, \cdots, v_{n-1}, v_n\}$. When n is odd, the non-split dominating set is $D_2 = \{v_1, v_2, \cdots, v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor + 3}, v_{\lfloor \frac{n}{2} \rfloor + 4}, \cdots, v_{n-1}, v_n\}$.

We observe that $V(< V - D_1 >) = \{v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor + 1}\}$ and $V(< V - D_2 >) = \{v_{\lfloor \frac{n}{2} \rfloor + 1}, v_{\lfloor \frac{n}{2} \rfloor + 2}\}$. Notice that there exist two paths from v_1 to $v_{\lfloor \frac{n}{2} \rfloor - 1}$ and from v_1 to $v_{\lfloor \frac{n}{2} \rfloor + 2}$. Let them be $P_{\lfloor \frac{n-1}{2} \rfloor}$ and $P_{\lfloor \frac{n}{2} \rfloor}$, when n is even. When n is odd, there exist two paths from v_1 to $v_{\lfloor \frac{n}{2} \rfloor}$ and from v_1 to $v_{\lfloor \frac{n}{2} \rfloor + 3}$. Let them be $P_{\lfloor \frac{n}{2} \rfloor}$ and $P_{\lfloor \frac{n}{2} \rfloor}$ respectively.

Case 1: n is even.

Placing $2^{\lfloor \frac{n-1}{2} \rfloor} - 1 + 2^{\lfloor \frac{n}{2} \rfloor} - 3$ pebbles on the source vertex v_1 we cannot put one pebble each on all the vertices of D_1. Hence, $\psi_{ns}(C_n) \geq 2^{\lfloor \frac{n-1}{2} \rfloor} - 1 + 2^{\lfloor \frac{n}{2} \rfloor} - 2$.

Now we prove the sufficient condition. Consider a configuration of $2^{\lfloor \frac{n-1}{2} \rfloor} - 1 + 2^{\lfloor \frac{n}{2} \rfloor} - 2$ pebbles. Let the source vertex be v_1. Using Theorem 3, we can cover the non-split dominating set D_1. Since D_1 has two paths of length $(\left\lfloor \frac{n}{2} \right\rfloor - 2)$ and $(\left\lfloor \frac{n}{2} \right\rfloor - 1)$, using $2^{\lfloor \frac{n}{2} \rfloor - 1} - 1$ pebbles we can cover all the vertices of the path of length $\left\lfloor \frac{n}{2} \right\rfloor - 2$ and using $2^{\lfloor \frac{n}{2} \rfloor} - 2$ pebbles we cover all the vertices of the path of length $\left\lfloor \frac{n}{2} \right\rfloor - 1$. The total number of pebbles used to cover the vertices of D_1 is $2^{\lfloor \frac{n}{2} \rfloor} - 1 + 2^{\lfloor \frac{n}{2} \rfloor} - 2$.

If we distribute all the pebbles on the vertices of $< V - D_1 >$, we need at least $2^{\lfloor \frac{n}{2} \rfloor} - 2$ pebbles whose pebbling process will result in another non-split dominating set D. The new $< V - D >$ is also connected. Similarly, we can prove the result for other source vertices. Hence, $\psi_{ns}(C_n) = 2^{\lfloor \frac{n-1}{2} \rfloor} - 1 + 2^{\lfloor \frac{n}{2} \rfloor} - 2$.

Case 2: n is odd.

Placing $2^{\lfloor \frac{n}{2} \rfloor + 1} - 4$ pebbles on the source vertex v_1 we cannot put one pebble each on all the vertices of D_2. Hence, $\psi_{ns}(C_n) \geq 2^{\lfloor \frac{n}{2} \rfloor + 1} - 3$.

Now we prove the sufficient condition. Consider a configuration C with $2^{\lfloor \frac{n}{2} \rfloor + 1} - 3$ pebbles on the vertices of C_n. Let the source vertex be v_1. Using Theorem 3 of the cover pebbling number of path, we can cover the non-split dominating set D_2. Note that D_2 has two paths of equal length $(\left\lfloor \frac{n}{2} \right\rfloor - 1)$. Using $2^{\lfloor \frac{n}{2} \rfloor} - 1$ pebbles we can cover all the vertices of the path of length $\left\lfloor \frac{n}{2} \right\rfloor - 1$ and

$2^{\lfloor \frac{n}{2} \rfloor} - 2$ pebbles we cover all the vertices of path of length $\lfloor \frac{n}{2} \rfloor - 1$. The total number of pebbles used to cover the D_2 is $2^{\lfloor \frac{n}{2} \rfloor + 1} - 3$.
If we distribute all the pebbles on the vertices of $< V - D_2 >$, we need at least $2^{\lfloor \frac{n}{2} \rfloor - 1} - 1 + 2^{\lfloor \frac{n}{2} \rfloor} - 1$ pebbles to form another non-split dominating set D. The new $< V - D >$ is also connected. Similarly, we can prove for the rest of the vertices assuming as source vertices. Hence, $\psi_{ns}(C_n) = 2^{\lfloor \frac{n}{2} \rfloor + 1} - 3$.

□

Theorem 8. *The NDC pebbling number for a friendship graph Fr_n is given by, $\psi_{ns}(Fr_n) = 4(n-1)+1$.*

Proof. Let the hub vertex be denoted by v_0 and the vertices which are adjacent to v_0 be denoted by $v_1, v_2, ..., v_{2n}$(clockwise manner).

Consider the non-split dominating set $D_1 = \{v_1, v_3, \cdots, v_{2n-1}\}$ or $D_2 = \{v_2, v_4, \cdots, v_{2n}\}$. Clearly the induced sub-graph $< V - D_1 >$ and $< V - D_2 >$ are connected. Without loss of generality, let us consider the non-split dominating set D_1. Consider the configuration of $4(n-1)$ pebbles on the vertex of v_1. Shifting $2(n-1)$ pebbles to the hub vertex we could cover maximum $n-1$ vertices of the non-split domination set. We are left with a vertex v_1 without a cover. Hence, $\psi_{ns}(Fr_n) \geq 4(n-1)+1$.

Now we prove the sufficient condition by distributing $4(n-1)+1$ pebbles on the vertices of Fr_n, that is, Hub vertex has minimum of $2(n-1)+2$ pebbles on it.

Using $2(n-1)$ pebbles we can cover dominate $n-1$ vertices of the non-split dominating set D_1. Further, using 2 pebbles we could cover the remaining vertex. Suppose, the hub vertex has less than $2(n-1)+2$. Let it be p pebbles. Then using p pebbles we could cover $\lfloor \frac{p}{2} \rfloor$ vertices. Assume that p' is the number of vertices receiving pebbles in this way. Keep a maximum of two pebbles on each vertex and transfer the remaining to the hub vertex. Thus, Thus using at least $2n$ pebbles we can cover dominate D_1. Hence, $\psi_{ns}(Fr_n) = 4(n-1)+1$.

□

3.2. NDC Pebbling Number for Some Families of Trees

Theorem 9. *The NDC pebbling number of a comb graph $P_n \odot K_1$ is given by, $\psi_{ns}(P_n \odot K_1) = \sum_{i=0}^{n+1} 2^i - 6$.*

Proof. Let the vertices of the path $P_n \odot K_1$ be denoted by $v_1, v_2, ..., v_n$ and the pendant vertices attached to each vertex $v_i, 1 \leq i \leq n$ on the path be $u_i, 1 \leq i \leq n$.

Consider the non-split dominating set $D = \{u_1, u_2, \cdots, u_n\}$. Clearly, $< V - D >$ is connected. Consider the configuration of $\sum_{i=0}^{n+1} 2^i - 7$ pebbles placed on the vertex u_1. Then, a minimum of $1 + 2^3 + 2^4 + \cdots + 2^{n-1} + 2^n + 2^{n+1}$ pebbles are required in order to produce a non-split dominating set cover. But we lack one pebble to cover u_1. Therefore, $\psi_{ns}(P_n \odot K_1) \geq \sum_{i=0}^{n+1} 2^i - 6$.

Case 1: Let the source vertex be v_1 or v_n.
Without loss of generality, let the source vertex be v_1. To cover the vertex u_1 we require 2 pebbles. Then, to cover the remaining vertices of $\{D - u_1\}$ we need $\sum_{i=0}^{n} 2^i - 3$ pebbles. Thus, the total number of pebbles used to cover the vertices of the non-split dominating set is $\sum_{i=0}^{n} 2^i - 1 \leq \sum_{i=0}^{n+1} 2^i - 6$. By symmetry, we can prove when v_n is the source vertex.

Case 2: Let the source vertex be v_k, $1 \leq k \leq n$.
Let v_k be the source vertex. To cover the adjacent vertex u_k we need 2 pebbles. Now to cover the remaining vertices $u_{k-1}, u_{k-2}, \cdots, u_2, u_1, u_{k+1}, \cdots, u_{n-1}, u_n$ of D we need $\sum_{i=1}^{k} 2^i + \sum_{i=1}^{n-(k-1)} 2^i$ pebbles, which is less than the total number of available pebbles.

Case 3: Let the source vertex be u_k, $1 \leq k \leq n$.
Let u_k be the source vertex. To cover the vertex u_k we need 1 pebble. Now to cover the remaining vertices $u_{k-1}, u_{k-2}, \cdots, u_2, u_1, u_{k+1}, \cdots, u_{n-1}, u_n$ of D we need $\sum_{i=1}^{k+1} 2^i + \sum_{i=1}^{n-(k-2)} 2^i + 1 \leq \sum_{i=0}^{n+1} 2^i - 6$ pebbles. Hence, $\psi_{ns}(P_n \odot K_1) = \sum_{i=0}^{n+1} 2^i - 6$.

□

Theorem 10. *The NDC pebbling number for Banana tree $B_{n,k}$ is, $\psi_{ns}(B_{n,k}) = 64(n-1)(k-2)+32(n-1)+4(k-2)+3$.*

Proof. Let v_0 be the vertex that joins all the k- star graphs. Let $V(B_{n,k}) = \{v_0,\ a_1^j,\ a_2^j,\ \cdots,\ a_{k-1}^j, a_0^j\}$, where $j = \{1,\ 2,\ 3,\ 4, \cdots,\ n\}$ and $E(B_{n,k}) = \{v_0 a_{k-1}^j, a_0^j a_i^j\}$, where $j = \{1,\ 2,\ 3,\ 4, \cdots,\ n\}$ and $1 = \{1,\ 2,\ 3,\ 4, \cdots,\ k-1\}$. Let the non-split dominating set $D = \{v_0,\ a_1^j,\ a_2^j,\ \cdots,\ a_{k-2}^j, a_0^j, a_{k-1}^1\}$, where $j = \{1,\ 2,\ 3,\ 4, \cdots,\ n\}$. Clearly, the induced sub-graph $< V - D >$ is connected.

Consider the distribution of $64(n-1)(k-2) + 32(n-1) + 4(k-2) + 2$ pebbles on any one of the pendant vertices. Let it be a_1^1. Then we could cover all the vertices of the dominating set D except one vertex. Hence, $\psi_{ns}(B_{n,k}) \geq 64(n-1)(k-2) + 32(n-1) + 4(k-2) + 3$.

Now consider distributing $64(n-1)(k-2) + 32(n-1) + 4(k-2) + 3$ pebbles on the vertices of $(B_{n,k})$.

Case 1: Let v_0 be the source vertex.
There are n copies of $(k-2)$ star graph pendant vertices at a distance of 3 from v_0 and n copies of the hub vertex of the k-star at a distance of 2. Thus, using $8n(k-2)+4n$ pebbles, we could cover all the vertices of D except a_{k-1}^1 which requires further 2 pebbles. Hence, using $8n(k-2)+4n+2$ pebbles, we can dominate the set D.

Case 2: Let any one of the hub vertex of the k- star graph be the source vertex.
Without loss of generality, let a_0^1 be the source vertex. To cover the dominating set of vertices that are adjacent to a_0^1 we require $2(k-1)$ pebbles. The rest of the vertices are at distances of 5 and 4. There are $(n-1)$ copies of $(k-2)$ star graph of pendant vertices is at a distance of 5 and $n-1$ copies of the hub vertex of the star graph are at a distance of 4. Thus, using $32(n-1)(k-2) + 16(n-1) + 2(k-1)$ pebbles, we can cover the non-split dominating set D. Hence, $\psi_{ns}(B_{n,k}) = 64(n-1)(k-2) + 32(n-1) + 4(k-2) + 3$.

□

Theorem 11. *The NDC pebbling number for $F_{n,k}$- fire cracker tree is, $\psi_{ns}(F_{n,k}) = 4(k-3)+3+16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right) + 8\left(\sum_{i=1}^{n-1} 2^i\right)$.*

Proof. Let $V(F_{n,k}) = \{a_i, b_i, c_{ij} | i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, k-2\}$ *and* $E(F_{n,k}) = \{a_i a_{i+1}|\ i = 1, 2, \cdots, n-1\} \cup \{a_i b_i,\ b_i c_{ij}\ | i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, k-2\}$. Consider the non-split dominating set $D = \{b_i, c_{ij}\ | i = 1, 2, \cdots, n; j = 1, 2, \cdots,\ k-2\}$. Clearly, $< V - D >$ is connected.

Consider the distribution of placing $4(k-3)+2+16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+8\left(\sum_{i=1}^{n-1} 2^i\right)$ pebbles on a pendant vertex of the first k star graph of the $(F_{n,k})$. Then, we require a minimum of $16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+8\left(\sum_{i=1}^{n-1} 2^i\right)$ pebbles to cover vertices of the non-split dominating set in $(n-1)$ k-star graphs. And the vertices in the first $(k-1)$ star graph are not covered. There are $k-3$ vertices at a distance 2 and one vertex at a distance one in the first $k-1$ star graph. Hence, we require further $4(k-3)+3$ pebbles. But the number of pebbles remaining is $4(k-3)+2$. Thus, $\psi_{ns}(F_{n,k} \geq 4(k-3)+3+16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+8\left(\sum_{i=1}^{n-1} 2^i\right)$.

Now we prove the sufficient condition. Consider a configuration of $4(k-3)+3+16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+ 8\left(\sum_{i=1}^{n-1} 2^i\right)$ pebbles.

Case 1: Let b_1 or b_n be a source vertex.

Without loss of generality, consider b_1 be the source vertex. Let us place $4(k-3)+3+16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+8\left(\sum_{i=1}^{n-1} 2^i\right)$ pebbles on b_1. Since there are $k-2$ vertices of the non-split dominating set that are adjacent to b_1, we need $2(k-2)$ pebbles to cover them and one more pebble needed to cover b_1. Now we are left with vertices in the $(n-1)$ parts of $(k-1)$ star graphs that are to be covered. To cover all those vertices we require $8(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+4\left(\sum_{i=1}^{n-1} 2^i\right)$ pebbles. Thus, the total number of pebbles used to cover the vertices of the non-split dominating set is $2(k-2)+1+8(k-2)+\left(\sum_{i=1}^{n-1} 2^i\right)+4\left(\sum_{i=1}^{n-1} 2^i\right) \leq 4(k-3)+3+16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+8\left(\sum_{i=1}^{n-1} 2^i\right)$. By symmetry, we can prove this for the source vertex b_n.

Case 2: Let b_s $(2 \leq s \leq n-1)$ be a source vertex.

Let us place $4(k-3)+3+16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+8\left(\sum_{i=1}^{n-1} 2^i\right)$ pebbles on b_s. Since there are $k-2$ vertices of the non-split dominating set that are adjacent to b_s, we need $2(k-2)$ pebbles to cover them and one more pebble to cover b_s. Now we are left with the vertices that are not covered in the $n-1$ parts of $(k-1)$ star graphs. To cover all those vertices we require $8(k-2)\left(\sum_{i=1}^{n-s} 2^i\right)+4\left(\sum_{i=1}^{n-s} 2^i\right)+8(k-2)\left(\sum_{i=1}^{s-1} 2^i\right)+4\left(\sum_{i=1}^{s-1} 2^i\right)$ pebbles. Thus, the total number of pebbles used to cover the vertices of non-split dominating set is $2(k-2)+1+8(k-2)+8(k-2)\left(\sum_{i=1}^{n-s} 2^i\right)+4\left(\sum_{i=1}^{n-s} 2^i\right)+8(k-2)\left(\sum_{i=1}^{s-1} 2^i\right)+4\left(\sum_{i=1}^{s-1} 2^i\right) \leq 4(k-3)+3+16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+8\left(\sum_{i=1}^{n-1} 2^i\right)$.

Case 3: Let a_1 or a_n be a source vertex.

Without loss of generality, consider a_1 the source vertex. Let us place $4(k-3)+3+16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+8\left(\sum_{i=1}^{n-1} 2^i\right)$ pebbles on a_1. Since there are $k-2$ vertices of the non-split dominating set that are at distance 2 and one vertex that is adjacent to a_1, we need $4(k-2)+2$ pebbles to cover them. Now we are left with the vertices that are not covered in the $n-1$ parts of $(k-1)$ star graphs. To cover all those vertices, we require $4(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+2\left(\sum_{i=1}^{n-1} 2^i\right)$ pebbles. Thus, the total number of pebbles used to cover the vertices of the non-split dominating set is $4(k-2)+2+4(k-2)+\left(\sum_{i=1}^{n-1} 2^i\right)+2\left(\sum_{i=1}^{n-1} 2^i\right) \leq 4(k-3)+3+16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+8\left(\sum_{i=1}^{n-1} 2^i\right)$. By symmetry, we can prove the result for the source vertex a_n.

Case 4: Let a_s $(2 \leq s \leq n-1)$ be a source vertex.

Let us place $4(k-3)+3+16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right)+8\left(\sum_{i=1}^{n-1} 2^i\right)$ pebbles on a_s. Since there are $k-2$ vertices of the non-split dominating set that are at distance 2 and one vertex that is adjacent to a_s, we need $4(k-2)+2$ pebbles to cover them. Now we are left with vertices that are not covered in the $n-1$ parts of the $(k-1)$ star graphs. To cover all those vertices, we require $4(k-2)\left(\sum_{i=1}^{n-s} 2^i\right)+2\left(\sum_{i=1}^{n-s} 2^i\right)+4(k-2)\left(\sum_{i=1}^{s-1} 2^i\right)+2\left(\sum_{i=1}^{s-1} 2^i\right)$ pebbles. Thus, the total number of pebbles used to cover the vertices of non-split dominating set is $4(k-2)+2+4(k-2)+8(k-2)\left(\sum_{i=1}^{n-s} 2^i\right)+$

$2\left(\sum_{i=1}^{n-s} 2^i\right) + 4(k-2)\left(\sum_{i=1}^{s-1} 2^i\right) + 2\left(\sum_{i=1}^{s-1} 2^i\right) \leq 4(k-3) + 3 + 16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right) + 8\left(\sum_{i=1}^{n-1} 2^i\right)$. Hence, $\psi_{ns}(F_{n,k} = 4(k-3) + 3 + 16(k-2)\left(\sum_{i=1}^{n-1} 2^i\right) + 8\left(\sum_{i=1}^{n-1} 2^i\right)$.

□

4. Conclusion

In this paper, we introduced the graph invariant, namely, the ‘non-split domination cover pebbling number’. Some basic results are discussed. Also, the NDC pebbling numbers for certain families of graphs, such as the complete graph, wheel graph, path, cycle, friendship graph, comb graph, banana tree, and ($F_{n,k}$)- fire cracker tree, are determined. Finding the NDC pebbling numbers for other families of graphs is still open.

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Conflict of Interest

The authors declare no conflict of interests.

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